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FRACTURE OF A SEMI-BOUNDED COMPOSITE MATERIAL WITH A NEAR-SURFACE PENNY-SHAPED CRACK UNDER COMPRESSION*

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The nonclassical problem of fracture mechanics for a near-surface crack is solved in the case of small distances between the free surface and the crack plane. The problem is axisymmetric for a penny-shaped crack. A numerical study for a composite material is carried out as an example.

Keywords: near-surface crack, compression along crack, critical stress, composite material

Introduction. When the forces acting on a body with a plane crack are parallel to the crack plane, the stress intensity factors predicted by linear fracture mechanics are equal to zero and the Irwin–Griffith failure criteria are inapplicable. In the case of tension and compression of the material along the crack plane, the approach first proposed in [1] is applied. The failure criterion, in this case, is the local loss of stability near the crack described by the three-dimensional linearized theory of elastic stability. According to this approach, the fracture is initiated by local loss of stability near cracks, and the critical compressive loads are determined by solving the appropriate eigenvalue problems using the three-dimensional linearized theory of stability of deformable bodies. The reviews [9–14] and monographs [2–4] provide detailed information on the fracture of materials under compression along the plane of cracks for various locations of interacting cracks. Note that a detailed analysis of approaches to the problems of fracture of materials along cracks was for the first time performed in [9] with a detailed bibliography.

The relationship between the critical compressive stress and the distance from the free surface to the plane of a near-surface penny-shaped crack (Fig. 1) was presented in [5, 7, 13, 14] for composite and highly elastic materials.

The case where the distance between the free surface and the crack plane tends to zero is of particular interest. This case has been investigated in detail for highly elastic materials [5, 7], but not for composite materials. This issue is of theoretical and applied interest for design of thin interlayers formed after spraying, thermal shock, etc.

Here we will use the combined numerical-analytical method proposed in [7] to solve the problem of fracture of a composite half-space compressed along a penny-shaped near-surface crack for small distances between the free surface and the crack.

1. Problem Formulation and Generalized Solution. We consider a penny-shaped crack of radius a in half-space $x_3 \geq -h$ located in the plane $x_3 = 0$ centered on axis Ox_3. The initial stresses acting along the crack cause biaxial tension and compression [14]:

$$S_{33}^0 = 0, \quad S_{11}^0 = S_{22}^0 \neq 0, u_m^0 = \delta_{jm}(\lambda_j - 1)x_j,$$

$$\lambda_1 = \lambda_2 \neq \lambda_3, \quad \lambda_j = \text{const},$$

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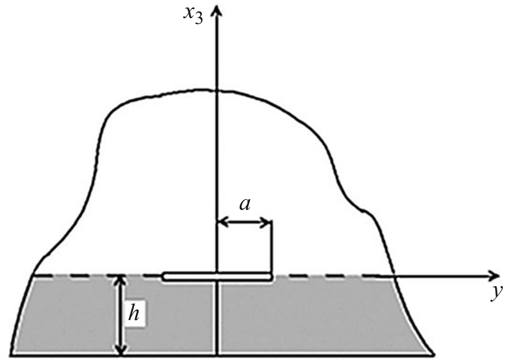

Fig. 1

where λ_j are the elongations along the axes; x_j are the Lagrange coordinates that coincide with the Cartesian coordinates in the undeformed state; S_{ij}^0 are the components of the symmetric stress tensor; u_j^0 are the displacements corresponding to the prestresses S_{ij}^0.

For the axisymmetric linearized problem, we will use the following boundary conditions on the crack faces $x_3 = \pm 0$ and on the free surface $x_3 = -h$ [14]:

$$t_{33} = 0, \quad x_3 = \pm 0, \quad 0 \le r < a; \quad t_{3r} = 0, \quad x_3 = \pm 0, \quad 0 \le r < a,$$

$$t_{33} = 0, \quad x_3 = -h, \quad 0 \le r < \infty; \quad t_{3r} = 0, \quad x_3 = -h, \quad 0 \le r < \infty,$$

where t_{ij} is the asymmetrical Kirchhoff stress tensor; (r, θ, x_3) are the cylindrical coordinates corresponding to the Cartesian coordinates x_j.

In the case of a body with a macrocrack (its size is much larger than the size of microstructures), the composite is considered as an anisotropic medium with induced macrocharacteristics [14].

Using the technique used in [14], we reduce the problem to the system of Fredholm equations with the additional condition

$$f(\xi) + \frac{1}{\pi k}\int_0^1 M_1(\xi,\eta) f(\eta) d\eta - \frac{2}{\pi k}\int_0^1 N_1(\xi,\eta) g(\eta) d\eta = 0,$$

$$g(\xi) + \frac{1}{\pi k}\int_0^1 M_2(\xi,\eta) g(\eta) d\eta - \frac{2}{\pi k}\int_0^1 N_2(\xi,\eta) f(\eta) d\eta + \widetilde{C}_1 = 0,$$

$$\int_0^1 g(\xi) d\xi = 0 \quad (0 \le \xi \le 1, 0 \le \eta \le 1),$$

$$f(\xi) \equiv \varphi(a\xi), \quad g(\xi) \equiv \psi(a\xi). \tag{1}$$

All the quantities in Eqs. (1) are dimensionless; $\widetilde{C}_1$ is an unknown constant related to the additional condition.

The kernels of the integral equations (1) are given by

$$M_1(\xi,\eta) = R_1(\eta+\xi) - R_1(1+\xi) + R_1(\eta-\xi) - R_1(1-\xi),$$

$$N_1(\xi,\eta) = S_1(\eta+\xi) + S_1(\eta-\xi),$$

$$M_2(\xi,\eta) = S_2(\eta+\xi) + S_2(\eta-\xi),$$

$$N_2(\xi,\eta)=R_2(\eta+\xi)-R_2(1+\xi)+R_2(\eta-\xi)-R_2(1-\xi),$$

$$R_1(\zeta)=2\left\{2\frac{k_2}{k}I_0(\beta_1+\beta_2,\zeta)-\frac{1}{2}\frac{(k_1+k_2)}{k}\left[\frac{k_2}{k_1}I_0(2\beta_2,\zeta)+I_0(2\beta_1,\zeta)\right]\right\},$$

$$S_1(\zeta)=\frac{(k_1+k_2)}{k}\left\{I_1(\beta_1+\beta_2,\zeta)-\frac{1}{2}[I_1(2\beta_1,\zeta)+I_1(2\beta_2,\zeta)]\right\},$$

$$S_2(\zeta)=2\left\{2\frac{k_1}{k_2}I_0(\beta_1+\beta_2,\zeta)-\frac{1}{2}\frac{(k_1+k_2)}{k}\left[\frac{k_1}{k_2}I_0(2\beta_2,\zeta)+I_0(2\beta_1,\zeta)\right]\right\},$$

$$R_2(\zeta)=\frac{(k_1+k_2)}{k}\left\{I_{-1}(\beta_1+\beta_2,\zeta)-\frac{1}{2}[I_{-1}(2\beta_1,\zeta)+I_{-1}(2\beta_2,\zeta)]\right\},$$

$$I_0(\rho,\zeta)=\rho(\zeta^2+\rho^2)^{-1},\quad I_{-1}(\rho,\zeta)=-\frac{1}{2\beta}\log(\zeta^2+\rho^2),$$

$$I_1(\rho,\zeta)=\beta(\rho^2-\zeta^2)(\zeta^2+\rho^2)^{-2},$$

$$\beta=ha^{-1},\quad \beta_i=\beta(n_i^0)^{-1/2},\quad i=1,2. \tag{2}$$

2. Procedure of Analysis. To establish the relationship between the critical shortening-elongations (stresses) and the dimensionless distance between the free surface and the crack plane β from integral equations (1), we used a procedure based on the Bubnov–Galerkin method. Let the coordinate functions be power functions. For N coordinate functions, we have

$$f(x)=\sum_{i=0}^{N}F_i x^i,\quad g(x)=\sum_{i=0}^{N}G_i x^i. \tag{3}$$

Unlike the previous studies [13, 14] where system (1) was numerically integrated after substitution of the coordinate functions (3), we use the method proposed in [7], which allows us to obtain new results for highly elastic materials [5–8]. It allows us, using a computer algebra software, to analytically evaluate the integrals of functions (2) included in the kernels of system (1) for the system of coordinate functions. This increased the accuracy of computation by excluding the error of numerical integration.

To accelerate the evaluation of integrals, we use an algorithm based on recurrence formulas:

$$L(n)=\int_0^1\frac{x^n}{(a^2+(x+y)^2)^2}dx,$$

$$L(n)=\frac{1}{n-3}\left(\frac{1}{a^2+(1+y)^2}-2y(n-2)L(n-1)-(a^2+y^2)(n-1)L(n-2)\right)\quad (n\neq 3),$$

$$V(n)=\int_0^1\frac{x^n}{(a^2+(x-y)^2)^2}dx,$$

$$V(n)=\frac{1}{n-3}\left(\frac{1}{a^2+(1-y)^2}+2y(n-2)V(n-1)-(a^2+y^2)(n-1)V(n-2)\right)\quad (n\neq 3). \tag{4}$$

The use of recurrence formulas (4) makes it possible to speed up the analytical evaluation of integrals from kernels (2) of the integral equation (1).

As a result, the system of integral equations (1) is reduced to the system of $2(N+1)+1$ equations:

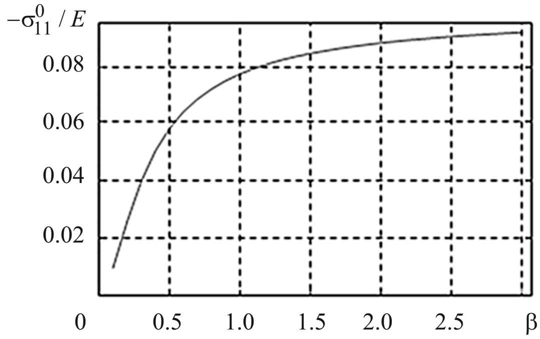

Fig. 2

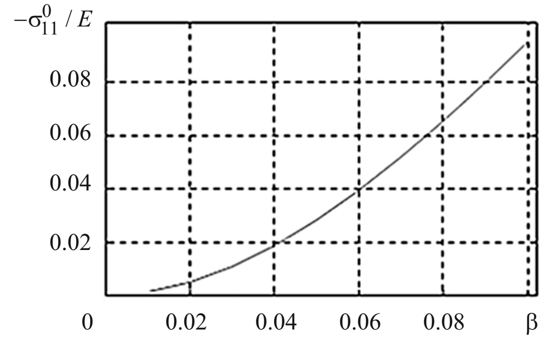

Fig. 3

$$\sum_{i=0}^{N} F_i F_{1ji} + \sum_{i=0}^{N} G_i G_{1ji} = 0,$$

$$\sum_{i=0}^{N} F_i F_{2ji} + \sum_{i=0}^{N} G_i G_{2ji} + \widetilde{C}_1 = 0,$$

$$\sum_{i=0}^{N} \frac{1}{i+1} G_i = 0 \quad (0 \le j \le N), \tag{5}$$

where F_{kji} and G_{kji} are the exact expressions calculated using computer algebra software and depending on the material constants and dimensionless distance between the crack and the free surface.

3. Numerical Results. As an example, we studied a composite with given characteristics of a transversely isotropic medium:

$$\nu = 0.3, \quad \nu' = 0.2, \quad G'/E = 0.1, \quad E'/E = 0.5. \tag{6}$$

Substituting (6) into (5) we get a system of equations whose coefficients F_{kji} and G_{kji} depend on the parameters β and σ_{11}^0. Analyzing this system numerically, we can determine the minimum critical stresses at which the system loses stability for different values of the dimensionless distance β between the crack and the free surface.

Using 10 coordinate functions, we obtain the dependences of the critical stresses on the dimensionless distance $\sigma_{11}^0(\beta)$ shown in Fig. 2 for large distances and in Fig. 3 for small distances. The results obtained for large dimensionless distances (Fig. 2) are in good agreement with the data obtained in [14], which means that the proposed procedure provides good accuracy. Figure 2 contains new data that could not be obtained using the methods proposed previously.

The table collects the critical stresses for very small dimensionless distances. Assuming that $\sigma_{11}^0 / E = A\beta^2$, the table gives the value of coefficient A.

Conclusions. We have analyzed, for the first time, the critical parameters defining the fracture of a half-space with a near-surface penny-shaped crack in a composite material under compression for a wide range of distances between the crack and the free surface. For the first time, results were obtained for the values of the relative distance between the crack and the free surface up to $\beta = 10^{-9}$, which are several orders of magnitude lower than those obtained earlier.

From the analysis of the results, it can be determined that for small dimensionless distances, the critical stresses σ_{11}^0 / E are quadratically dependent on the dimensionless distance with coefficient $A = -1.28$.

TABLE

β	σ_{11}^0 / E	A
9×10^{-2}	-7.997×10^{-3}	–0.987
8×10^{-2}	-6.553×10^{-3}	–1.024
7×10^{-2}	-5.197×10^{-3}	–1.061
6×10^{-2}	-3.951×10^{-3}	–1.097
5×10^{-2}	-2.834×10^{-3}	–1.134
4×10^{-2}	-1.871×10^{-3}	–1.169
3×10^{-2}	-1.084×10^{-3}	–1.204
2×10^{-2}	-4.947×10^{-4}	–1.237
1×10^{-2}	-1.264×10^{-4}	–1.268
1×10^{-3}	-1.290×10^{-6}	–1.290
1×10^{-4}	-1.287×10^{-8}	–1.287
1×10^{-5}	-1.285×10^{-10}	–1.285
1×10^{-6}	-1.284×10^{-12}	–1.284
1×10^{-9}	-1.284×10^{-18}	–1.284

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