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AXISYMMETRIC WAVES IN PRESTRESSED HIGHLY ELASTIC COMPOSITE MATERIAL. LONG WAVE APPROXIMATION*

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The statement and method of solving problems on the propagation of axisymmetric harmonic waves in a highly elastic laminated composite material are considered within the framework of the linearized theory of elasticity for prestressed bodies. The case of wave propagation along the layers of a composite material is investigated. The dispersion equations for quasi-transverse waves and their long-wave approximations are obtained in the cases of perfect bonding and free sliding of the layers. The phase velocity of long waves is studied for a material with a Treloar elastic potential.

Keywords: incompressible laminated composite material, elastic wave, initial stresses, dispersion equation, long wave approximation

Introduction. Numerous articles [1–4, 8–14, etc.] are devoted to dynamic processes in prestressed bodies, including layered materials. The results are also presented in different monographs [5–7, etc.].

Problems on the radial propagation of axisymmetric waves and torsion waves in layered composite materials of periodic structure with perfectly bonded layers were considered in [5, 7, 10, 14].

Using the general solutions [5, 6, 11], we will obtain the general dispersion equations for compressible and incompressible materials and the long-wave approximation for quasi-longitudinal and quasi-transverse waves, axisymmetric waves and symmetric and antisymmetric torsional waves. The dispersion equations were analyzed numerically for relatively rigid materials with Murnaghan potential.

These results are best exposed in the monographs [5, 7].

Similar analytical results for axisymmetric waves in the case of free sliding of layer were obtained in [1–4, 9]. Numerical analysis in a wide range of frequencies was performed for highly elastic materials with Treloar potential.

The method described in [5, 7] was used in [1–5, 7, 9–11, 14].

In this paper, the problem of axisymmetric wave propagation in a prestressed laminated incompressible composite material is solved using the three-dimensional linearized theory of elasticity for prestressed bodies. We will numerically analyze the dispersion relations for long quasi-transverse waves and study the influence of the initial stresses on the speed of propagation.

1. Problem Statement. When studying dynamic axisymmetric processes in prestressed laminated composite materials, we use Lagrangian coordinates $y_n \equiv y^n$, which, in the initial stress–strain state, coincide with the Cartesian and Lagrangian coordinates r', θ, y_3, which, in the initial stress–strain state, coincide with cylindrical coordinates. The coordinates $y_n \equiv y^n$ and r', θ, y_3 are related in an ordinary manner. We will use an extended problem statement, taking into account that in the initial

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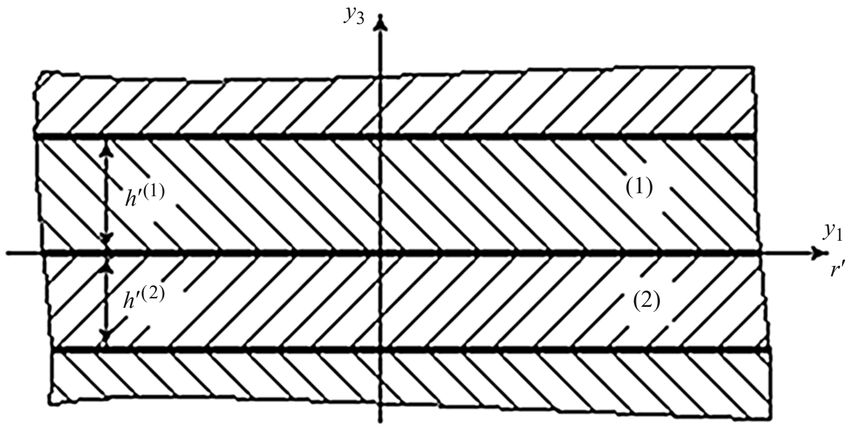

Fig. 1

stress–strain state the individual components of the composite material can be deformed differently; i. e., we assume that the individual components of the composite material are not bonded to each other in the initial stress–strain state and come into complete contact only after the occurrence of the stress–strain state. We introduce a Cartesian coordinate system in the initial stress–strain state so that the axis y_3 is normal to the interfaces between the layers (Fig. 1).

The materials of the layers are considered to be isotropic hyperelastic and to have arbitrary elastic potentials; in the case of transversely isotropic hyperelastic layers, we assume that the axis of isotropy is directed along the axis Oy_3. In addition, we consider prestressed laminated composite materials that consist of alternating layers of two types. Each of these types is characterized by the same material and initial stress–strain state.

We believe that the initial stress state is homogeneous and is determined by the following relations:

$$u_m^{0(j)} = (\lambda_m^{(j)} - 1)x_m, \quad \lambda_m^{(j)} = \text{const}, \quad j = 1, 2. \tag{1.1}$$

We also assume that the following relations are valid for each of the layers:

$$S_{11}^{0(j)} = S_{22}^{0(j)} \neq S_{33}^{0(j)}, \quad \sigma_{11}^{0(j)} = \sigma_{22}^{0(j)} \neq \sigma_{33}^{0(j)},$$

$$\lambda_1^{(j)} = \lambda_2^{(j)}, \quad h'^{(j)} = \lambda_3^{(j)} h^{(j)}, \quad j = 1, 2. \tag{1.2}$$

In (1.1) and (1.2), the indices in parentheses ($j = 1, 2$) refer to the two types of layers; $S_{tt}^{0(j)}$ are the components of the generalized Lagrangian strain tensor; $h'^{(j)}$ is the thickness of the jth layer in the initial stress–strain state; $h^{(j)}$ is the thickness of the jth layer in the natural state; $\lambda_m^{(j)}$ are the elongation coefficients along the respective axes; $u_m^{0(j)}$ are the displacements.

We assume that in the general solutions of the spatial dynamic linearized problem of elasticity for prestressed bodies:

$$u_{r'}^{(j)} = u_{r'}^{(j)}(r', y_3, \tau), \quad u_\theta^{(j)} \equiv 0, \quad u_3^{(j)} = u_3^{(j)}(r', y_3, \tau),$$

$$u_4^{(j)} \equiv p^{(j)} = p^{(j)}(r', y_3, \tau), \quad j = 1, 2. \tag{1.3}$$

In this case, we can also assume that in the general solutions of spatial dynamic linearized problems of elasticity with respect to the general solution of the axisymmetric problem in cylindrical coordinates

$$\Psi'^{(j)} \equiv 0, \quad \chi'^{(j)} = \chi'^{(j)}(r', y_3, \tau), \quad j = 1, 2. \tag{1.4}$$

Taking into account (1.4), let us write down the basic relations for incompressible bodies.

For determining the displacements taking into account (1.4), we obtain

$$u_{r'}^{(j)} = -\frac{\partial^2}{\partial r' \partial y_3}\chi'^{(j)}, \quad u_3^{(j)} = \Delta_1' \chi'^{(j)}, \quad \rho'^{(j)} = \rho^{(j)},$$

$$u_4^{(j)} \equiv p^{(j)} = \left[\left(\kappa'^{(j)}_{1111} - \kappa'^{(j)}_{1133} - \kappa'^{(j)}_{1313} \right) \Delta_1' + \kappa'^{(j)}_{3113} \frac{\partial^2}{\partial y_3^2} - \rho'^{(j)} \frac{\partial^2}{\partial \tau^2} \right] \frac{\partial}{\partial y_3} \chi'^{(j)}, \quad j = 1, 2. \tag{1.5}$$

Taking into account (1.4), the components of the stress tensor $Q'^{(j)}$, when $y_3 = \text{const}$, can be written as

$$Q'^{(j)}_{33} = \left[\left(\kappa'^{(j)}_{1111} + \kappa'^{(j)}_{3333} - 2\kappa'^{(j)}_{1133} - \kappa'^{(j)}_{1313} \right) \Delta_1' + \kappa'^{(j)}_{3113} \frac{\partial^2}{\partial y_3^2} - \rho'^{(j)} \frac{\partial^2}{\partial \tau^2} \right] \frac{\partial}{\partial y_3} \chi'^{(j)},$$

$$Q'^{(j)}_{3r'} = \left(\kappa'^{(j)}_{1313} \Delta_1' - \kappa'^{(j)}_{3113} \frac{\partial^2}{\partial y_3^2} \right) \frac{\partial}{\partial r'} \chi'^{(j)}, \quad j = 1, 2. \tag{1.6}$$

To determine the functions $\chi'^{(j)}$ under condition (1.4) we have

$$\left[\left(\Delta_1' + \xi_2'^{(j)2} \frac{\partial^2}{\partial y_3^2} \right) \left(\Delta_1' + \xi_3'^{(j)2} \frac{\partial^2}{\partial y_3^2} \right) - \frac{\rho'^{(j)}}{\kappa'^{(j)}_{1331}} \left(\Delta_1' + \frac{\partial^2}{\partial y_3^2} \right) \frac{\partial^2}{\partial \tau^2} \right] \chi'^{(j)} = 0, \quad j = 1, 2. \tag{1.7}$$

The values of $\xi_2'^{(j)}$ and $\xi_3'^{(j)}$ in (1.7) are determined in the same way as in [9], for the materials of each of the layers under consideration; $\rho'^{(j)}$ is the density of the material of each layer.

Consider two cases of contact between layers of composite material:

perfect bonding

$$u'^{(1)}_{r'}(0) = u'^{(2)}_{r'}(0), \quad u'^{(1)}_3(0) = u'^{(2)}_3(0),$$

$$Q'^{(1)}_{33}(0) = Q'^{(2)}_{33}(0), \quad Q'^{(1)}_{3r'}(0) = Q^{(2)}_{3r'}(0), \tag{1.8}$$

and free sliding

$$u'^{(1)}_3(0) = u'^{(2)}_3(0), \quad Q'^{(1)}_{33}(0) = Q'^{(2)}_{33}(0),$$

$$Q'^{(1)}_{3r'}(0) = 0, \quad Q^{(2)}_{3r'}(0) = 0. \tag{1.9}$$

Due to the periodicity of the structure, the conditions of the Floquet theorem must also be satisfied:

in the case of perfect bonding

$$u'^{(1)}_{r'}(h^{(1)}) = u'^{(2)}_{r'}(-h^{(2)}), \quad u'^{(1)}_3(h^{(1)}) = u'^{(2)}_3(-h^{(2)}),$$

$$Q'^{(1)}_{33}(h^{(1)}) = Q'^{(2)}_{33}(-h^{(2)}), \quad Q'^{(1)}_{3r'}(h^{(1)}) = Q^{(2)}_{3r'}(-h^{(2)}), \tag{1.10}$$

in the case of free sliding

$$u'^{(1)}_3(h^{(1)}) = u'^{(2)}_3(-h^{(2)}), \quad Q'^{(1)}_{33}(h^{(1)}) = Q'^{(2)}_{33}(-h^{(2)}),$$

$$Q'^{(1)}_{3r'}(h^{(1)}) = 0, \quad Q^{(2)}_{3r'}(-h^{(2)}) = 0. \tag{1.11}$$

Thus, the study of the propagation laws of axisymmetric elastic waves in prestressed incompressible laminated composite materials is reduced to the solutions of Eqs. (1.7) subject to the interface conditions (1.8) or (1.9) and periodicity conditions (1.10) or (1.11), corresponding to the Floquet theorem. When satisfying the boundary conditions and the periodicity conditions, it is necessary to use expressions (1.5) and (1.6).

2. Solution Method. Consider the radial propagation of an axisymmetric wave in prestressed incompressible laminated composite materials. We will only consider the case of a wave that goes "to infinity." To determine the "true" phase velocity of axisymmetric waves in a prestressed laminated composite material, as in [5, 7], we assume

$$\chi'^{(j)}\left(r', y_3, \tau\right) = \chi'^{(j)(0)}\left(y_3\right) H_0^{(1)}\left(r'k\right) e^{-i\omega\tau}, \quad C = \omega k^{-1}, \quad j = 1, 2. \tag{2.1}$$

In (2.1), k and ω are the wave number and angular frequency; C is a "true" phase velocity of axisymmetric waves; $H_0^{(1)}(x)$ is the Hankel function of the first kind and zero order, the function ensures the propagation of axisymmetric waves going "to infinity;" $\chi'^{(j)(0)}(y_3)$ is the amplitude function.

Substituting (2.1) into (1.5), we obtain the following expressions for the displacements:

$$u_{r'}^{(j)} = u_{r'}^{(j)(0)} \frac{d}{dr'} H_0^{(1)}\left(r'k\right) e^{-i\omega\tau}, \quad u_3^{(j)} = u_3^{(j)(0)} H_0^{(1)}\left(r'k\right) e^{-i\omega\tau},$$

$$u_4^{(j)} \equiv p^{(j)} = p^{(j)(0)} H_0^{(1)}\left(r'k\right) e^{-i\omega\tau},$$

$$u_{r'}^{(j)(0)} = -\frac{d}{dy_3} \chi'^{(j)(0)}\left(y_3\right), \quad u_3^{(j)(0)} = -k^2 \chi'^{(j)(0)}\left(y_3\right),$$

$$p^{(j)(0)} = \left[-k^2\left(\kappa'^{(j)}_{1111} - \kappa'^{(j)}_{1133} - \kappa'^{(j)}_{1313}\right) + \kappa'^{(j)}_{3113} \frac{d^2}{dy_3^2} + \omega^2 \rho'^{(j)}\right] \frac{\partial}{\partial y_3} \chi'^{(j)(0)}\left(y_3\right), \quad j = 1, 2. \tag{2.2}$$

Similarly, substituting (2.1) into (1.6), to determine the components of the stress tensor $Q'^{(j)}$ when $y_3 =$ const we get

$$Q'^{(j)}_{33} = Q'^{(j)(0)}_{33} H_0^{(1)}\left(r'k\right) e^{-i\omega\tau}, \quad Q'^{(j)}_{3r'} = Q'^{(j)(0)}_{3r'} \frac{\partial}{\partial r'} H_0^{(1)}\left(r'k\right) e^{-i\omega\tau},$$

$$Q'^{(j)(0)}_{33} = \left[-k^2\left(\kappa'^{(j)}_{1111} + \kappa'^{(j)}_{3333} - 2\kappa'^{(j)}_{1133} - \kappa'^{(j)}_{1313}\right) + \kappa'^{(j)}_{3113} \frac{d^2}{dy_3^2} + \omega^2 \rho'^{(j)}\right] \frac{d}{dy_3} \chi'^{(j)(0)}\left(y_3\right),$$

$$Q'^{(j)(0)}_{3r'} = -\left(k^2 \kappa'^{(j)}_{1313} + \kappa'^{(j)}_{3113} \frac{d^2}{dy_3^2}\right) \chi'^{(j)(0)}\left(y_3\right), \quad j = 1, 2. \tag{2.3}$$

Substituting (2.1) into (1.7), we obtain the following equation for determining the functions $\chi'^{(j)(0)}(y_3)$:

$$\left[\left(\xi'^{(j)2}_2 \frac{d^2}{dy_3^2} - k^2\right)\left(\xi'^{(j)2}_3 \frac{d^2}{dy_3^2} - k^2\right) + \frac{\omega^2 \rho'^{(j)}}{\kappa'^{(j)}_{1331}}\left(\frac{d^2}{dy_3^2} - k^2\right)\right] \chi'^{(j)(0)}\left(y_3\right) = 0, \quad j = 1, 2. \tag{2.4}$$

Since in (2.2)–(2.4), all relations are written for amplitudes, the conditions of continuity and periodicity are formulated for amplitudes as well. In this regard, we select adjacent layers (Fig. 1) and assume that the layer denoted by index "1" occupies the domain $0 \le y_3 \le h'^{(1)}$ along the axis Oy_3 and the layer denoted by index "2" occupies the domain $-h'^{(2)} \le y_3 \le 0$ along the axis Oy_3. The conditions of continuity of the stress and displacement vectors, which are formulated for the corresponding amplitudes, must be satisfied in the following form in the case of perfect bonding and $y_3 = 0$:

$$u'^{(1)(0)}_{r'}(0) = u'^{(2)(0)}_{r'}(0), \quad u'^{(1)(0)}_3(0) = u'^{(2)(0)}_3(0),$$

$$Q'^{(1)(0)}_{33}(0)=Q'^{(2)(0)}_{33}(0), \quad Q'^{(1)(0)}_{3r'}(0)=Q'^{(2)(0)}_{3r'}(0), \tag{2.5}$$

and in the case of free sliding:

$$u'^{(1)(0)}_{3}(0)=u'^{(2)(0)}_{3}(0), \quad Q'^{(1)(0)}_{33}(0)=Q'^{(2)(0)}_{33}(0),$$

$$Q'^{(1)(0)}_{3r'}(0)=0, \quad Q^{(2)(0)}_{3r'}(0)=0. \tag{2.6}$$

Due to the periodicity of this structure, according to the Floquet theorem, the periodicity conditions for amplitude must also be satisfied:

in the case of perfect bonding:

$$u'^{(1)(0)}_{r'}(h^{(1)})=u'^{(2)(0)}_{r'}(-h^{(2)}), \quad u'^{(1)(0)}_{3}(h^{(1)})=u'^{(2)(0)}_{3}(-h^{(2)}),$$

$$Q'^{(1)(0)}_{33}(h^{(1)})=Q'^{(2)(0)}_{33}(-h^{(2)}), \quad Q'^{(1)(0)}_{3r'}(h^{(1)})=Q^{(2)(0)}_{3r'}(-h^{(2)}), \tag{2.7}$$

and in the case of free sliding:

$$u'^{(1)(0)}_{3}(h^{(1)})=u'^{(2)(0)}_{3}(-h^{(2)}), \quad Q'^{(1)(0)}_{33}(h^{(1)})=Q'^{(2)(0)}_{33}(-h^{(2)}),$$

$$Q'^{(1)(0)}_{3r'}(h^{(1)})=0, \quad Q^{(2)(0)}_{3r'}(-h^{(2)})=0. \tag{2.8}$$

Thus, to solve the problem, it is necessary to find the solution of ordinary differential equations (2.4) that satisfies the continuity conditions (2.5) or (2.6) and the periodicity conditions (2.7) and (2.8), taking into account (2.2) and (2.3).

Consider the patterns of radial propagation of axisymmetric elastic waves along the layers (Fig. 1) in a prestressed incompressible laminated composite material. The same as the results in the monographs [5, 7], the solution of equations (2.4) is presented in the form

$$\chi'^{(j)(0)}(y_3)=A^{(j)}_1 e^{ik\alpha^{(j)}_1 y_3}+A^{(j)}_2 e^{-ik\alpha^{(j)}_1 y_3}+A^{(j)}_3 e^{ik\alpha^{(j)}_2 y_3}+A^{(j)}_4 e^{-ik\alpha^{(j)}_2 y_3},$$

$$A^{(j)}_n=\mathrm{const}, \quad j=1,2. \tag{2.9}$$

To determine the functions $\alpha^{(j)^2}_1$ and $\alpha^{(j)^2}_2$ from (2.4) and (2.9), we have the characteristic equations

$$\kappa'^{(j)}_{3113}\alpha^{(j)^4}-\alpha^{(j)^2}\left[C^2\rho'^{(j)}-\kappa'^{(j)}_{3333}-\kappa'^{(j)}_{1111}+2(\kappa'^{(j)}_{1133}+\kappa'^{(j)}_{1313})\right]$$

$$-(C^2\rho'^{(j)}-\kappa'^{(j)}_{1331})=0, \quad C=\frac{\omega}{k}, \quad j=1,2. \tag{2.10}$$

Substituting the amplitudes of displacements (2.2) and stresses (2.3) into the interface conditions (2.5) or (2.6) and the periodicity conditions (2.7) or (2.8), respectively, we obtain a homogeneous system of eight algebraic equations for the constants $B^{(j)}_m$ $(m=\overline{1,4};\ j=1,2)$. The system is omitted because of awkwardness. Equating the matrix determinants of this system to zero, we obtain the dispersion equation for an arbitrary axisymmetric wave that propagates along the layers of the layered composite material with layers free to slide.

Let us represent solution (2.9) relative to the center of each layer:

$$\chi'^{(1)(0)}(y_3)=B^{(1)}_1 e^{ik\alpha^{(1)}_1\left(y_3-\frac{1}{2}h^{(1)}\right)}+B^{(1)}_2 e^{-ik\alpha^{(1)}_1\left(y_3-\frac{1}{2}h^{(1)}\right)}$$

$$+B_3^{(1)} e^{ik\alpha_2^{(1)}\left(y_3-\frac{1}{2}h'^{(1)}\right)}+B_4^{(1)} e^{-ik\alpha_2^{(1)}\left(y_3-\frac{1}{2}h'^{(1)}\right)},$$

$$\chi'^{(2)(0)}(y_3)=B_1^{(2)} e^{ik\alpha_1^{(2)}\left(y_3+\frac{1}{2}h'^{(2)}\right)}+B_2^{(2)} e^{-ik\alpha_1^{(2)}\left(y_3+\frac{1}{2}h'^{(2)}\right)}$$

$$+B_3^{(2)} e^{ik\alpha_2^{(2)}\left(y_3+\frac{1}{2}h'^{(2)}\right)}+B_4^{(2)} e^{-ik\alpha_2^{(2)}\left(y_3+\frac{1}{2}h'^{(2)}\right)}. \tag{2.11}$$

For a prestressed incompressible composite material (as with [5, 7]), we consider a quasi-transverse (shear) wave propagating along the axis Or' and polarized in the plane $r'Oy_3$. For such a wave, the displacements $u_{r'}^{(j)}$ will be antisymmetric and $u_3^{(j)}$ will be symmetric about the center of the respective layers. According to solutions (2.11), we use

$$B_1^{(j)}=B_2^{(j)}, \quad B_3^{(j)}=B_4^{(j)}, \quad j=1,2. \tag{2.12}$$

Considering notation (2.2) and (2.3), substituting (2.11) and (2.12) into the boundary conditions (2.5) or (2.6) and the periodicity conditions (2.7) or (2.8), respectively, and performing some transformations, we obtain a homogeneous system of algebraic equations of the fourth order. From the condition of of the existence of its nontrivial solutions we obtain the dispersion equation

$$\det\|\alpha_{rs}\|=0, \quad r,s=\overline{1,4} \tag{2.13}$$

where

in the case of perfect bonding

$$\alpha_{1m}=k\alpha_m^{(1)}\sin\frac{1}{2}k\alpha_m^{(1)}h'^{(1)}, \quad \alpha_{1,2+m}=k\alpha_m^{(2)}\sin\frac{1}{2}k\alpha_m^{(2)}h'^{(2)},$$

$$\alpha_{2m}=k^2\cos\frac{1}{2}k\alpha_m^{(1)}h'^{(1)}, \quad \alpha_{2,2+m}=-k^2\cos\frac{1}{2}k\alpha_m^{(2)}h'^{(2)},$$

$$\alpha_{3m}=k^3\alpha_m^{(1)}(C^2\rho^{(1)}-\kappa'^{(1)}_{1111}-\kappa'^{(1)}_{3333}+2\kappa'^{(1)}_{1133}+\kappa'^{(1)}_{1313}-\alpha_m^{(1)2}\kappa'^{(1)}_{3113})\sin\frac{1}{2}k\alpha_m^{(1)}h'^{(1)},$$

$$\alpha_{3,2+m}=k^3\alpha_m^{(2)}(C^2\rho^{(2)}-\kappa'^{(2)}_{1111}-\kappa'^{(2)}_{3333}+2\kappa'^{(2)}_{1133}+\kappa'^{(2)}_{1313}-\alpha_m^{(2)2}\kappa'^{(2)}_{3113})\sin\frac{1}{2}k\alpha_m^{(2)}h'^{(2)},$$

$$\alpha_{4m}=k^2(\kappa'^{(1)}_{1313}-\alpha_m^{(1)2}\kappa'^{(1)}_{3113})\cos\frac{1}{2}k\alpha_m^{(1)}h'^{(1)},$$

$$\alpha_{4,2+m}=-k^2(\kappa'^{(2)}_{1313}-\alpha_m^{(2)2}\kappa'^{(2)}_{3113})\cos\frac{1}{2}k\alpha_m^{(2)}h'^{(2)}, \quad m=1,2. \tag{2.14}$$

in the case of free sliding

$$\alpha_{1m}=\cos\frac{1}{2}k\alpha_m^{(1)}h'^{(1)}, \quad \alpha_{1,2+m}=-\cos\frac{1}{2}k\alpha_m^{(2)}h'^{(2)},$$

$$\alpha_{2m}=\alpha_1^{(1)}(C^2\rho^{(1)}-\kappa'^{(1)}_{1111}-\kappa'^{(1)}_{3333}+2\kappa'^{(1)}_{1133}+\kappa'^{(1)}_{1313}-\alpha_m^{(1)2}\kappa'^{(1)}_{3113})\sin\frac{1}{2}k\alpha_m^{(1)}h'^{(1)},$$

$$\alpha_{2,2+m}=\alpha_m^{(2)}(C^2\rho^{(2)}-\kappa'^{(2)}_{1111}-\kappa'^{(2)}_{3333}+2\kappa'^{(2)}_{1133}+\kappa'^{(2)}_{1313}-\alpha_m^{(2)2}\kappa'^{(2)}_{3113})\sin\frac{1}{2}k\alpha_m^{(2)}h'^{(2)},$$

$$\alpha_{3m}=(\kappa'^{(1)}_{1313}-\alpha^{(1)2}_{m}\kappa'^{(1)}_{3113})\cos\frac{1}{2}k\alpha^{(1)}_{m}h'^{(1)},\qquad \alpha_{33}=\alpha_{34}=0,$$

$$\alpha_{41}=\alpha_{42}=0,\qquad \alpha_{4,2+m}=(\kappa'^{(2)}_{1313}-\alpha^{(2)2}_{m}\kappa'^{(2)}_{3113})\cos\frac{1}{2}k\alpha^{(2)}_{m}h'^{(2)},\qquad m=1,2. \tag{2.15}$$

In expressions (2.14) and (2.15), C is the "true" phase velocity of the quasi-transverse wave along the axis Or' polarized in the plane $r'Oy_3$ for the corresponding interface conditions between the layers of the composite material. The results in the form (2.14) for perfect bonding of the layers were obtained in [5, 7].

We introduce the notation

$$\xi'^{(j)}=\kappa'^{(j)}_{1111}+\kappa'^{(j)}_{3333}-2\kappa'^{(j)}_{1133}-\kappa'^{(j)}_{1313},\qquad \beta'^{(j)}=(\alpha^{(j)2}_{2}-\alpha^{(j)2}_{1})\kappa'^{(j)}_{3113},$$

$$\theta'^{(j)}_{m}=\kappa'^{(j)}_{1313}-\alpha^{(j)2}_{m}\kappa'^{(j)}_{3113},\qquad \zeta'^{(j)}_{m}=C^2\rho^{(j)}-\xi'^{(j)}-\alpha^{(j)2}_{m}\kappa'^{(j)}_{3113},\quad j,m=1,2. \tag{2.16}$$

Using notation (2.16), we can write the dispersion equation (2.13) for a quasi-transverse wave in an incompressible laminated composite material as

in the case of perfect bonding:

$$-\beta'^{(1)}\beta'^{(2)}\Big(\alpha^{(1)}_{1}\alpha^{(1)}_{2}\sin\frac{1}{2}k\alpha^{(1)}_{1}h'^{(1)}\sin\frac{1}{2}k\alpha^{(1)}_{2}h'^{(1)}\cos\frac{1}{2}k\alpha^{(2)}_{1}h'^{(2)}\cos\frac{1}{2}k\alpha^{(2)}_{2}h'^{(2)}$$

$$+\alpha^{(2)}_{1}\alpha^{(2)}_{2}\cos\frac{1}{2}k\alpha^{(1)}_{1}h'^{(1)}\cos\frac{1}{2}k\alpha^{(1)}_{2}h'^{(1)}\sin\frac{1}{2}k\alpha^{(2)}_{1}h'^{(2)}\sin\frac{1}{2}k\alpha^{(2)}_{2}h'^{(2)}\Big)+\alpha^{(1)}_{1}\sin\frac{1}{2}k\alpha^{(1)}_{1}h'^{(1)}$$

$$\times\cos\frac{1}{2}k\alpha^{(1)}_{2}h'^{(1)}\Big[\alpha^{(2)}_{1}(\theta'^{(1)}_{2}-\theta'^{(2)}_{2})(\zeta'^{(2)}_{1}-\zeta'^{(1)}_{1})\sin\frac{1}{2}k\alpha^{(2)}_{1}h'^{(2)}\cos\frac{1}{2}k\alpha^{(2)}_{2}h'^{(2)}$$

$$-\alpha^{(2)}_{2}(\theta'^{(1)}_{2}-\theta'^{(2)}_{1})(\zeta'^{(2)}_{2}-\zeta'^{(1)}_{1})\sin\frac{1}{2}k\alpha^{(2)}_{2}h'^{(2)}\cos\frac{1}{2}k\alpha^{(2)}_{1}h'^{(2)}\Big]-\alpha^{(1)}_{2}\sin\frac{1}{2}k\alpha^{(1)}_{2}h'^{(1)}$$

$$\times\cos\frac{1}{2}k\alpha^{(1)}_{1}h'^{(1)}\Big[\alpha^{(2)}_{1}(\theta'^{(1)}_{1}-\theta'^{(2)}_{2})(\zeta'^{(2)}_{1}-\zeta'^{(1)}_{2})\sin\frac{1}{2}k\alpha^{(2)}_{1}h'^{(2)}\cos\frac{1}{2}k\alpha^{(2)}_{2}h'^{(2)}$$

$$-\alpha^{(2)}_{2}(\theta'^{(1)}_{1}-\theta'^{(2)}_{1})(\zeta'^{(2)}_{2}-\zeta'^{(1)}_{2})\sin\frac{1}{2}k\alpha^{(2)}_{2}h'^{(2)}\cos\frac{1}{2}k\alpha^{(2)}_{1}h'^{(2)}\Big]=0, \tag{2.17}$$

in the case of free sliding:

$$\beta'^{(2)}\cos\frac{1}{2}k\alpha^{(2)}_{1}h'^{(2)}\cos\frac{1}{2}k\alpha^{(2)}_{2}h'^{(2)}\Big(\alpha^{(1)}_{1}\zeta'^{(1)}_{1}\theta'^{(1)}_{2}\sin\frac{1}{2}k\alpha^{(1)}_{1}h'^{(1)}\cos\frac{1}{2}k\alpha^{(1)}_{2}h'^{(1)}$$

$$-\alpha^{(1)}_{2}\zeta'^{(1)}_{2}\theta'^{(1)}_{1}\sin\frac{1}{2}k\alpha^{(1)}_{2}h'^{(1)}\cos\frac{1}{2}k\alpha^{(1)}_{1}h'^{(1)}\Big)$$

$$-\beta'^{(1)}\cos\frac{1}{2}k\alpha^{(1)}_{1}h'^{(1)}\cos\frac{1}{2}k\alpha^{(1)}_{2}h'^{(1)}\Big(\alpha^{(2)}_{2}\zeta'^{(2)}_{2}\theta'^{(2)}_{1}\sin\frac{1}{2}k\alpha^{(2)}_{2}h'^{(2)}\cos\frac{1}{2}k\alpha^{(2)}_{1}h'^{(2)}$$

$$-\alpha^{(2)}_{1}\zeta'^{(2)}_{1}\theta'^{(2)}_{2}\sin\frac{1}{2}k\alpha^{(2)}_{1}h'^{(2)}\cos\frac{1}{2}k\alpha^{(2)}_{2}h'^{(2)}\Big)=0. \tag{2.18}$$

Note that with condition (2.12) and notation (2.16), the continuity conditions (2.5), (2.6) and periodicity conditions (2.7), (2.8) independently lead to the dispersion equations (2.17) and (2.18), respectively, i. e., under condition (2.12) for solution (2.11) we have the same continuity and periodicity conditions.

Analysis of formulas (2.17) and (2.18) shows that during the propagation of axisymmetric waves, the layers of the composite interact.

3. Long-wave (Low-Frequency) Approximation. For the long wavelength (low frequency) approximation, we assume that

$$\frac{h'^{(1)}+h'^{(2)}}{\Lambda} << 1, \tag{3.1}$$

where $\Lambda=\dfrac{2\pi}{k}$ is the wavelength.

In the case of perfect bonding of the layers for the long-wave (low-frequency) approximation under condition (3.1), restricting the analysis to the one-term approximation, we obtain the limit estimate for the phase velocity of the quasi-transverse wave:

$$C^2=\frac{1}{\rho^{(1)}h'^{(1)}+\rho^{(2)}h'^{(2)}}\left[\kappa'^{(2)}_{1331}h'^{(2)}+h'^{(1)}\kappa'^{(1)}_{1331}-\frac{h'^{(1)}h'^{(2)}(\kappa'^{(1)}_{1313}-\kappa'^{(2)}_{1313})^2}{\kappa'^{(1)}_{3113}h'^{(2)}+\kappa'^{(2)}_{3113}h'^{(1)}}\right]. \tag{3.2}$$

If $h'^{(1)}>>h'^{(2)}$, then

$$C^2=\frac{\kappa'^{(1)}_{1331}}{\rho^{(1)}}. \tag{3.3}$$

If $h'^{(1)}<<h'^{(2)}$, then

$$C^2=\frac{\kappa'^{(2)}_{1331}}{\rho^{(2)}}. \tag{3.4}$$

It follows from formula (3.2) that the phase velocity will be non-negative if

$$\kappa'^{(2)}_{1331}h'^{(2)}+h'^{(1)}\kappa'^{(1)}_{1331}-\frac{h'^{(1)}h'^{(2)}(\kappa'^{(1)}_{1313}-\kappa'^{(2)}_{1313})^2}{\kappa'^{(1)}_{3113}h'^{(2)}+\kappa'^{(2)}_{3113}h'^{(1)}}\geq 0. \tag{3.5}$$

The phase velocity will be minimal ($C=0$) provided that

$$m^2\frac{\lambda_3^{(1)2}}{\lambda_3^{(2)2}}+m\frac{\lambda_3^{(1)}}{\lambda_3^{(2)}}\left[\frac{\kappa'^{(1)}_{3113}}{\kappa'^{(2)}_{3113}}+\frac{\kappa'^{(2)}_{1331}}{\kappa'^{(1)}_{1331}}-\frac{(\kappa'^{(1)}_{1313}-\kappa'^{(2)}_{1313})^2}{\kappa'^{(1)}_{1331}\kappa'^{(2)}_{3113}}\right]+\frac{\kappa'^{(1)}_{3113}}{\kappa'^{(2)}_{3113}}\frac{\kappa'^{(2)}_{1331}}{\kappa'^{(1)}_{1331}}=0, \tag{3.6}$$

where $m=\dfrac{h^{(1)}}{h^{(2)}}$ is the lamination parameter.

In the case of free sliding of the layers for the long-wave (low-frequency) approximation under condition (3.1), restricting the analysis to the one-term approximation, we obtain the limit estimate for the phase velocity of the quasi-transverse wave:

$$C^2=\frac{(\kappa'^{(1)}_{1331}-\kappa'^{(1)2}_{1313}\kappa'^{(1)-1}_{3113})h'^{(1)}+(\kappa'^{(2)}_{1331}-\kappa'^{(2)2}_{1313}\kappa'^{(2)-1}_{3113})h'^{(2)}}{\rho^{(1)}h'^{(1)}+\rho^{(2)}h'^{(2)}}. \tag{3.7}$$

If $h'^{(1)}>>h'^{(2)}$, then

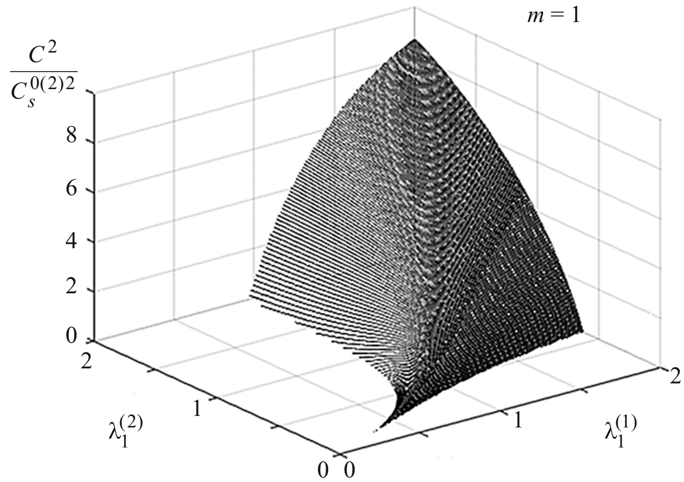

Fig. 2

$$C^2 = \rho^{(1)-1}(\kappa'^{(1)}_{1331} - \kappa'^{(1)2}_{1313}\kappa'^{(1)-1}_{3113}). \tag{3.8}$$

If $h'^{(1)} << h'^{(2)}$, then

$$C^2 = \rho^{(2)-1}(\kappa'^{(2)}_{1331} - \kappa'^{(2)2}_{1313}\kappa'^{(2)-1}_{3113}). \tag{3.9}$$

The phase velocity in the case of free sliding will be non-negative if

$$(\kappa'^{(1)}_{1331} - \kappa'^{(1)2}_{1313}\kappa'^{(1)-1}_{3113})h'^{(1)} + (\kappa'^{(2)}_{1331} - \kappa'^{(2)2}_{1313}\kappa'^{(2)-1}_{3113})h'^{(2)} \ge 0. \tag{3.10}$$

It follows from formula (3.7) that the phase velocity will be zero if

$$m = -\frac{\lambda_3^{(2)}}{\lambda_3^{(1)}} \frac{\kappa'^{(2)}_{1331} - \kappa'^{(2)2}_{1313}\kappa'^{(2)-1}_{3113}}{\kappa'^{(1)}_{1331} - \kappa'^{(1)2}_{1313}\kappa'^{(1)-1}_{3113}}. \tag{3.11}$$

From formulas (3.3), (3.4), and (3.8), (3.9), it follows that when long axisymmetric waves propagate, the phase velocity of a quasi-transverse wave, when the layers of the composite material are perfectly bonded, is always greater than in the case of free sliding.

4. Numerical Results. For numerical analysis, we specify the type of elastic potential. Consider the effect of initial stresses on the velocity of long quasi-transverse waves in an incompressible composite material with Treloar potential [5]:

$$w^{(j)} = 2c_{10}^{(j)} A_1^{(j)}, \quad j = 1, 2, \tag{4.1}$$

where $c_{10}^{(j)}$ is the elastic constant, $A_1^{(j)}$ is the algebraic invariant.

Let the initial state be

$$S_{11}^{0(j)} = S_{22}^{0(j)} \ne 0, \quad S_{33}^{0(j)} = 0, \quad \lambda^{(j)} = \lambda_1^{(j)} = \lambda_2^{(j)}, \quad \lambda_3^{(j)} = \lambda^{(j)-2}, \quad j = 1, 2 \tag{4.2}$$

and we will use the following relations of the mechanical characteristics of the layers:

$$\mu \equiv c_{10}^{(1)} / c_{10}^{(2)} = 5, \quad \rho \equiv \rho^{(1)} / \rho^{(2)} = 10/7. \tag{4.3}$$

Using (4.1) and (4.2), we obtain the components of the tensor $\kappa'^{(j)}$ in (3.2)–(3.11):

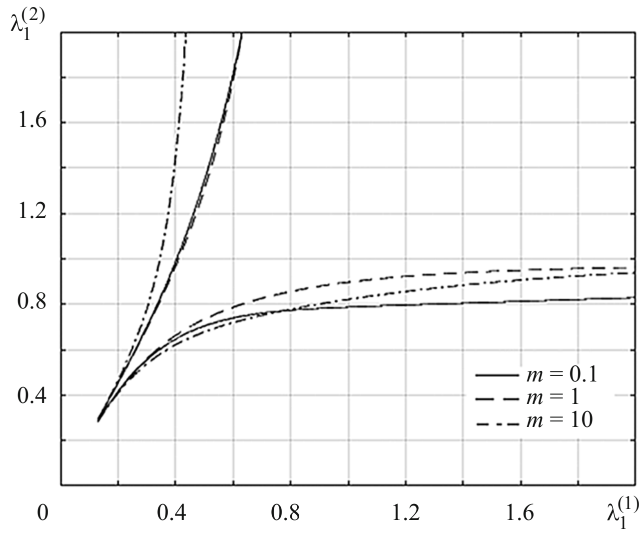

Fig. 3

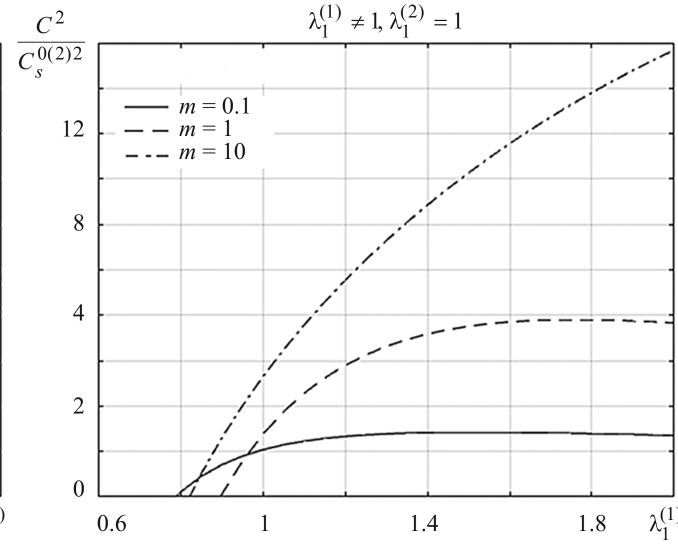

Fig. 4

$$\kappa'^{(j)}_{1331} = \frac{2\mu^{(j)}\lambda_1^{(j)2}}{\lambda_1^{(j)} + \lambda_3^{(j)}}, \qquad \kappa'^{(j)}_{3113} = \kappa'^{(j)}_{1313} = \frac{2\mu^{(j)}\lambda_3^{(j)2}}{\lambda_1^{(j)} + \lambda_3^{(j)}}, \qquad j = 1, 2. \tag{4.4}$$

The case of perfect bonding (3.2). With (4.2) and (4.4), formula (3.2) can be written as

$$\frac{C^2}{C_s^{0(2)2}} = 2\lambda_1^{(1)-2}\lambda_1^{(2)-2}(\rho m\lambda_1^{(2)2} + \lambda_1^{(1)2})^{-1}\Big[\mu(\lambda_1^{(2)3}+1) + m(\lambda_1^{(1)3}+1)\Big]^{-1}$$
$$\times\Big\{m^2\mu\lambda_1^{(1)6}\lambda_1^{(2)4} + m\Big[2\mu\lambda_1^{(1)2}\lambda_1^{(2)2} + \lambda_1^{(1)4}(\lambda_1^{(2)3}-1)(\lambda_1^{(1)3}+1)$$
$$+\mu^2\lambda_1^{(2)4}(\lambda_1^{(1)3}-1)(\lambda_1^{(2)3}+1)\Big] + \mu\lambda_1^{(1)4}\lambda_1^{(2)6}\Big\}, \qquad C_s^{0(2)} = \sqrt{\mu_2/\rho_2}. \tag{4.5}$$

Figure 2 shows the surface corresponding to $\frac{C^2}{C_s^{0(2)2}} \geq 0$ when $m = 1$ in the range of $0 \leq \lambda_1^{(1)} \leq 2$ and $0 \leq \lambda_1^{(2)} \leq 2$. Figure 3 shows the curves corresponding to $C = 0$ when $m = 0.1$ (solid line), when $m = 1$ (dashed line), and when $m = 10$ (dash-and-dot line). Figures 2 and 3 demonstrate that at certain values of $\lambda_1^{(1)}, \lambda_1^{(2)}$, and the lamination parameter m the phase velocity of long waves in the composite material can be zero.

Let us consider some cases of the prestressed composite material.

1) $\lambda_1^{(1)} = \lambda_1^{(2)} = 1$ (the initial state of the composite material is stress-free).

In this case

$$\frac{C}{C_s^{0(2)}} = \sqrt{\frac{\mu(m+1)^2}{(\rho m+1)(\mu+m)}} > 0 \tag{4.6}$$

for any values of ρ, μ, and m.

2) $\lambda_1^{(1)} \neq 1, \lambda_1^{(2)} = 1$ (the first layer is prestressed, the second one is free of stresses).

In this case, to determine the phase velocity, we obtain the formula

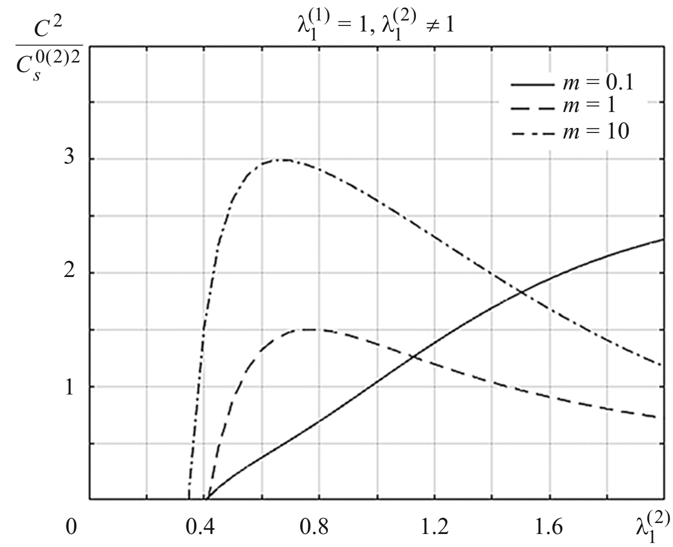

Fig. 5

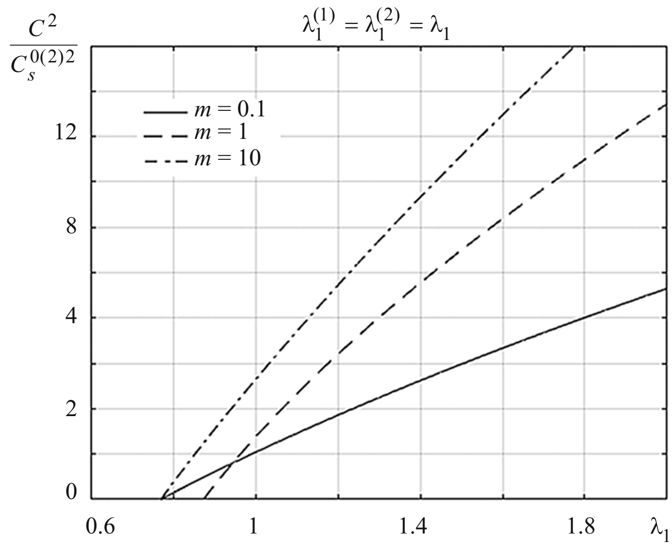

Fig. 6

$$\frac{C^2}{C_s^{0(2)2}} = 2\mu \frac{\lambda_1^{(1)2}\left[\lambda_1^{(1)2}(1+\lambda_1^{(1)2}m^2)+2m\right]+2\mu m(\lambda_1^{(1)3}-1)}{\lambda_1^{(1)2}(\rho m+\lambda_1^{(1)2})\left[2\mu+m(\lambda_1^{(1)3}+1)\right]}. \tag{4.7}$$

In the range of non-negative values of the phase velocity, the curves corresponding to formula (4.7) at certain values of the lamination parameter m are shown in Fig. 4.

3) $\lambda_1^{(1)} = 1, \lambda_1^{(2)} \neq 1$ (the first layer is free of stresses, the second one is prestressed).

For the phase velocity we have

$$\frac{C^2}{C_s^{0(2)2}} = 2\frac{\mu\lambda_1^{(2)2}\left[\lambda_1^{(2)2}(\lambda_1^{(2)2}+m^2)+2m\right]+2m(\lambda_1^{(2)3}-1)}{\lambda_1^{(2)2}(\rho m\lambda_1^{(2)2}+1)\left[\mu(\lambda_1^{(2)3}+1)+2m\right]}. \tag{4.8}$$

Figure 5 shows the curves corresponding to the non-negative values of the phase velocity at certain values of the lamination parameter m.

4) $\lambda_1^{(1)} = \lambda_1^{(2)} = \lambda_1 \neq 1$ (the layers of the composite material are equally stressed).

The phase velocity of a long wave can be determined by the formula

$$\frac{C^2}{C_s^{0(2)2}} = 2\frac{\mu\left[\lambda_1^6\left(1+m^2\right)+2m\right]+m\left(1+\mu^2\right)\left(\lambda_1^6-1\right)}{\lambda_1^2(\rho m+1)(\lambda_1^3+1)(\mu+m)}. \tag{4.9}$$

The graphs for the phase velocity of the long wave are shown in Fig. 6.

We can conclude that the phase velocity of the long wave in the case of perfect bonding of the layers can tend to zero at the appropriate values of the initial stresses and the lamination parameter. In the considered range of initial strains, the effect occurs if one of the layers (cases 2 and 3) or all layers (case 4) are precompressed.

The case of free sliding (3.7). Taking into account relations (4.2) and (4.4), we represent the formula for determining the phase velocity (3.7) as

$$\frac{C^2}{C_s^{0(2)2}} = 2\frac{\mu m\lambda_1^{(2)4}(\lambda_1^{(1)3}-1)+\lambda_1^{(1)4}(\lambda_1^{(2)3}-1)}{\lambda_1^{(1)2}\lambda_1^{(2)2}(\rho m\lambda_1^{(2)2}+\lambda_1^{(1)2})}. \tag{4.10}$$

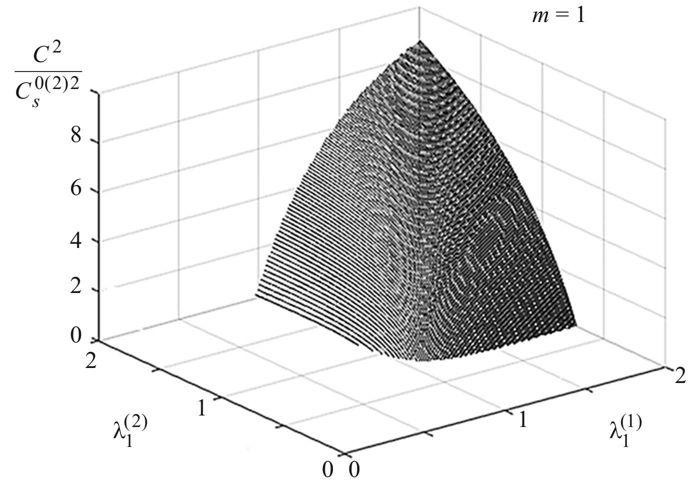

Fig. 7

Figure 7 shows the surface corresponds to the non-negative values of $\frac{C^2}{C_s^{0(2)2}}$ in the case of free sliding of the layers.

Traces of the corresponding surfaces on the plane $\lambda_1^{(1)}O\lambda_1^{(2)}$ at different values of the lamination parameter m are shown in Fig. 8.

Figure 8 demonstrates that the phase velocity of a long wave in the case of free sliding can be zero if one of the layers is compressed and the other one is stretched.

In the case of free sliding, we consider the same examples as in the case of perfect bonding:

1) $\lambda_1^{(1)} = \lambda_1^{(2)} = 1.$

$$C = 0. \tag{4.11}$$

2) $\lambda_1^{(1)} \neq 1; \lambda_1^{(2)} = 1$ (Fig. 9).

$$\frac{C^2}{C_s^{0(2)2}} = \frac{2\mu m(\lambda_1^{(1)3} - 1)}{\lambda_1^{(1)2}(\rho m + \lambda_1^{(1)2})}. \tag{4.12}$$

3) $\lambda_1^{(1)} = 1; \lambda_1^{(2)} \neq 1$ (Fig. 10).

$$\frac{C^2}{C_s^{0(2)2}} = \frac{2\left(\lambda_1^{(2)3} - 1\right)}{\lambda_1^{(2)2}\left(\rho m \lambda_1^{(2)2} + 1\right)}. \tag{4.13}$$

4) $\lambda_1^{(1)} = \lambda_1^{(2)} = \lambda_1 \neq 1$ (Fig. 11.

$$\frac{C^2}{C_s^{0(2)2}} = \frac{2(\mu m + 1)\left(\lambda_1^3 - 1\right)}{\lambda_1^2(\rho m + 1)}. \tag{4.14}$$

From formulas (4.12)–(4.14) and Figs. 9–11, it follows that the phase velocity of the long wave is zero only in the case of the stress-frere initial state $\lambda_1^{(1)} = \lambda_1^{(2)} = 1.$

The above formulas, graphs, and conclusions are obtained under condition (4.2).

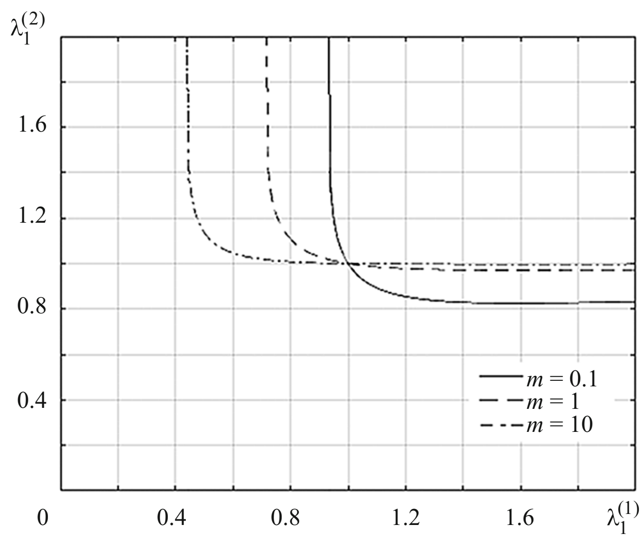

Fig. 8

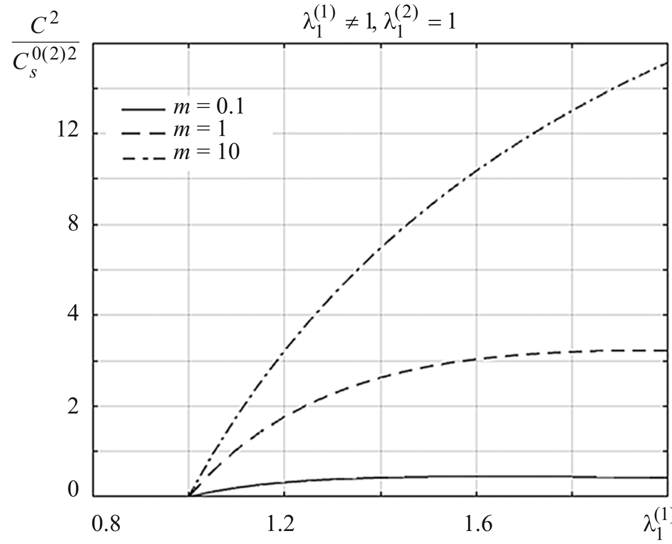

Fig. 9

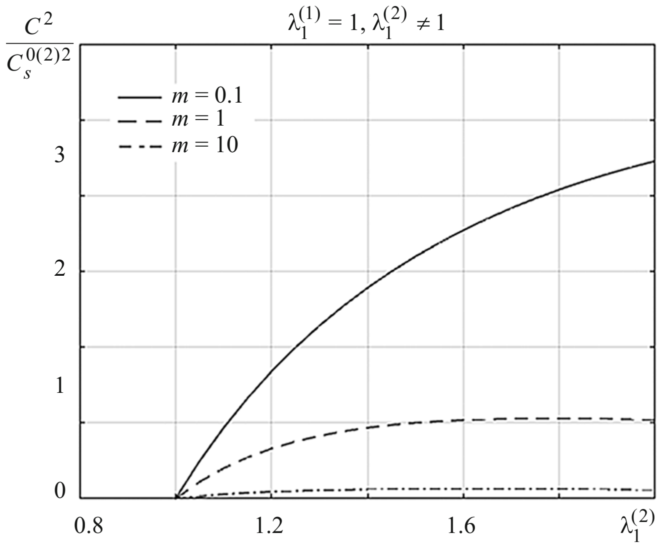

Fig. 10

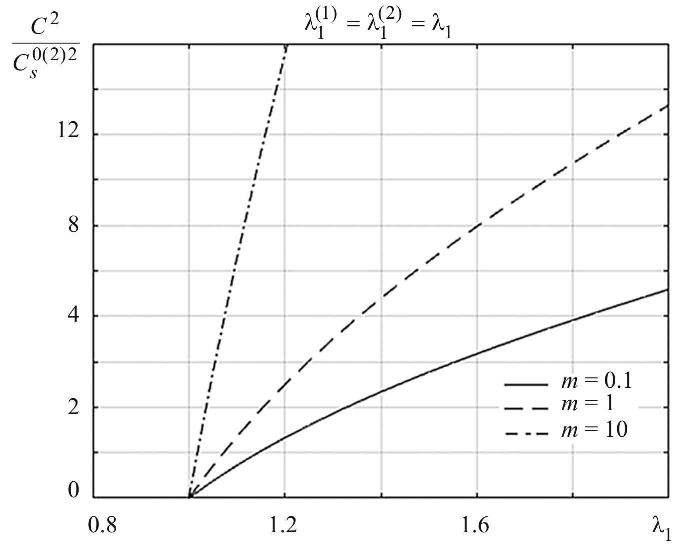

Fig. 11

Conclusions. Analyzing the effect of the initial stresses, the contact conditions of the layers, and the lamination parameter on the phase velocity of a long quasi-transverse wave, the following conclusions can be drawn:

The initial stresses significantly affect the phase velocities of long axisymmetric waves in an incompressible laminated composite material.

The phase velocity of a long wave between perfectly bonded layers of the composite material can tend to zero at certain values of the initial stresses and the lamination parameter. In the considered range of initial strains, the effect occurs if one of the layers or all layers are precompressed.

The phase velocity of a long wave in the case of free sliding can be zero if one of the layers of the composite material is compressed and the other one is stretched. Otherwise, the phase velocity of the long wave is zero only if the initial state is stress-free.

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