DOI 10.1007/s10778-021-01060-y
International Applied Mechanics, Vol. 57, No. 1, January, 2021

STRESS STATE OF AN ORTHOTROPIC ELECTROELASTIC MEDIUM WITH AN ARBITRARILY ORIENTED ELLIPTIC CRACK UNDER UNIAXIAL TENSION*

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The problem of electric and stress state in a piezoelectric space with an arbitrary orientated elliptical crack under homogeneous force and electric loading is considered. The solution to this problem is obtained on the basis of the triple Fourier transformation and the Fourier transform of Green?s function for an infinite electroelastic space. Testing the approach against particular cases confirms its effectiveness. The numerical study is carried out, and the stress intensity factors along the elliptical crack front are studied for different crack orientations in the orthotropic electroelastic space under tension.

Keywords: orthotropic piezoelectric material, flat elliptical crack, arbitrary orientation, stress intensity factors

Introduction. The use of piezoelectric materials in the creation of power converters and elements of measuring devices for various purposes is of great interest in the study and analysis of the concentration of force and electric fields in electroelastic bodies with defects such as cavities, inclusions, and cracks. The solution of electroelasticity problems in three-dimensional statement, which takes into account the coupling of the force and electric fields, is associated with significant mathematical difficulties because, in this case, the initial system of equations of electroelasticity is a coupled system of differential equations of complex structure [1, 4]. At present, two-dimensional electroelasticity problems have been sufficiently studied. We can note the studies [8, 9, 20, 23] concerned with the stress state near single cavities, inclusions, cracks, and during the interaction of concentrators of electric and mechanical fields in piezoelectric material. Structurally similar approaches have been proposed in [17, 21] to solve three-dimensional coupled equations of electroelasticity for transversely isotropic bodies. The exact solutions of the electroelasticity problems with the special orientation of the concentrator of force and electric fields relative to the axis of symmetry of the piezoelectric material were found using the approaches. Thus, when using these approaches, it was usually assumed that the axis of symmetry of the electroelastic material is oriented along the axis of rotation of the stress concentrator or it is perpendicular to the plane where the flat crack is located [5–7, 12–14, 17–19, 21, 23]. At the same time, with other orientations of the concentrators of force and electric fields relative to the axis of symmetry of the piezoelectric material, these approaches are ineffective in solving spatial problems. Note also that the results of studies on stress intensity factors (SIF) for elastic isotropic bodies with circular and elliptical cracks are sufficiently studied in the monographs [3, 10]. For electroelastic bodies (with the mentioned restrictions on the orientation of cracks), similar studies were conducted in [5, 6, 11, 12, 18]; for

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* This study was sponsored by the budget program “Support for Priority Areas of Scientific Research” (KPKVK 6541230).

magnetoelastic bodies in [15, 16]. The stress distribution in an orthotropic electroelastic space with a triaxial ellipsoidal inclusion under tension was studied in [14].

We consider the problem of an arbitrarily oriented elliptical crack in an orthotropic electroelastic medium subject to uniaxial tension, based on the generalization of the approach [22] (for an anisotropic purely elastic medium with an elliptical crack), in the case of an orthotropic piezoelectric material. The research uses the triple Fourier transformation of spatial variables, the Fourier transform of Green's function for an electroelastic anisotropic medium, as well as Cauchy's residue theorems and Gaussian quadrature formulas. For partial cases (where the elliptical crack is in the isotropy plane of the electroelastic transversely isotropic material), the results coincide with the data obtained by other approaches. The stress intensity factors (SIF) along the elliptic crack boundary at different orientations in orthotropic electroelastic material were found.

1. Basic Equations and Problem Statement. Let an orthotropic electroelastic material have a flat elliptical crack. We assume that the electroelastic material is under uniaxial tension directed perpendicular to the plane of the elliptical crack, and the electric displacement in the same direction is equal to zero. The presence of the crack in the material, as a concentrator of force and electric fields, leads to the perturbation of electric and stress states.

The complete system of equations of electroelasticity statics has the following form:

equilibrium equation in the absence of body forces

$$\sigma_{ij,j} = 0, \tag{1}$$

equation of forced electrostatics

$$D_{i,i} = 0, \quad E_i = -\Psi_{,i}, \tag{2}$$

kinematic equations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

equation of state

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn} + e_{nij}\Psi_{,n},$$
$$D_i = e_{imn}\varepsilon_{mn} - k_{in}\Psi_{,n}, \tag{3}$$

where σ_{ij}, ε_{ij}, u_i, D_i, E_i, Ψ are the components of stresses, strains, displacements, electric displacements (induction), electric field, and electric potential, respectively.

The following notation of tensors is also introduced: C_{ijmn}, e_{imn}, k_{ij} are the elastic moduli, piezomoduli, dielectric constants. For piezoelectric orthotropic bodies, the elastic properties of the material are described by nine independent constants c_{11}, c_{22}, c_{33}, c_{12}, $c_{13}, c_{23}, c_{44}, c_{55}, c_{66}$; the piezomoduli are e_{15}, e_{24}, e_{31}, e_{32}, e_{33}; the dielectric constants are three independent constants k_{11}, k_{22}, k_{33}. The components of the tensor of elastic moduli, piezomoduli, and dielectric constants are related to the mentioned independent constants as follows:

$$C_{1111} = c_{11}, \quad C_{2222} = c_{22}, \quad C_{3333} = c_{33}, \quad C_{1122} = C_{2211} = c_{12},$$
$$C_{1133} = C_{3311} = c_{13}, \quad C_{2233} = C_{3322} = c_{23}, \quad C_{2323} = C_{2332} = C_{3232} = C_{3223} = c_{44},$$
$$C_{3131} = C_{3113} = C_{1331} = C_{1313} = c_{55}, \quad C_{1212} = C_{1221} = C_{2121} = C_{2112} = c_{66},$$
$$e_{113} = e_{131} = e_{15}, \quad e_{223} = e_{232} = e_{24}, \quad e_{311} = e_{31}, \quad e_{322} = e_{32},$$
$$e_{333} = e_{33}, \quad k_{11}, \quad k_{22}, \quad k_{33}. \tag{4}$$

The other components of these three tensors are equal to zero.

The equations of statics of electroelasticity with respect to displacements and electric potential for an orthotropic electroelastic body follow from relations (1)–(3) and the components of tensors (4).

When studying the problem, it is convenient to introduce a new coordinate system where one of the axes coincides with the normal to the crack plane. Assume that the original coordinate system $Oxyz$ is related to new (local) system $Ox^1y^1z^1$ in such a way that the system can be obtained from the original system by rotating around the axis Ox by angle α. Then the tensors of elastic moduli, piezomoduli, and dielectric constants C^{α}_{ijkl}, e^{α}_{ijk}, k^{α}_{ij} in the new coordinate system are $C^{\alpha}_{ijkl}=C_{mnpk}\alpha_{im}\alpha_{jn}\alpha_{kp}\alpha_{lq}$, $e^{\alpha}_{ijk}=e_{mnp}\alpha_{im}\alpha_{jn}\alpha_{kp}$, $k^{\alpha}_{ij}=k_{mn}\alpha_{im}\alpha_{jn}$, where α_{ij} is the transformation matrix of the following form:

$$\alpha_{ij}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix}. \tag{5}$$

An arbitrary orientation of the flat elliptical crack can be obtained by sequential rotation at angles α, β, γ around the axes of the coordinate system $0x$, $0y$, $0z$, respectively. The transformation matrix T_{ij} is found as follows:

$$T_{ij}=\begin{bmatrix}\cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta\\ \sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\gamma & -\sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\gamma & -\sin\alpha\cos\beta\\ -\cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma & \cos\alpha\sin\beta\sin\gamma+\sin\alpha\cos\gamma & \cos\alpha\cos\beta\end{bmatrix}.$$

This matrix is the result of successive multiplication of three matrices reflecting the right rotation around each of the coordinate axes obtained similarly to expression (5)

$$\alpha_{ij}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix},\quad \beta_{ij}=\begin{bmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{bmatrix},\quad \gamma_{ij}=\begin{bmatrix}\cos\gamma & -\sin\gamma & 0\\ \sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Then we obtain new tensors of elastic moduli, piezomoduli, and dielectric constants $C^{(\alpha,\beta,\gamma)}_{ijkl}$, $e^{(\alpha,\beta,\gamma)}_{ijk}$, $k^{(\alpha,\beta,\gamma)}_{ij}$ using transformations of tensors:

$$C^{(\alpha,\beta,\gamma)}_{ijkl}=C_{mnpk}T_{im}T_{jn}T_{kp}T_{lq},\qquad e^{(\alpha,\beta,\gamma)}_{ijk}=e_{mnp}T_{im}T_{jn}T_{kp},$$

$$k^{(\alpha,\beta,\gamma)}_{ij}=k_{mn}T_{im}T_{jn},$$

where the repeated indices are summed.

Note that we will use the conventional tensor notation of expressions below, i. e., we will mean that the repeated indices in the expressions are summed. Note also that without fundamentally changing anything in the scheme of solving the problem, instead of the transformation T_{ij} associated with the rotation around the coordinate axes $0x, 0y, 0z$, we could enter other transformation, for example, corresponding to rotations through Euler angles. But for clarity, we chose transformation that corresponds to successive rotations around three different coordinate axes. To describe the stress and electrical states, we use more unified notation [7]. We present in the following form:

elastic displacements and electric potential

$$U_M=\begin{cases}u_m, & M=1,\ 2,\ 3,\\ \Psi, & M=4,\end{cases} \tag{6}$$

elastic deformation or electric field

$$Z_{Mn}=\begin{cases}\varepsilon_{mn}, & M=1,\ 2,\ 3,\\ \Psi_{,n}, & M=4,\end{cases} \tag{7}$$

stress or electrical displacements

$$\Sigma_{iJ}=\begin{cases}\sigma_{ij}, J=1,\ 2,\ 3,\\ D_i, J=4,\end{cases} \tag{8}$$

electroelastic moduli

$$E_{iJMn}^{(\alpha,\beta,\gamma)}=\begin{cases}C_{ijmn}^{(\alpha,\beta,\gamma)},\ J,M=1,\ 2,\ 3,\\ e_{nij}^{(\alpha,\beta,\gamma)},\ J=1,\ 2,\ 3;\ M=4,\\ e_{imn}^{(\alpha,\beta,\gamma)},\ J=4;\ M=1,\ 2,\ 3,\\ -k_{in}^{(\alpha,\beta,\gamma)},\ J,\ M=4.\end{cases} \tag{9}$$

With notation (6)–(9), the equation of state (3) can be written as

$$\Sigma_{iJ}=E_{iJMn}^{(\alpha,\beta,\gamma)}Z_{Mn}. \tag{10}$$

Note that the problem for an orthotropic electroelastic material with an arbitrarily oriented elliptic crack is not divided into two independent symmetric and antisymmetric problems. The problem is considered in the general statement when the boundary conditions include both normal and tangential forces, as well as the normal component of the electric displacement vector on the crack surface:

$$\tau_{13}^{\pm}=f^{(\alpha,\beta,\gamma)},\quad \tau_{23}^{\pm}=g^{(\alpha,\beta,\gamma)},\quad \sigma_{33}^{\pm}=-p^{(\alpha,\beta,\gamma)}, \tag{11}$$

$$D_3^{\pm}=-D^{(\alpha,\beta,\gamma)},\quad (x_1,x_2)\in S,\quad U_M(\vec{x})\to 0\quad \text{as}\quad |\vec{x}|\to\infty,$$

where S is the crack surface related to the new coordinate system (which is obtained by successive rotation at angles α,β,γ around the axes of the old system), and the loads must be recorded in the new coordinate system. In the given principal stress state and electrical displacement in the material, as well as the crack surface that is free from force and electrical influences, representing the stress and electrical state satisfying equation (10), the superposition of the principal and perturbed states, we obtain boundary conditions for determination of the disturbed state using (11).

2. Solution Method. When considering the problem, we use Green's function $G_{IJ}(\vec{x}-\vec{x}')$ for an infinite electroelastic anisotropic space that satisfies the equation

$$E_{kJMn}^{(\alpha,\beta,\gamma)}G_{JM,kn}+\delta_{JM}\delta(\vec{x}-\vec{x}')=0, \tag{12}$$

where $\delta(\vec{x}-\vec{x}')$ is the Dirac delta function; δ_{JM} is the Kronecker symbol, and the comma after the index indicates differentiation with respect to the corresponding variable. The triple Fourier transformation can be used to represent Green's function that satisfies (12), using as

$$G_{JM}(\vec{x}-\vec{x}')=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_{JM}(\vec{\xi})D^{-1}(\vec{\xi})e^{i\vec{\xi}\cdot(\vec{x}-\vec{x}')}d\xi_1 d\xi_2 d\xi_3. \tag{13}$$

In expression (13) $A_{JM}(\vec{\xi})$ the corresponding algebraic additions of matrix elements are denoted by

$$\{K_{JM}(\vec{\xi})\}=\{E_{iJMn}^{(\alpha,\beta,\gamma)}\xi_i\xi_n\}, \tag{14}$$

and $D(\vec{\xi})$ is the determinant of the matrix (14) and is a polynomial of the eighth order.

Let us represent the perturbed electric and stress state, generalizing the purely elastic case [22], in terms of unknown jumps of displacements and electric potential through the crack surface in the form of

$$U_I(\vec{x}) = \frac{1}{4\pi^2} \sum_{N=1}^{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{E_{lJM3}^{(\alpha,\beta,\gamma)} \xi_l^N A_{IJ}(\vec{\xi}^N)}{\partial D(\vec{\xi}^N)/\partial \xi_3} \iint_S b_M(\vec{x}') e^{-i\vec{\xi}^N \cdot (\vec{x}-\vec{x}')} d\xi_1 d\xi_1 dx_1' dx_2',$$

where for the elliptical crack the unknown vector $\vec{b}(\vec{x})$, in the case of homogeneous force and electric loads in the material, takes the form

$$\vec{b}(\vec{x}) = \vec{b}(1 - x_1^2/a_1^2 - x_2^2/a_2^2)^{1/2}, \tag{15}$$

where a_1, a_2 are the semi-axes of the elliptical crack; $\vec{b}$ is a constant vector of the fourth order, the components of the vector are complex in the general case. Summation in the expressions is performed for ξ_3^M, which are the roots of the equation $D(\vec{\xi}) = 0$ with a negative imaginary part for $x_3 > 0$, and $\vec{\xi}^M$ is a vector of the form $\vec{\xi}^M = (\xi_1, \xi_2, \xi_3^M(\xi_1, \xi_2))$. The components of stress and electrical displacement are defined by the following expression:

$$\Sigma_{iJ}(\vec{x}) = E_{iJKl}^{(\alpha,\beta,\gamma)} U_{K,l}$$

$$= \left(\frac{-i}{4\pi^2}\right) \sum_{N=1}^{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \iint_S \frac{E_{iJKl}^{(\alpha,\beta,\gamma)} E_{pQM3}^{(\alpha,\beta,\gamma)} \xi_p^N \xi_l^N A_{KQ}(\vec{\xi}^N)}{\partial D(\vec{\xi}^N)/\partial \xi_3} b_M(\vec{x}') e^{-i\vec{\xi}^M \cdot (\vec{x}-\vec{x}')} d\xi_1 d\xi_2 dx_1 dx_2. \tag{16}$$

Carrying out the transformation of expressions (16) similarly to the elastic case [22], the components of stresses and electrical displacement in the crack plane for homogeneous force and electric fields are obtained in the following form:

$$\Sigma_{iJ}(\vec{x}) = \left(\frac{-i}{4}\right) \int_0^{2\pi} \sum_{N=1}^{4} F_{iJM}^{(\alpha,\beta,\gamma)}(\eta_1/a_1, \eta_2/a_2, \xi_3^N(\eta_1/a_1, \eta_2/a_2)) b_M^{(0,0)} d\varphi, \tag{17}$$

where the function $F_{iJM}^{(\alpha,\beta,\gamma)}(\xi_1, \xi_2, \xi_3)$ is defined by the expression

$$F_{iJM}^{(\alpha,\beta,\gamma)}(\xi_1, \xi_2, \xi_3) = E_{iJKl}^{(\alpha,\beta,\gamma)} E_{pQM3}^{(\alpha,\beta,\gamma)} \xi_p^N \xi_l^N \frac{A_{KQ}(\vec{\xi}^N)}{\partial D(\vec{\xi}^N)/\partial \xi_3}.$$

After additional analysis of asymptotic expressions for stresses and electrical displacement in the crack plane, the stress intensity and electrical displacement factors K_I, K_{II}, K_{III}, and K_D are obtained as follows:

$$k_{iJ} = i\sqrt{\pi a}(x_1^2/a_1^4 + x_2^2/a_2^4)^{-1/4} \sum_{N=1}^{4} F_{JM}^{(\alpha,\beta,\gamma)}(x_1/a_1^2, x_2/a_2^2, \xi_3^N(x_1/a_1^2, x_2/a_2^2)) b_M^{(0,0)},$$

$$K_I = k_{33}, \quad K_{II} = k_{31}n_1 + k_{32}n_2, \quad K_{III} = k_{31}(-n_2) + k_{32}n_1, \quad K_{IV} = K_D = k_{34}. \tag{18}$$

For the flat elliptical crack, the components of the normal vector to the crack boundary have the form

$$n_1 = (x_1/a_1^2)/(x_1^2/a_1^4 + x_2^2/a_2^4)^{1/2},$$

$$n_2 = (x_2/a_2^2)/(x_1^2/a_1^4 + x_2^2/a_2^4)^{1/2}.$$

Satisfying the boundary conditions on the crack surface and evaluating the one-dimensional integrals (17) by Gaussian quadratures, we determine the unknown displacement jumps and electric displacement through the surface of the elliptical crack, and then, according to (18), find the stress intensity and electrical displacement factors.

We will test the approach against a partial case of the problem with known exact solution. We consider the problem of a flat elliptical crack in a transversely isotropic electroelastic space in the isotropy plane of the piezoelectric material, under

TABLE 1

Parameter	PZT-4	PXE-5	TsTS-19	PZT-7A	$BaTiO_3$	PZT-5H
ν_{piezo}	0.48513	0.48815	0.45958	0.47324	0.34369	0.37867
ν_{elast}	0.35034	0.34591	0.36359	0.35239	0.29768	0.30074
$\nu_{control}$	0.35034	0.34591	0.36359	0.35239	0.29768	0.30074

TABLE 2

a_2 / a_1	φ					
	0	π/10	π/5	3π/10	2π/5	π/2
0.7	0.922061 (0.922061)	0.944164 (0.944164)	0.995662 (0.995662)	1.049943 (1.049943)	1.088403 (1.088403)	1.102073 (1.102073)
0.5	0.731778 (0.731780)	0.779345 (0.779347)	0.874177 (0.874179)	0.960134 (960136)	1.015842 (1.015844)	1.034891 (1.034893)
0.3	0.484906 (0.484949)	0.574152 (0.574204)	0.705990 (0.706054)	0.805592 (0.805664)	0.865420 (0.865497)	0.885313 (0.885392)
0.1	0.174191 (0.174455)	0.313217 (0.313690)	0.424300 (0.424943)	0.496109 (0.496860)	0.537334 (0.538148)	0.550841 (0.551676)

uniaxial tension σ_{33}^0 and shear forces σ_{23}^0. The normal component of electrical displacement D_z^0 is assumed to be zero. In this case, the stress intensity factors K_I for the dielectrically impermeable crack, according to [11], do not depend on the properties of the material and coincide with their expression for a purely elastic isotropic material with the same crack shape and pressure value σ_{33}^0. In this case, the electric displacement intensity factor K_D becomes zero along the crack boundary. At the same time, the stress intensity factors K_{II}, K_{III} depend on both the elastic and electrical properties of the material during shear according to [12]. According to [12], we can take the expressions of K_{II}, K_{III} for isotropic elastic material at the same shear loads and the same shape of the flat crack, and instead of Poisson's ratio ν in the corresponding expressions, we need to use ν_{piezo}, which is calculated based on the electroelastic properties of the piezoelectric material. Since the procedure for calculating ν_{piezo} is detailed in [12], we show in Table 1 only the values found for the piezoceramic materials [12]. The initial data on the properties of piezoelectric materials used in the calculations are contained in [1, 5, 7, 12]. The second row of Table 1 shows the values of ν_{elast}, which are found only from the elastic properties of the transversely isotropic electroelastic material (without taking into account electrical properties) [10]. The values of $\nu_{control}$ in the third row of Table 1 is found from the expression for ν_{piezo} if we put electrical permeability and piezoelectric moduli of the electroelastic material close to zero. When calculating the initial values of the piezomoduli e_{31}, e_{15}, e_{33} and dielectric constants k_{11}, k_{33} of the corresponding materials, their values were multiplied by 10^{-12}. The values of ν_{elast} and $\nu_{control}$, which are calculated using two different expressions, coincide.

Based on the results [11, 12] for an electroelastic transversely isotropic space containing an internal flat elliptical crack located in the isotropy plane of the material, under uniaxial tension along the axis of symmetry σ_{33}^0 and shear σ_{23}^0, as well as when $D_3^0 = 0$ (there is no vector component of electrical displacement that is normal to the surface crack) we obtain the following expressions of the SIFs along the boundary of the crack:

TABLE 3

a_2 / a_1	φ					
	0	π/10	π/5	3π/10	2π/5	π/2
0.7	0 (0)	0.484247 (0.484247)	0.873452 (0.873452)	1.140050 (1.140050)	1.292851 (1.292852)	1.342522 (1.342523)
0.5	0 (0)	0.486011 (0.486020)	0.824163 (0.824176)	1.032808 (1.032825)	1.147555 (1.147574)	1.184402 (1.184421)
0.3	0 (0)	0.452036 (0.452178)	0.699259 (0.699478)	0.843453 (0.843717)	0.922991 (0.923281)	0.948683 (0.948981)
0.1	0 (0)	0.303017 (0.303738)	0.425474 (0.426487)	0.500851 (0.502043)	0.543613 (0.544907)	0.557573 (0.558898)

TABLE 4

a_2 / a_1	φ					
	0	π/10	π/5	3π/10	2π/5	π/2
0.7	0.578320 (0.578320)	0.537139 (0.537139)	0.433285 (0.433285)	0.298525 (0.298525)	0.151398 (0.151398)	0 (0)
0.5	0.431202 (0.431210)	0.385068 (0.385074)	0.292024 (0.292029)	0.193174 (0.193177)	0.095988 (0.095990)	0 (0)
0.3	0.267534 (0.267618)	0.214890 (0.214957)	0.148660 (0.148707)	0.094655 (0.094684)	0.046323 (0.046337)	0 (0)
0.1	0.090782 (0.090998)	0.048016 (0.048130)	0.030152 (0.030223)	0.018736 (0.018780)	0.009094 (0.009116)	0 (0)

$$K_I = \frac{\sigma_{33}^0}{E(k)} \left(\frac{\pi b}{a} \right)^{1/2} (a^2 \sin^2 \beta + b^2 \cos^2 \beta)^{1/4}, \tag{19}$$

$$K_{II} = (\pi ab)^{1/2} \frac{k^2 \sigma_{23}^0 \sin \beta}{\left[(k^2 + \nu_{\text{piezo}} k_1^2) E(k) - \nu_{\text{piezo}} k_1^2 K(k) \right] (a^2 \sin^2 \beta + b^2 \cos^2 \beta)^{1/4}}, \tag{20}$$

$$K_{III} = \left(\frac{\pi b^3}{a} \right)^{1/2} \frac{(1 - \nu_{\text{piezo}}) k^2 \sigma_{23}^0 \cos \beta}{\left[(k^2 + \nu_{\text{piezo}} k_1^2) E(k) - \nu_{\text{piezo}} k_1^2 K(k) \right] (a^2 \sin^2 \beta + b^2 \cos^2 \beta)^{1/4}}, \tag{21}$$

$$K_D = 0,$$

where $k = (1 - b^2 / a^2)^{1/2}$, $k_1 = b / a$; $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind.

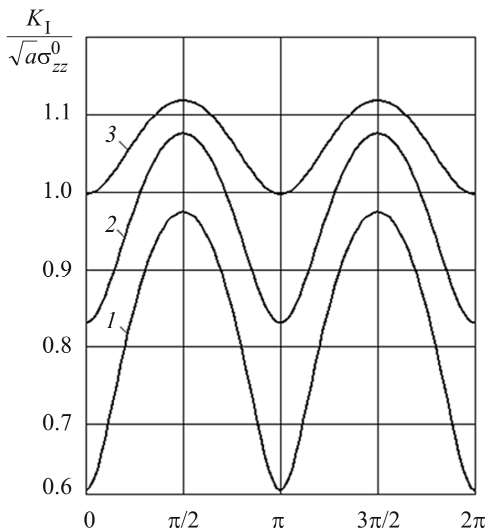

Fig. 1

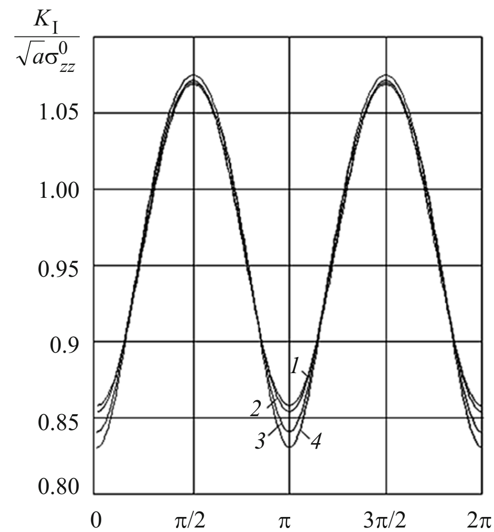

Fig. 2

The results of the comparison based on the two approaches (according to formulas (18) and according to expressions (19)–(21)) for material PZT-4 are shown in Tables 2–4. Tables 2, 3, and 4 show the calculated values of the SIFs K_I, K_{II}, and K_{III}, respectively. The values of the SIFs found using formulas (17), (18) are given without parentheses, and the values found on the basis of expressions (19)–(21) are given in parentheses. When finding the jumps of the displacement vector and the electric potential through the surface of an elliptic crack, the one-dimensional integrals in (17) were evaluated using 24-point Gaussian quadrature formulas.

Note that approximately the same accuracy of matching the results of calculations by the two approaches took place over the entire interval $[0, 2\pi]$ of the angle φ. To control the values, this interval was divided into 100 identical subintervals, the values were compared at the ends of the subintervals. When calculating the SIFs for the other piezoceramic materials from Table 1 (comparison was performed at the values of angles φ given in Tables 2–4), the results of calculations by the two approaches were consistent with the approximately the same accuracy as for the PZT-4 material.

Note also that when testing the calculation algorithm for an orthotropic electroelastic material with an arbitrarily oriented flat elliptical crack based on formulas (17), (18), the testing was against the partial case for an elliptical crack in a purely elastic orthotropic material.

3. Analysis of the Numerical Results. Consider an orthotropic piezoelectric material $Ba_2NaNb_5O_{15}$ with electroelastic properties (17 independent electroelastic parameters) that are given in [4].

Assume that an elliptical crack is located in the *xy* plane of the piezoelectric material. Figure 1 shows the distribution of the SIF K_I along the front of the elliptical crack under uniaxial tension σ_{zz}^0 in this orthotropic electrostatic material. Curves *1*, *2*, *3* correspond to the following ratios of the semi-axes: $b/a = 0.4$, 0.6, 0.8. The largest values of SIF are achieved at the points of the semi-minor axis of the elliptical crack.

Figure 2 shows the change of the SIF K_I along the crack boundary in the material $Ba_2NaNb_5O_{15}$ (under uniaxial tension σ_{zz}^0, $b/a = 0.6$) depending on the orientation of the crack in the material. Lines *1*, *2*, *3*, *4* correspond to the angles of rotation $\alpha = 0$, $\pi/6$, $\pi/3$, $\pi/2$.

Figure 3 shows the change of the SIF K_I for an elliptical crack in a transversely isotropic electrostatic material PZT-4 [1] at the same calculation parameters (crack geometry, load, and angles of rotation in the material) as in the previous case. It is seen that for PZT-4 the values of K_I more depend on the orientation of the crack in the material than for $Ba_2NaNb_5O_{15}$.

Figures 4–6 show the change of K_I, K_{II}, K_{III} along the boundary of the elliptical crack in the elastic orthotropic fiberglass material (orthogonally reinforced 2:1) according to [2, p. 64]. For this material, the distribution of SIF depends significantly on the orientation of the crack. Figure 4 shows that the value of K_I is maximum when $\alpha = \pi/3$. Figures 5 and 6 show the change in K_{II} and K_{III} along the front of the elliptical crack under internal pressure. Note that values of K_{II} and K_{III}

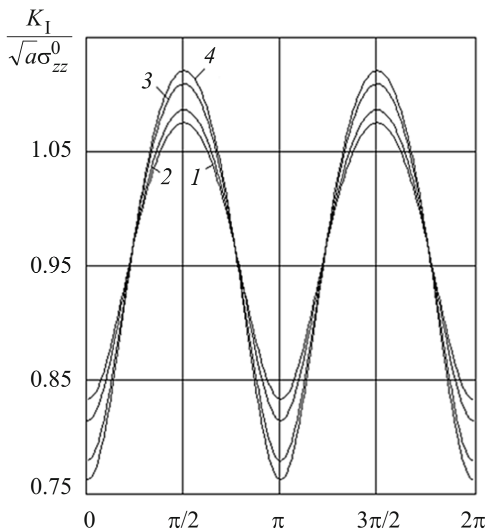

Fig. 3

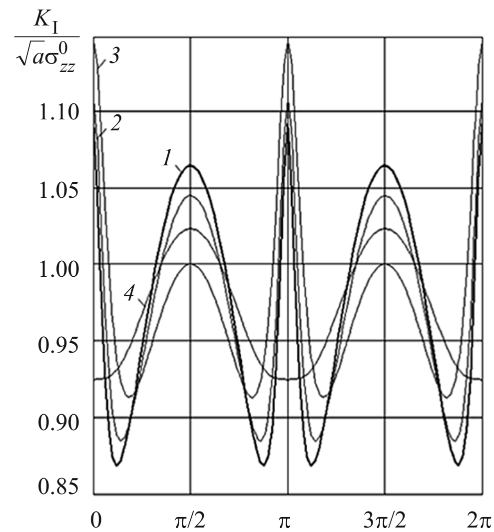

Fig. 4

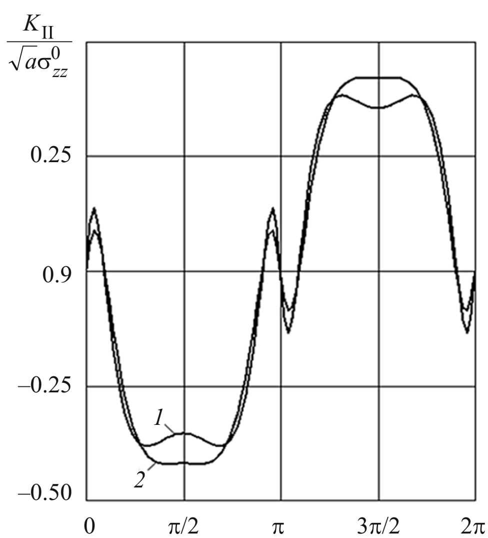

Fig. 5

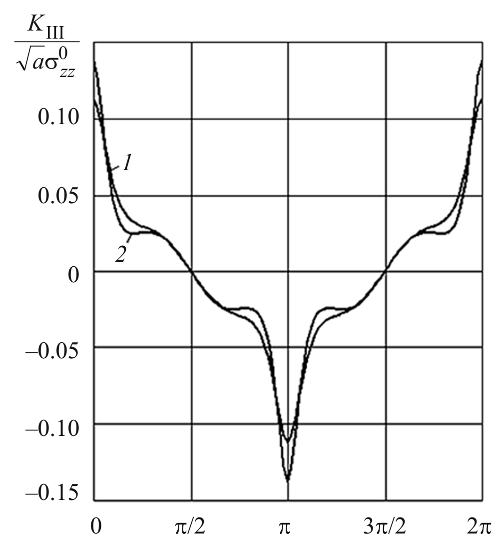

Fig. 6

under symmetric loads occur for the flat elliptical crack in the orthotropic material are nonzero when the crack is not in the plane of symmetry of the material. Curves *1* and *2* in these figures correspond to $\alpha = \pi/6$ and $\alpha = \pi/3$, and when $\alpha = 0$ and $\alpha = \pi/2$ the values of K_{II} and K_{III} are equal to zero along the entire boundary of the crack (in these cases, the crack is located in one of the planes of symmetry of the orthotropic material).

Conclusions. Thus, this paper has developed an approach to the study of the stress state in an orthotropic electroelastic space with an arbitrarily oriented flat elliptical crack under homogeneous force and electrical loads. The distribution of stress intensity factors along the boundary of the flat elliptical crack at different orientations in an orthotropic piezoelectric material under uniaxial tension has been studied.

REFERENCES

1. V. T. Grinchenko, A. F. Ulitko, and N. A. Shul'ga, *Electroelasticity*, Vol. 5 of the five-volume series *Mechanics of Coupled Fields in Structural Members* [in Russian], Naukova Dumka, Kyiv (1989).

2. S. G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Body*, Mir, Moscow (1981).
3. M. P. Savruk, *Stress Intensity Factors in Cracked Bodies*, Vol. 2 of the four-volume Handbook V. V. Panasyuk (ed.), *Fracture Mechanics and Strength of Materials* [in Russian], Naukova Dumka, Kyiv (1988).
4. M. O. Shul'ga and V. L. Karlash, *Resonant Electromechanical Vibrations of Piezoelectric Plates* [in Ukrainian], Naukova Dumka, Kyiv (2008).
5. W. Q. Chen and C. W. Lim, "3D point force solution for a permeable penny-shaped crack embedded in an infinite transversely isotropic piezoelectric medium," *Int. J. Fract.*, **131**, No. 3, 231–246 (2005).
6. C. R. Chiang and G. J. Weng, "The nature of stress and electric-displacement concentrations around a strongly oblate cavity in a transversely isotropic piezoelectric material," *Int. J. Fract.*, **134**, No. 3–4, 319–337 (2005).
7. M. L. Dunn and M. Taya, "Electroelastic field concentrations in and around inhomogeneities in piezoelectric solids," *J. Appl. Mech.*, **61**, No. 4, 474–475 (1994).
8. A. Y. Hodes and V. V. Loboda, "A contact zone approach for an arc crack at the interface between two electrostrictive materials," *Int. J. Solids Struct.*, **128**, No. 1, 262–271 (2017).
9. S. A. Kaloerov, "Determination of intensity factors for stresses, induction and field strength in multi-connected electro-elastic anisotropic media," *Int. Appl. Mech.*, **43**, No. 6, 631–637 (2007).
10. M. K. Kassir and G. Sih, *Three-dimensional Crack Problems*, Vol. 2, *Mechanics of Fracture*, Nordhoff International Publishing, Leyden (1975).
11. V. S. Kirilyuk, "On the stress state of a piezoceramic body with a flat crack under symmetric loads," *Int. Appl. Mech.*, **41**, No. 11, 1263–1271 (2005).
12. V. S. Kirilyuk, "Stress state of a piezoelectric ceramic body with a plane crack under antisymmetric loads," *Int. Appl. Mech.*, **42**, No. 2, 152–161 (2006).
13. V. S. Kirilyuk, "Stress state of a piezoceramic body with a plane crack opened by a rigid inclusion," *Int. Appl. Mech.*, **44**, No. 7, 757–768 (2008).
14. V. S. Kirilyuk and O. I. Levchuk, "Stress state of an orthotropic piezoelectric body with a triaxial ellipsoidal inclusion subject to tension," *Int. Appl. Mech.*, **55**, No. 3, 305–310 (2019).
15. L. V. Mol'chenko and I. I. Loos, "Thermomagnetoelastic deformation of flexible isotropic shells of revolution subject to Joule heating," *Int. Appl. Mech.*, **55**, No. 1, 68–78 (2019).
16. L. V. Mol'chenko and I. I. Loos, "Thermomagnetoelastic deformation of a flexible orthotropic conical shell with electrical conductivity and Joule heat taken into account," *Int. Appl. Mech.*, **55**, No. 5, 534–543 (2019).
17. Yu. N. Podil'chuk, "Representation of the general solution of statics equations of the electroelasticity of a transversally isotropic piezoceramic body in terms of harmonic functions," *Int. Appl. Mech.*, **34**, No. 7, 623–628 (1998).
18. Yu. N. Podil'chuk, "Electroelastic equilibrium of transversally isotropic, piezoceramic media containing cavities, inclusions, and cracks," *Int. Appl. Mech.*, **34**, No. 10, 1023–1034 (1998).
19. F. Shang, M. Kuna, and T. Kitamura, "Theoretical investigation of an elliptical crack in thermopiezoelectric material. Part 1: Analytical development," *Theor. Appl. Fract. Mech.*, **40**, No. 3, 237–246 (2003).
20. H. Sosa and N. Khutoryansky, "New developments concerning piezoelectric materials with defects," *Int. J. Solids Struct.*, **33**, No. 23, 3399–3414 (1996).
21. Z. K. Wang and B. L. Zheng, "The general solution of three-dimension problems in piezoelectric media," *Int. J. Solids Struct.*, **32**, No. 1, 105–115 (1995).
22. J. R. Willis, "The stress field around an elliptical crack in an anisotropic elastic medium," *Int. J. Eng. Sci.*, **6**, No. 5, 253–263 (1968).
23. T. Y. Zhang and C. F. Gao, "Fracture behaviors of piezoelectric materials," *Theor. Appl. Fract. Mech.*, **41**, No. 1 – 3, 339–379 (2004).