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PARAMETRIC VIBRATIONS OF A HINGED THERMOVISCOELASTIC RECTANGULAR PIEZOELECTRIC PLATE WITH SHEAR STRAINS AND DISSIPATIVE HEATING TAKEN INTO ACCOUNT*

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We presented the model of parametric vibrations of the hinged rectangular thermoviscoelastic piezoelectric plate taking into account the shear strains and dissipative heating. A solution to this problem is reduced to the classic Mathieu equation. We studied the effect of the temperature of dissipative heating on the parametric vibrations of the plate.

Keywords: viscoelastic piezoelectric plate, parametric vibrations, dissipative heating, shear strains, Mathieu equation

Introduction. A large number of publications are devoted to the forced vibrations of thin-walled elements made of passive (without piezoelectric effect) and active (with piezoelectric effect) materials [1–4]. A number of studies on the forced vibrations of thin-walled elements, taking into account the effect of dissipative heating were published in [5–15]. However, there are very few published studies of the parametric vibrations of structural elements made of piezoelectric materials. There are no studies of parametric vibrations taking into account the coupling of the electromechanical and temperature fields, such as the dissipative heating caused by hysteresis losses in an inelastic material. Meanwhile, at a certain value of the amplitude of the harmonic load, the temperature of dissipative heating can reach the point of material degradation when the active material loses the piezoelectric effect and becomes passive [8–15]. In this case, it becomes impossible to cause parametric vibrations in an element made of such a material by applying a harmonic potential difference. We will call such a load critical. For a passive material, the critical load is the melting point or temperature at which the performance of the structure deteriorates. To determine the critical load on a piezoelectric element, it is necessary to solve the related problem of thermoelectromechanics for various amplitudes of harmonic load and find the temperature amplitude when the dissipative heating becomes equal to the Curie point. In this case, a specific type of thermal destruction of an inelastic rectangular plate takes place, when it is not divided into parts, but, as indicated above, ceases to fulfill its functional purpose due to the transformation of the active material of the plate into a passive one.

The purpose of this article is to obtain a simple formula for the critical load. Here we will use the basic relations and notation of the article [16].

1. Problem Formulation and Solution. We consider a three-layer rectangular plate with an inner passive layer of thickness h_0 and two external identical piezoactive layers of thickness h_1. The total thickness of the plate is $H = h_0 + 2h_1$. In

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plan, the plate has dimensions $a \times b$. In the subcritical state, the planar displacements at the ends of the plate are taken to be zero. The piezoelectric layers are covered by electrodes to which a potential difference $\widehat{V}_0 = V_0 + V_1 \cos \omega t$ is applied. The governing equations for forces and moments taking into account shear strains for such a plate have the following form [5–8]:

$$T_{11} = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + \widehat{T}_0, \quad T_{22} = C_{12}\varepsilon_1 + C_{11}\varepsilon_2 + \widehat{T}_0, \quad T_{12} = C_{66}\varepsilon_{12},$$

$$M_{11} = D_{11}\kappa_1 + D_{12}\kappa_2, \quad M_{22} = D_{12}\kappa_1 + D_{11}\kappa_{yy}, \quad M_{12} = D_{66}\kappa_{12},$$

$$Q_1 = B_{13}\varepsilon_{13}, \quad Q_2 = B_{23}\varepsilon_{23}, \quad \widehat{T}_0 = 2\gamma_{31}\widehat{V}_0 = T_0 + T_1 \cos \omega t,$$

$$T_0 = 2\gamma_{31}V_0, \quad T_1 = 2\gamma_{31}V_1. \tag{1}$$

In (1), the stiffness characteristics are determined by the formulas:

$$C_{11} = h_0 B_{11}^0 + 2h_1 B_{11}^1, \quad D_{11} = \frac{h_0^3}{12} B_{11}^0 + h_1 B_{11}^1 \left(\frac{1}{2} h_0^2 + h_0 h_1 + \frac{2}{3} h_1^2 \right) + \frac{1}{6} h_1^3 \frac{\gamma_{31}^2}{\gamma_{33}},$$

$$C_{12} = h_0 B_{12}^0 + 2h_1 B_{12}^1, \quad D_{12} = \frac{h_0^3}{12} B_{12}^0 + h_1 B_{12}^1 \left(\frac{1}{2} h_0^2 + h_0 h_1 + \frac{2}{3} h_1^2 \right) + \frac{1}{6} h_1^3 \frac{\gamma_{31}^2}{\gamma_{33}},$$

$$C_{66} = h_0 B_{66}^0 + 2h_1 B_{66}^1, \quad D_{66} = \frac{h_0^3}{12} B_{66}^0 + h_1 B_{66}^1 \left(\frac{1}{2} h_0^2 + h_0 h_1 + \frac{2}{3} h_1^2 \right),$$

$$B_{13} = h_0 G_{13}^0 + 2h_1 G_{13}^1, \quad B_{23} = h_0 G_{23}^0 + 2h_1 G_{23}^1, \tag{2}$$

and $G_{13}^0, G_{23}^0, G_{13}^1, G_{23}^1$ are the shear moduli of the passive (superscript "0") and piezoactive (superscript "1") layers, respectively;

$$B_{11}^k = \frac{1}{(S_{11}^E)^k \left[1 - (\nu^k)^2\right]}, \quad B_{12}^k = \nu^k B_{11}^k, \quad \nu^k = -\frac{(S_{12}^E)^k}{(S_{11}^E)^k}, \quad \gamma_{31}^k = d_{11}^k \left[1 + (\nu^k)\right] B_{11}^k,$$

$$\gamma_{33}^k = e_{33}^k [1 - (k_p^2)^k], \quad (k_p^2)^k = 2(d_{31}^k)^2 \Big/ \left[(S_{11}^E)^k e_{33}^k (1 - \nu^k)\right], \quad G_{13}^k = G_{23}^k = \frac{e_{11}^k}{S_{55}^k e_{33}^k - (d_{15}^k)^2}, \tag{3}$$

$(S_{11}^E)^k, d_{ij}^k, e_{ij}^k$ are the compliance, piezomoduli, and dielectric constants of the material of the kth (k = 0, 1, 2) layer. As indicated above, it is hereinafter assumed that the piezoactive layers have the same electromechanical properties.

At zero tangential displacements of the ends of the plate in the subcritical momentless strain state $\varepsilon_1 = \varepsilon_2 = \varepsilon_{12} = 0$ and from the first three constitutive equations (1) we have: $T_{11} = T_{22} = \hat{T}_0, T_{12} = 0$.

Then, if inertia forces act only in the normal direction, the equations of motion in the postcritical state take the following form [13]:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \hat{T}_0 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + q = \rho \frac{\partial^2 w}{\partial t^2},$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \quad \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0. \tag{4}$$

The kinematic characteristics are determined by the following formulas [16]:

$$\varepsilon_{13} = \frac{\partial w}{\partial x} + \varphi_x, \qquad \varepsilon_{23} = \frac{\partial w}{\partial y} + \varphi_y,$$

$$\kappa_1 = \frac{\partial \varphi_x}{\partial x}, \qquad \kappa_2 = \frac{\partial \varphi_y}{\partial y}, \qquad \kappa_{12} = \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}, \tag{5}$$

where w, φ_x, φ_y are the normal deflection and angles of rotation.

Substituting (5) into the constitutive equations for the moments and transverse forces (1), and the obtained result into the equations of motion (4), we get three equations for the deflection w and transverse shears φ_x, φ_y:

$$D_{11}\frac{\partial^2 \varphi_x}{\partial x^2} + D_{12}\frac{\partial^2 \varphi_y}{\partial x \partial y} + D_{66}\left(\frac{\partial^2 \varphi_x}{\partial y^2} + \frac{\partial^2 \varphi_y}{\partial x \partial y}\right) - B_{13}\left(\frac{\partial w}{\partial x} + \varphi_x\right) = 0,$$

$$D_{11}\frac{\partial^2 \varphi_y}{\partial y^2} + D_{12}\frac{\partial^2 \varphi_x}{\partial x \partial y} + D_{66}\left(\frac{\partial^2 \varphi_x}{\partial x \partial y} + \frac{\partial^2 \varphi_y}{\partial x^2}\right) - B_{23}\left(\frac{\partial w}{\partial y} + \varphi_y\right) = 0,$$

$$B_{13}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \varphi_x}{\partial x}\right) + B_{23}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \varphi_y}{\partial y}\right) + (T_0 + T_1 \cos \omega t)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \tilde{\rho}\frac{\partial^2 w}{\partial t^2} = 0. \tag{6}$$

If the ends of the plate are hinged, the solution of the system of equations (6) is found by the formulas that automatically satisfy these boundary conditions:

$$w = W \sin(k_m x)\sin(p_n y), \qquad \varphi_x = X \cos(k_m x)\sin(p_n y),$$

$$\varphi_y = Y \sin(k_m x)\cos(p_n y), \qquad k_m = \frac{m\pi}{a}, \qquad p_n = \frac{n\pi}{b}. \tag{7}$$

Substituting (7) into (6), we get the system of equations

$$a_{11}X + a_{12}Y = -k_m B_{13} W, \qquad a_{12}X + a_{22}Y = -k_m B_{23} W,$$

$$k_m B_{13} X + p_n B_{23} Y + \left[k_m^2 B_{13} + p_n^2 B_{23} + (k_m^2 + p_n^2)(T_0 + T_1 \cos \omega t)\right] W + \tilde{\rho}\ddot{W} = 0. \tag{8}$$

Here

$$a_{11} = k_m^2 D_{11} + p_n^2 D_{66} + B_{13}, \qquad a_{12} = k_m p_n (D_{12} + D_{66}), \qquad a_{22} = p_n^2 D_{11} + k_m^2 D_{66} + B_{23}. \tag{9}$$

Solving the first two equations of (8) for X and Y, we obtain

$$X = x_{11} W, \qquad Y = y_{11} W,$$

$$x_{11} = \frac{\Delta_1}{\Delta}, \qquad y_{11} = \frac{\Delta_2}{\Delta}, \tag{10}$$

where

$$\Delta_1 = \begin{vmatrix} -k_m B_{13} & a_{12} \\ -p_n B_{23} & a_{22} \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} a_{11} & -k_m B_{13} \\ a_{12} & -p_n B_{23} \end{vmatrix}, \qquad \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{12} \end{vmatrix}. \tag{11}$$

Substituting (10) into the third of Eqs. (8), we get a differential equation for W

$$\ddot{W} + \omega^2 W + (P_0 + P_1 \cos \omega t) W = 0. \tag{12}$$

Here

$$\omega^2 = \left[B_{13}(k_m^2 + k_m x_{11}) + B_{23}(p_n^2 + p_n y_{11})\right]/\rho,$$

$$P_0 = T_0(k_m^2 + p_n^2)/\rho, \quad P_1 = T_1(k_m^2 + p_n^2)/\rho. \tag{13}$$

Let us represent Eq. (12) in the form

$$\frac{d^2 w}{dt^2} + \Omega^2(1 - 2\mu\cos\omega t)w = 0, \tag{14}$$

where

$$\Omega^2 = \omega^2 + P_0, \quad 2\mu = -P_1 / P_0. \tag{15}$$

Equation (14) is the classical Mathieu equation well studied in the literature [1]. It provides information on the areas of dynamic instability (ADI) of oscillations.

According to [1], the boundaries of the ADI are concentrated around the frequencies

$$\omega = 2\Omega / k \quad (k = 1,\ 2,\ 3,\ \ldots). \tag{16}$$

In this case, the boundaries of the first (main) ADI ($k = 1$) are determined by the formula

$$\omega = 2\Omega(1 \pm \mu)^{1/2}, \tag{17}$$

the second one ($k = 2$):

$$\omega = 2\Omega\left(1 + \frac{1}{3}\mu^2\right)^{1/2}, \quad \omega = \Omega(1 - 2\mu^2)^{1/2}, \tag{18}$$

the third one:

$$\omega = \frac{2}{3}\Omega\left(1 - \frac{9\mu^2}{1 \pm 9\mu}\right). \tag{19}$$

We will use the simplest damping model, assuming that this model is proportional to the rate of change in the normal deflection $\dot{w}$. Then Eq. (14) is replaced by the equation

$$\frac{d^2 w}{dt^2} + 2\varepsilon\frac{dw}{dt} + \Omega^2(1 - 2\mu\cos\omega t)w = 0. \tag{20}$$

Here ε characterizes the damping.

In this case, the main ADI is found by the formula

$$\omega = 2\Omega\left[1 \pm (\mu^2 - \Delta^2)^{1/2}\right], \quad \Delta = 2\varepsilon / \Omega. \tag{21}$$

Hence we have the minimum critical force (MCU) for the first (main) ADI:

$$\mu_* = \Delta. \tag{22}$$

The second ADI is determined by the ratio

$$\omega = \Omega\left[1-\mu^2 \pm \left(\mu^2 - \frac{\Delta^2}{1-\mu^2}\right)^{1/2}\right]^{1/2}, \tag{23}$$

in this case, the MCU is determined by the formula

$$\mu_* = \Delta^{1/2} \tag{24}$$

the MCU for the third ADI is determined by the ratio

$$\mu_* = \Delta^{1/3}. \tag{25}$$

Calculations [1] show that when the damping of the MCU is taken into account, it sharply increases with increase in the ADI number. Therefore, in the study of parametric oscillations, only the first (main) ADI is of practical importance.

2. Effect of the Temperature of Dissipative Heating on Parametric Vibrations. For the boundary conditions considered above, due to the equality of strains to zero in the subcritical state, dissipative heating will be caused only by dielectric losses in the material. In this case, the stationary temperature of dissipative heating in the piezoelectric layer is determined from the solution of the energy equation of the form

$$\frac{d^2\theta}{dz^2} + w = 0, \qquad w = \frac{1}{2\lambda}\omega^2 \varepsilon\left[(E')^2 + (E'')^2\right]. \tag{26}$$

Here the origin is selected at the center of the inner passive layer; $\theta = T - T_0$, T_0 is the initial temperature of the layer; ε is the dielectric constant; λ is the coefficient of thermal conductivity. Let a constant temperature equal to T_0 be maintained on the outer surfaces of the plate so that $\theta = 0$ when $z = \pm h_2$, $h_2 = (h_0/2) + h_1$ with the indicated electrical load

$$E' = V_1 / h_1, \qquad E'' = 0, \qquad w = \frac{1}{2\lambda}\omega^2 \varepsilon (V_1 / h_1)^2. \tag{27}$$

Then the solution to the energy equation has the form

$$\theta = (1/2) w (h_2^2 - z^2). \tag{28}$$

The maximum temperature of dissipative heating in the upper piezoactive layer is achieved when $z = h_0/2$ and is equal to $\theta_{\max} = \frac{1}{2} w\left(h_2^2 - \frac{h_0^2}{4}\right)$. Equating the maximum temperature to the point of material degradation (for example, the Curie point) θ_C of the active material, we find the critical value of the potential difference V_C

$$V_C = \frac{2h}{\omega}\sqrt{\frac{\lambda\theta_K}{\varepsilon\left(h_2^2 - \frac{h_0^2}{4}\right)}}. \tag{29}$$

When the potential difference exceeds the critical value, parametric vibratilns cannot be excited due to the loss of functional capacity by the piezoactive material.

Thus, when $V > V_C$ there is a specific type of thermal destruction of a viscoelastic plate during its parametric vibrations when it retains its integrity but loses its functional purpose.

In conclusion, we note that, as a rule, piezoelectric elements operate at temperatures much lower than the Curie point. Therefore, under θ_C it is necessary to assume the maximum temperature allowed by the conditions of their performance.

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