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CRITICAL ELECTRIC LOAD ON A HINGED THERMOVISCOELASTIC RECTANGULAR PLATE WITH PIEZOELECTRIC SENSORS AND ACTUATORS*

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We proposed a failure criterion for the system of controlling the forced vibration of a hinged rectangular thermoviscoelastic plate with piezoelectric sensors and actuators. This criterion is associated with the self-heating temperature reaching some critical value. The critical electric load is found on the basis of this criterion.

Keywords: thermoviscoelastic plate, sensors and actuators, vibrations, self-heating, critical electric load

Introduction. Thin plates made of polymer-based composite materials are the most common structural members of modern technology [1, 3, 12, 20, 30–32]. One of the basic operating modes of such elements is forced harmonic vibrations, including resonance [1, 2, 5–8, 18, 19, 26]. Many polymeric materials have significant hysteresis losses due to harmonic vibrations. This effect is widely used for damping the forced resonant vibrations of elements. To this end, elements with high hysteresis losses are incorporated into the structure of an element with low hysteresis losses. Such type of damping is called passive damping. The passive damping of vibrations is widely discussed in the literature (encyclopedias, monographs, and journal articles) [3, 4, 11, 16, 18, 23]. However, an increase in hysteresis losses can be accompanied by a significant increase in the self-heating temperature (SHT). It affects all aspects of the mechanical and thermal state of a structure: stress–strains distribution, the dynamic characteristics of resonant vibrations (amplitude, amplitude– and temperature–frequency characteristics, frequency dependence of damping factor), the dynamic and static, mechanical and thermal stability, and creep behavior of thin-walled elements [1, 5–9, 21, 24, 27].

In addition to passive methods for vibration damping, recent trends have been toward the use of active methods that incorporate piezoelectric elements into passive thin-walled metallic, polymeric, or composite structures. In most cases, piezoelectric components act as active elements. There are several basic approaches to the active damping of vibrations of thin-walled elements under a mechanical load. One approach is the usage of piezoelectric actuators to which voltage is applied to balance the mechanical load. The other approach is based on the joint use of piezoelectric sensors and actuators. A voltage proportional to the rate of variation in the sensor voltage is applied to the actuator while terms proportional to the rate of variation of displacements appear in the equations of motion for displacements. As a result, damping caused by hysteresis losses in the structural material results in additional damping proportional to the rate of variation of displacements. The third approach is to combine passive and active damping. Foreign studies on the active control of stationary and nonstationary vibrations of structural members are reviewed in [20, 22, 31, 32]. It shows the importance of studying the influence of various factors on the

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efficiency of sensors and actuators in the harmonic loading of structural members. A particularly important task is to develop criteria for the loss of the functionality of sensors and actuators.

There are various criteria for the loss of load-carrying capacity of thin-walled structural members under mechanical and thermal loading (static and dynamic buckling, unacceptable strain level, etc.). Any structural member with piezoelectric sensors and actuators under harmonic electromechanical loading has a limiting operating temperature θ_{cr}. The performance of the piezoelectric element is sharply worsened when it reaches that temperature. This point will be called critical. As a rule, it is much lower than the Curie point. When approaching that point, the electromechanical-coupling coefficient tends to zero [2, 7, 9, 21, 22, 24, 27]. The load under which the self-heating temperature becomes critical will also be called critical. To determine it, it is necessary to solve the coupled problem of thermoelectromechanics for various amplitudes of the harmonic loading and to find the amplitude at which the self-heating temperature becomes equal to θ_{cr}. An analytical solution to the problem of vibrations and self-heating of a hinged plate with sensors and actuators is presented in [1].

This solution will be used here to obtain an analytical expression for the critical load. We will use the basic relations, notation, and results of [1].

1. Problem Formulation and Solution. We will consider a three-layer rectangular plate with a passive core layer of thickness h_0 and two piezoelectric face layers with opposite polarization, each of thickness h_1. In plan, the plate has dimensions $a \times b$. The constitutive equations for the forces and moments, taking into account shear strains in the plate, have the following form [6, 7, 10, 28]:

$$N_{xx} = A_{11}\varepsilon_{xx} + A_{12}\varepsilon_{yy}, \quad N_{yy} = A_{12}\varepsilon_{xx} + A_{22}\varepsilon_{yy}, \quad N_{xy} = A_{66}\varepsilon_{xy},$$

$$M_{xx} = D_{11}\kappa_{xx} + D_{12}\kappa_{yy} - M_9, \quad M_{yy} = D_{12}\kappa_{xx} + D_{22}\kappa_{yy} - M_9, \quad M_{xy} = D_{66}\kappa_{xy},$$

$$Q_x = K_S A_{55}\varepsilon_{xz}, \quad Q_y = K_S A_{44}\varepsilon_{yz}.$$

For a hinged rectangular plate, the dissipation function represented in [10] in terms of strains, using kinematic relations and the solution of the electromechanics problem obtained in [10], can be reduced to the following form (x, y are Cartesian coordinates):

$$D = d_0 + d_1\cos(2k_m x) + d_2\cos(2p_n y) + d_3\cos(2k_m x)\cos(2p_n y), \tag{1}$$

where

$$d_0 = \frac{\omega}{8}(D_1 + D_2 + D_3 + D_4), \quad d_1 = \frac{\omega}{8}(-D_1 + D_2 + D_3 - D_4),$$

$$d_2 = \frac{\omega}{8}(D_1 + D_2 - D_3 - D_4), \quad d_3 = \frac{\omega}{8}(D_1 + D_2 - D_3 - D_4),$$

$$D_1 = (A''_{11} + D''_{11})|U|^2 k_m^2 + 2(A''_{12} + D''_{12})|UV| k_m p_n + (A''_{22} + D''_{22})|V|^2 p_2^n,$$

$$D_2 = 2(A''_{66} + D''_{66})[(p_n U' + k_m V')^2 + (p_n U'' + k_m V'')^2],$$

$$D_3 = K_S A''_{44}[(p_n w' + U')^2 + (p_n w'' + U'')^2],$$

$$D_4 = K_S A''_{55}[(k_m w' + U')^2 + (p_n w'' + U'')^2], \quad |UV| = U'V' + U''V'',$$

$$U' = x'_{11}M + (w'_1 w' - w''_1 w''), \quad U'' = x''_{11}M + (w'_1 w'' + w''_1 w'),$$

$$V' = y'_{11}M + (w'_2 w' - w''_2 w''), \quad V'' = y''_{11}M + (w'_2 w'' + w''_2 w'),$$

$$k_m = \frac{m\pi}{a}, \quad p_n = \frac{n\pi}{b}\ (m,\ n = 1, 3, 5, \ldots), \quad M = \frac{16}{mn\pi^2}M_0, \quad M_0 = m_0 V_0,$$

$$m_0 = \frac{h_0 + h_1}{S'_{11}(1-\nu)}d'_{31}, \quad \nu = -\frac{S'_{12}}{S'_{11}}, \tag{2}$$

where the indices mn are omitted everywhere. For example, M is used instead of M_{mn}, etc. The complex quantities $x_{11} = x'_{11} + ix''_{11}$, $y_{11} = y'_{11} + iy''_{11}$, $w_1 = w'_1 + iw''_1$, $w_2 = w''_2 + iw''_2$ are defined by the expressions

$$(x_{11}, y_{11}) = \frac{(a_{12} p_n - a_{22} k_m,\ a_{12} k_m - a_{11} p_n)}{\Delta},$$

$$(w_1, w_2) = \frac{[K_S (a_{12} A_{44} p_n - a_{22} A_{55} k_m),\ K_S (a_{12} A_{55} k_m - a_{11} A_{44} p_n)]}{\Delta},$$

$$\Delta = a_{11} a_{22} - a_{12}^2, \quad a_{11} = \left(\frac{m\pi}{a}\right)^2 D_{11} + \left(\frac{n\pi}{b}\right)^2 D_{66} + K_S A_{55},$$

$$a_{22} = \left(\frac{n\pi}{b}\right)^2 D_{22} + \left(\frac{m\pi}{a}\right)^2 D_{66} + K_S A_{44}, \quad a_{12} = \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)(D_{12} + D_{66}).$$

The self-heating temperature determined from the energy equation is

$$\theta = \theta_0 + \theta_1 \cos(2k_m x) + \theta_2 \cos(2p_n y) + \theta_3 \cos(2k_m x)\cos(2p_n y). \quad (3)$$

The maximum temperature is defined by the formula

$$\theta_{\max} = \theta_0 + \theta_1 + \theta_2 + \theta_3. \quad (4)$$

The following notation is used in (3), (4):

$$\theta_0 = \frac{d_0}{h}, \quad \theta_1 = \frac{d_1}{4h\lambda k_m^2 + 2\delta}, \quad \theta_2 = \frac{d_2}{4h\lambda p_n^2 + 2\delta}, \quad \theta_3 = \frac{d_3}{4h\lambda(k_m^2 + p_n^2) + 2\delta}, \quad (5)$$

where h is the total thickness of the plate; λ is the reduced thermal conductivity factor; δ is the heat-transfer coefficient to the environment.

The critical load is found by equating the maximum temperature (4) to the critical point θ_{cr}:

$$\theta_{\max} = \theta_{cr}.$$

As a result, we obtain a quadratic equation for determining M:

$$M^2 + \frac{m_1}{m_2} M - \frac{1}{m_2}\theta_{cr} = 0, \quad (6)$$

where

$$m_1 = \frac{d_{01}}{h} + \frac{d_{11}}{4h\lambda k_m^2 + 2\delta} + \frac{d_{21}}{4h\lambda p_m^2 + 2\delta} + \frac{d_{31}}{4h\lambda(k_m^2 + p_m^2) + 2\delta},$$

$$m_2 = \frac{d_{02}}{h} + \frac{d_{12}}{4h\lambda k_m^2 + 2\delta} + \frac{d_{22}}{4h\lambda p_m^2 + 2\delta} + \frac{d_{32}}{4h\lambda(k_m^2 + p_m^2) + 2\delta}.$$

The solution of Eq. (6) has the form

$$M = -\frac{1}{2}\frac{m_1}{m_2} \pm \sqrt{\frac{1}{4}\frac{m_1^2}{m_2^2} + \frac{1}{m_2}\theta_{cr}}. \quad (7)$$

Then the critical value M is

$$M_{cr} = \frac{1}{2}\frac{m_1}{m_2}\left(\sqrt{1+\frac{4m_2}{m_1^2}\theta_{cr}} - 1\right). \tag{8}$$

The critical voltage applied to the plate can found using the last of formulas (2):

$$V_{0cr} = \frac{mn\pi^2}{16m_0} M_{cr} = \frac{mn\pi^2 m_1}{32m_0 m_2}\left(\sqrt{1+\frac{4m_2}{m_1^2}\theta_{cr}} - 1\right). \tag{9}$$

Conclusions. Thin-walled elements such as rods, plates, and shells made of polymer-based composite materials are widely used in various fields of modern technology. The vibration mode of their operation, in particular, resonance one, is one of the most common. In recent years, piezoelectric sensors and actuators have been intensively used to actively control the vibrations of the mentioned elements. Due to the hysteresis losses inherent in polymeric materials, such elements can show a significant increase in the self-heating temperature. It has a significant effect on the performance of sensors and actuators. When the temperature reaches the critical point, the performance of the plate vibration control system significantly worsens. We have obtained a simple formula for the critical electrical load under which the temperature reaches the critical point.

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