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ON EXISTENCE OF ATTRACTORS IN DISSIPATIVE THREE-DIMENSIONAL SYSTEMS

N. V. Nikitina

The theorems on the birth of attractors from homoclinic loops are proved. Applications of the theorems are considered.

Keywords: nonlinear systems, symmetry principles, limit cycle, bifurcation, chaos

Introduction. In dynamic systems, homoclinic trajectories suggest the presence of trajectories with complex behavior. These trajectories play an important role in bifurcation theory. The birth of periodic orbits from homoclinic loops (HL) in three-dimensional systems was discussed in [1–7, 9–16].

We will prove theorems of the existence of periodic orbits in dissipative three-dimensional systems and consider applications of these theorems to the birth of attractors. The birth of strange attractors is associated with orbital instability and the conditions of the theorems below.

Consider the system

$$\frac{dx}{dt} = F(x, \mu), \qquad x(t) \in R^n, \qquad \mu \in R^m, \tag{1}$$

where $n = 3$, $F(x, \mu)$ is a smooth function; R^m is the parameter space. Introduce a small deviation in the neighborhood of the partial solutions $\bar{x}_i$ $(i = 1, 2, \ldots, n)$ of Eqs. (1): $\delta x_i = x_i(t) - \bar{x}_i(t)$ $(i = 1, 2, \ldots, n)$. Let δx_i be new coordinates. The linear system corresponding to system (1) in the coordinates δx_i

$$d\delta x / dt = A(\bar{x})\delta x, \qquad \delta x \in R^n, \tag{2}$$

where $A(\bar{x}) = \partial F / \partial x|_{x=\bar{x}}$, is called a system of variational equations [16]. By analyzing the roots of the characteristic equation of the matrix $A(\bar{x})$, we can study the mechanism of formation of periodic and complex motions. We represent the matrix $A(\bar{x})$ of system (2) as the sum of two matrices:

$$A(\bar{x}) = N + M(\bar{x}), \tag{3}$$

where the matrix N corresponds to the spectrum of the linear system (2) which do not have partial solutions. The matrix $M(\bar{x})$ corresponds to the portion of the spectrum of Eqs. (2) that contains the partial solutions $\bar{x}_1, \bar{x}_2, \bar{x}_3$.

Let us introduce some notation and formulate Shilnikov's theorem [17]. Denote the characteristic numbers (CN) of the singular point O at the origin of coordinates of system (1) by $\gamma, \lambda_1, \lambda_2$, $\gamma > 0 > \mathrm{Re}\,\lambda_{1,2}$.

The unstable manifold W^u of the saddle O is one-dimensional, and the stable manifold W^s is two-dimensional. The unstable manifold consists of three orbits: saddle O and two separatrices Γ_1 and Γ_2. Assume that the system has a separatrice loop, i.e., Γ_1 tends to O as $t \to \infty$. Consider a negative saddle value at the saddle-focus: $\sigma = \gamma + \mathrm{Re}\,\lambda_1 + \mathrm{Re}\,\lambda_2 < 0$.

Theorem 13.6 (Shilnikov [17]). When the saddle value σ is negative, a unique stable periodic orbit is born from the homoclinic loop L.

S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterova Str., Kyiv, Ukraine 03057; e-mail: center@inmech.kiev.ua. Translated from Prikladnaya Mekhanika, Vol. 55, No. 5, pp. 121–134, September–October 2019. Original article submitted April 13, 2018.

Here we also deal with a homoclinic loop. Shilnikov's theorem is focused on the existence of HL on which the saddle value is negative. The principle of symmetry in three-dimensional systems was considered in [7]. It was established that there exist HLs that close due to symmetry (antisymmetry). Thus, the principle of symmetry can be used to establish the existence of a periodic orbit in a homoclinic loop and the sign of the saddle value. One approach is used to establish the sign of the saddle values and the attraction of all points of the loop. The theorems below prove the existence of an attractor. In one case, it is a closed trajectory. Changing the parameters makes the trajectory close as $t \to \infty$; then the strange attractor is born.

Shilnikov's theorem 13.6 [17] was discussed in [1]. We will consider a constructive principle of determining the sign of the saddle value of a homoclinic loop. This is due to the attraction by points of the trajectory and the formation of limit cycles or a strange attractor in three-dimensional systems. This principle imposes certain conditions on the parameter values, making the theorems sufficient. The theorems below cover some basic models generating attractors.

The principles of symmetry (antisymmetry) for two-dimensional systems [6, 13] can be applied to analyze bifurcations and establish the symmetry of attractors and closing of trajectories in three-dimensional systems. The stability criteria for orbits (closing) in two-dimensional systems are formulated in [13]. Let us represent the two-dimensional system in the form

$$dx_1 dt = F_1(x),$$
$$dx_2 dt = F_2(x), \tag{4}$$

where $x_1, x_2 \in R$ and $F_1 \in C(R^2, R), F_2 \in C(R^2, R)$, and $F_i(0) = 0\,(i = 1, 2)$.

Let us formulate the geometrical principle of symmetry that can be used to establish the conditions of closeness of a phase trajectory.

The trajectory of system (4) is closed if the function $F_1(x)$ is even in x_1 and the function $F_2(x)$ is odd in x_1:

$$F_1(-x_1, x_2) = F_1(x_1, x_2),$$
$$F_2(-x_1, x_2) = -F_2(x_1, x_2). \tag{5}$$

This statement is based on the fact that Ox_2 is an axis of symmetry on the plane Ox_1x_2 and any integral curve on the left of the Ox_2-axis is the mirror image of the curve on its right.

The principle of symmetry suggests that there exists a closed trajectory in system (4) if the function $F_2(x)$ is even in x_2 and the function $F_1(x)$ is odd in x_2:

$$F_1(x_1, -x_2) = -F_1(x_1, x_2),$$
$$F_2(x_1, -x_2) = F_2(x_1, x_2). \tag{6}$$

Here Ox_1 is the axis of symmetry, according to [13].

The principle of antisymmetry for nonlinear systems was formulated in [6]. There exists a closed trajectory in system (4) if the functions on the right-hand side of (4) are related by

$$F_1(x_1, -x_2) = -F_1(-x_1, x_2),$$
$$F_2(x_1, -x_2) = -F_2(-x_1, x_2). \tag{7}$$

Antisymmetry is associated with two axes of coordinates; i.e., if the Ox_1-axis is the axis of antisymmetry, then the Ox_2-axis is the axis of antisymmetry as well.

There exists a closed trajectory in system (4) if the functions on the right-hand side of (1) are related by

$$F_1(-x_1, x_2) = -F_1(x_1, -x_2),$$
$$F_2(-x_1, x_2) = -F_2(x_1, -x_2). \tag{8}$$

Linear oscillators with positive and negative dissipation are antisymmetric, but trajectories of such systems are not closed.

1. Theorems on Existence of Attractors in Three-Dimensional System. We will prove theorems for smooth systems. We make the following assumptions on system (1):

Assumption 1. In system (1), there exists a singular point $O(0, 0, 0)$ (saddle-node with negative saddle value) such that the characteristic numbers of the point O are $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 > 0$. There exists a neighborhood of the singular point O filled with saddle-nodes points that, according to the Hartman–Grobman theorem [15], go over into saddle-foci, so that $\mathrm{Re}\lambda_1 < 0, \mathrm{Re}\lambda_2 < 0,$ $\lambda_3 > 0$ and $\mathrm{Re}\lambda_1 + \mathrm{Re}\lambda_2 + \lambda_3 < 0$.

Assumption 2. The characteristic equation for the system $|M(\bar{x}) - \lambda E| = 0$ where E is a unit matrix, has one zero, and two imaginary eigenvalues. When $\bar{x} \neq 0$, the differential system of equations (1) forms an HL.

Theorem 1. If the differential system (1) satisfies the conditions of Assumptions 1 and 2 and the sum of eigenvalues of the characteristic equation of the system $|M(\bar{x}) - \lambda E| = 0$ is equal to zero $(\bar{\lambda}_1(\bar{x}) + \bar{\lambda}_2(\bar{x}) + \bar{\lambda}_3(\bar{x}) = 0)$, then the HL forms an attractor in the vicinity of the singular point $O(0, 0, 0)$ of system (1).

Proof. If the conditions of Assumptions 1 and 2 are satisfied, the saddle value of the points on the trajectory is determined by the system $|A(\bar{x}) - \lambda E| = 0|$ from the equation (according to (3)), i.e., $|A(\bar{x}) - \lambda E| = |N - \lambda E| = 0$.

The bifurcation process in the field of the three-dimensional system (1) follows the Hartman–Grobman theorem. The neighborhood filled with saddle-nodes so that $\sigma = \lambda_1 + \lambda_2 + \lambda_3 < 0$ goes over into a saddle-focus continuum with a negative saddle value. For the saddle-focus domain $\mathrm{Re}\lambda_1 < 0, \mathrm{Re}\lambda_2 < 0, \lambda_3 > 0$, the saddle-focus loop has saddle value $\sigma = \mathrm{Re}\,\lambda_1 + \mathrm{Re}\,\lambda_2 + \lambda_3 < 0$ at all points of the trajectory. This value is determined by the roots of the linear system; then $\sigma = \sigma_O = \sigma_A = \sigma_B$, where σ_A, σ_B are the values of any points A, B, including singular ones. The HL has a negative saddle value at all points. The homoclinic loop with negative value σ such that $\mathrm{Re}\,\lambda_1 < 0, \mathrm{Re}\,\lambda_2 < 0, \lambda_3 > 0$ gives rise to an attractor.

Corollary. Let the differential system (1) satisfy Assumptions 1 and 2, and two saddle loops form in system (1). Then attractors are born from the loops if the orbits of the loops do not cross.

Theorem 2. If the differential system (1) forms an HL, the singular point $O(0, 0, 0)$ of system (1) is a saddle-focus with zero saddle value $(\sigma_O = 2\mathrm{Re}\lambda_{1,2} + \lambda_3 = 0)$, and the saddle value of the points defined by the characteristic equation of the system $|M(\bar{x}) - \lambda E| = 0$ is negative for $\bar{x} \neq 0$, then there exists an attractor in the neighborhood of the singular point $O(0, 0, 0)$ of system (1).

Proof. If the conditions of the theorem are satisfied, the matrix $A(\bar{x})$ has eigenvalues satisfying the equation $|A(\bar{x}) - \lambda E| = |M(\bar{x}) - \lambda E| = 0$. Then an HL with a negative saddle value and attraction at each point forms. Such a loop gives rise to an attractor.

Theorem 3. If the differential system (1) forms an HL, the singular point $O(0, 0, 0)$ of system (1) is a saddle-focus with saddle value $\sigma_O = 2\mathrm{Re}\lambda_{1,2} + \lambda_3 > 0$, and the saddle value of the points defined by the characteristic equation of the system $|M(\bar{x}) - \lambda E| = 0$, $\sigma_{\bar{x}}$ is negative, then there exists an attractor in system (1) if $\sigma = \sigma_O + \sigma_{\bar{x}} < 0$.

Proof. If the conditions of the theorem, then the singular point $O(0, 0, 0)$ is a saddle-focus. The saddle value of the equations $|N - \lambda E| = 0$ is positive. Then, in view of (3), the HL with a negative saddle value and attraction at each point forms an attractor.

The above theorems are based on Shilnikov's theorem 13.6 [17] and their proofs are similar in structure. In what follows, we will show how to establish whether the saddle value is negative and analyze the mechanism of orbital instability in the case of a strange attractor.

APPLICATIONS

Application of Theorem 1. *Example 1.* Consider the Lorentz system

$$dx/dt = s(-x + y),$$

$$dy/dt = rx - y - xz,$$

$$dz/dt = -bz + xy, \tag{9}$$

where b, r, s are positive parameters $(r > 1)$. Introducing small deviations $\delta x, \delta y, \delta z$ from the partial solutions $\bar{x}, \bar{y}, \bar{z}$ in (9), we set up variational equations:

$$\delta\dot{x} = -s\delta x + s\delta y,$$

$$\delta\dot{y} = (r - \bar{z})\delta x - \delta y - \bar{x}\delta z,$$

$$\delta\dot{z} = -b\delta z + \bar{y}\delta x + \bar{x}\delta y. \tag{10}$$

System (9) has the following singular points:

$$O(0, 0, 0), \quad A(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1), \quad B(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1).$$

The characteristic equation of system (10)

$$\lambda^3 + \lambda^2(b+s+1) + \lambda(s(1-r+\bar{z}) + b(s+1) + \bar{x}^2) + s(b(1-r+\bar{z}) + \bar{x}(\bar{x}+\bar{y})) = 0 \tag{11}$$

can be used to determine the characteristic numbers of the points in the field of the three-dimensional space of system (9). At the point O, Eq. (11) becomes

$$(\lambda + b)(\lambda^2 + \lambda(1+s) + s(1-r)) = 0,$$

whence

$$\lambda_{1,2} = -(s+1)/2 \pm \sqrt{((s+1)/2)^2 + s(r-1)},$$

$$\lambda_3 = -b. \tag{12}$$

The singular point O is a saddle-node with saddle value $\sigma = -(s+1) - b$. Let us write the matrix equality based on the variational equations (10). Let us rearrange (3) into the form

$$\begin{pmatrix} -s & s & 0 \\ r-\bar{z} & -1 & -\bar{x} \\ \bar{y} & \bar{x} & -b \end{pmatrix} = \begin{pmatrix} -s & s & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -\bar{z} & 0 & -\bar{x} \\ \bar{y} & \bar{x} & 0 \end{pmatrix}. \tag{13}$$

The matrix of the variation system (10) is on the left-hand side of Eq. (13). The first matrix on the right-hand side of (13) is the spectrum of the linear system corresponding to system (9). The second matrix $M(\bar{x})$ corresponds to the part of Eqs. (10) that contains the partial solutions $\bar{x}$, $\bar{y}$, $\bar{z}$ and has the roots of the characteristic equation:

$$\bar{\lambda}_{1,2} = \pm i\sqrt{\bar{x}^2}, \quad \bar{\lambda}_3 = 0.$$

Note that the partial solutions $\bar{x}$, $\bar{y}$, $\bar{z}$ of the system of equations (9) are unknown. System (9) has several attractors. Let us represent system (9) as three two-dimensional equations:

$$\frac{dx}{dt} = s(-x+y), \quad \frac{dy}{dt} = rx - y, \tag{14}$$

$$\frac{dx}{dt} = sx, \quad \frac{dz}{dt} = -bz, \tag{15}$$

$$\frac{dy}{dt} = -y, \quad \frac{dz}{dt} = -bz. \tag{16}$$

According to Theorem 1, the linear system consisting of subsystems (14)–(16) determined the closing of system (9). Subsystems (14)–(16) determined the symmetry of system (9) (for some parameter values), which is identical to the closing of the trajectory. According to (14), there is a stable singular point with the following CN at zero on the plane xy: $\lambda_1 = 0$, $\lambda_2 = -(s+1)$. Subsystem (14) satisfies conditions (7) (or (8)). Denote the right-hand sides in subsystem (15) by $F_x = sx$, $F_z = -bz$. In subsystem (15), the function F_x is even in z and the function F_z is odd in z, i.e.,

$$F_x(x, -z) = F_x(x, z),$$

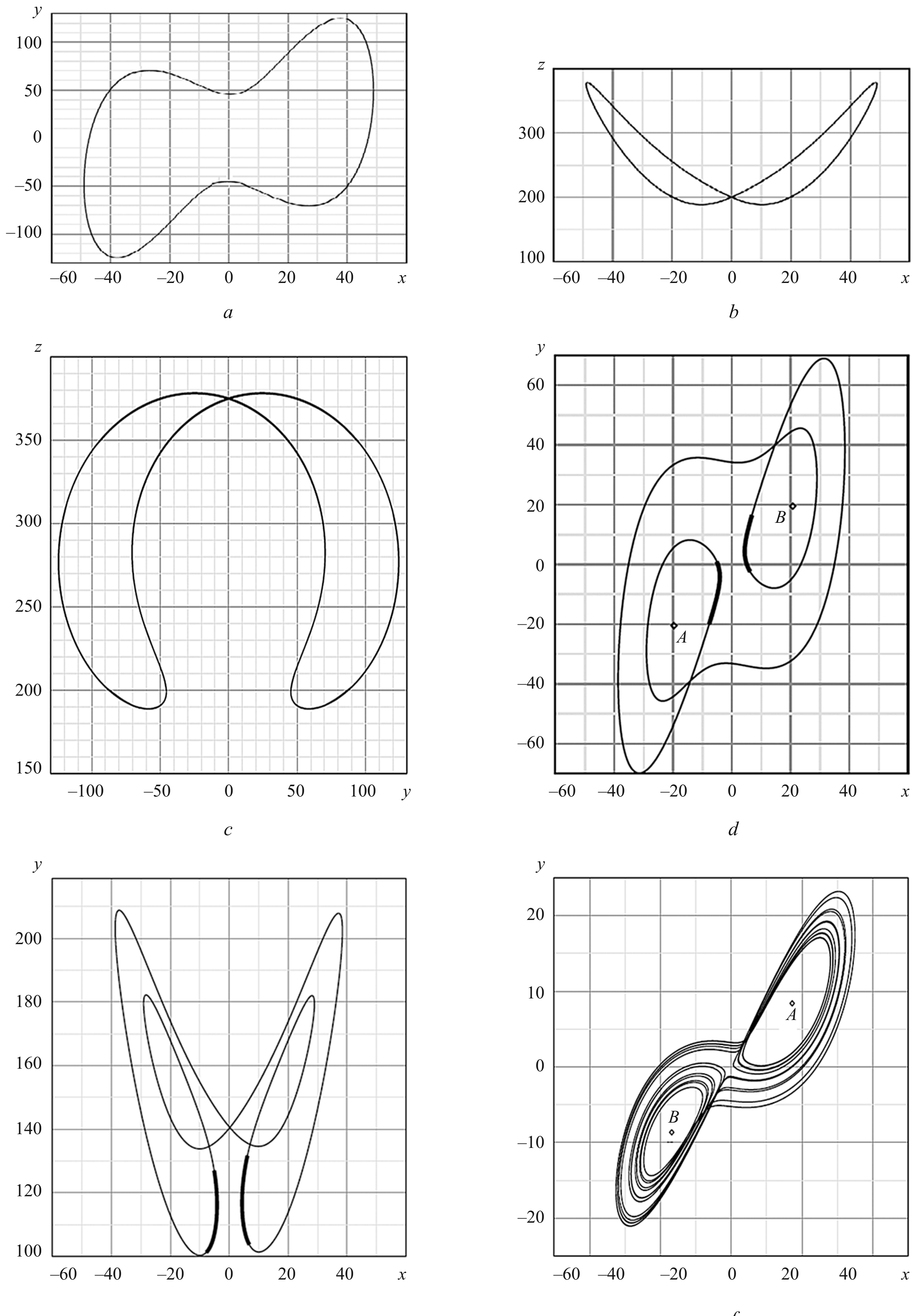
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Fig. 1

$$F_z(x,-z) = -F_x(x,z).$$

In system (15), the Oz-axis on the plane Oxz is the axis of symmetry (see (6)). Denote the right-hand sides in system (16) by $F_y = -y, F_z = -bz$. In system (16), the function F_y is even in z and the function F_z is odd in z, i.e.,

$$F_y(x,-z) = F_y(x,z),$$

$$F_z(y,-z) = -F_z(x,z).$$

In system (16), the Oz-axis on the plane Oyz is the axis of symmetry (see (6)). On the planes xy, xz, yz, there are stable singular points, according to (14)–(16).

In system (14), conditions (7) (or (8)) are satisfied. In systems (14)–(15), conditions (8) are satisfied and the Oz-axis is the axis of symmetry. The conditions of symmetry and antisymmetry are satisfied on the three coordinate planes. This means that in the three-dimensional system, there are parameter values corresponding three-dimensional attractors with symmetry.

When $(b, r, s) = (9.5/3;\ 280;\ 7)$ (Fig. 1*a*,*b*,*c*), the Lorentz trajectory turns into a "butterfly." The butterfly is closed in view of Theorems 1. At certain parameter values, the butterfly is symmetric or antisymmetric on each coordinate plane. Then, the trajectory closes, according to the principle of symmetry. A butterfly also forms at other parameter values such as $(b, r, s) =$ (8/3; 280; 7). Thus, there is a range of parameter values in which the trajectory closes and forms regular attractors with symmetric (or nearly symmetric) portraits on the coordinate planes.

In Fig. 1*d*,*e* for $(b, r, s) = (8/3;\ 153;\ 10)$, the heavy lines represent saddle-node points of the trajectory, while the thin lines represent node-focus points for which $\mathrm{Re}\lambda_1 < 0, \mathrm{Re}\lambda_2 < 0, \lambda_3 < 0$. A decrease in the parameter r leads to the occurrence of additional periodic motions (period multiplication). The parameter r is associated with the change in the topology of three-dimensional space in the neighborhood of zero and singular points O, A, B. The representative point closes trajectory; The Lorentz model self-controls the process using additional regular motions (Fig. 1*d*,*e*).

An alternative of a regular attractor is a strange attractor that arises upon loss of stability of the orbit of the representative point.

In this case, the following condition of Shilnikov's theorem is also satisfied: the saddle value σ is negative. Let $(b,r,s) =$ (8/3; 28; 10). The singular point O is a saddle-node. The plane xy is divided into domains of influence of the singular points A and B. Figure 1*f* shows a fragment of the revolution of the trajectory around the singular point A and the transition to and the revolution around the point B. In the case of a regular attractor, the revolution around a singular point must have a certain repeatability around the singular point B. The behavior of the representative point is determined by the topology of the space in the neighborhood of the singular points A, B, and $O(0, 0, 0)$. At the given parameter values, however, the topology demonstrates chaotic oscillations. The semiloop tends to use periodic oscillations to balance the inhomogeneity of the solution in the neighborhood of the singular points A and B. Thus, the way to chaos lies through period multiplication. If periodic trajectories do not compensate the instability of motion, then infinite nonrepeated periodic motion occurs, forming a chaotic attractor. The chaos caused by orbital instability is preceded by regular motion with period multiplication (Fig. 1*d*,*e*), and the following parameter values $(b, r, s) = (9/3;\ 200;\ 10)$; Fig. 2*a*,*b*,*c*.

Thus, period multiplication can occur, which is identified as the reaction of the self-control of the model to a certain topology in the neighborhood of the singular points. Chaotic motions occur when the multiple periodic motions cannot balance the nonuniformity of the motion as a whole. The topology of a multidimensional (three-dimensional in our case) space defines the programs of multiple period and chaos. In the large, regular and chaotic attractors are formed.

The system of Lorentz equations obtained by simplifying the Navier–Stokes equations describes the thermal convection in a thin film of viscous fluid. This system is not very realistic, but instructive model of turbulence in a fluid. It is possibly realized in lasers and spin systems [8]. The Lorentz model in the form of a butterfly can also be used to model a stable moderate-sized firm [3].

Application of Theorem 2. *Example 2.* Consider a system of three differential nonlinear Chua equations:

$$\frac{dx}{dt} = \alpha(ax - bx^3 + y), \qquad \frac{dy}{dt} = x - y + z, \qquad \frac{dz}{dt} = -\beta y, \tag{17}$$

where a, b, α, β are positive parameters. Despite the fact that the dynamics of the Chua circuit was described in detail in [5, 17], we will use Theorem 2 to establish whether an attractor forms from a homoclinic trajectory. System (12) has three equilibrium

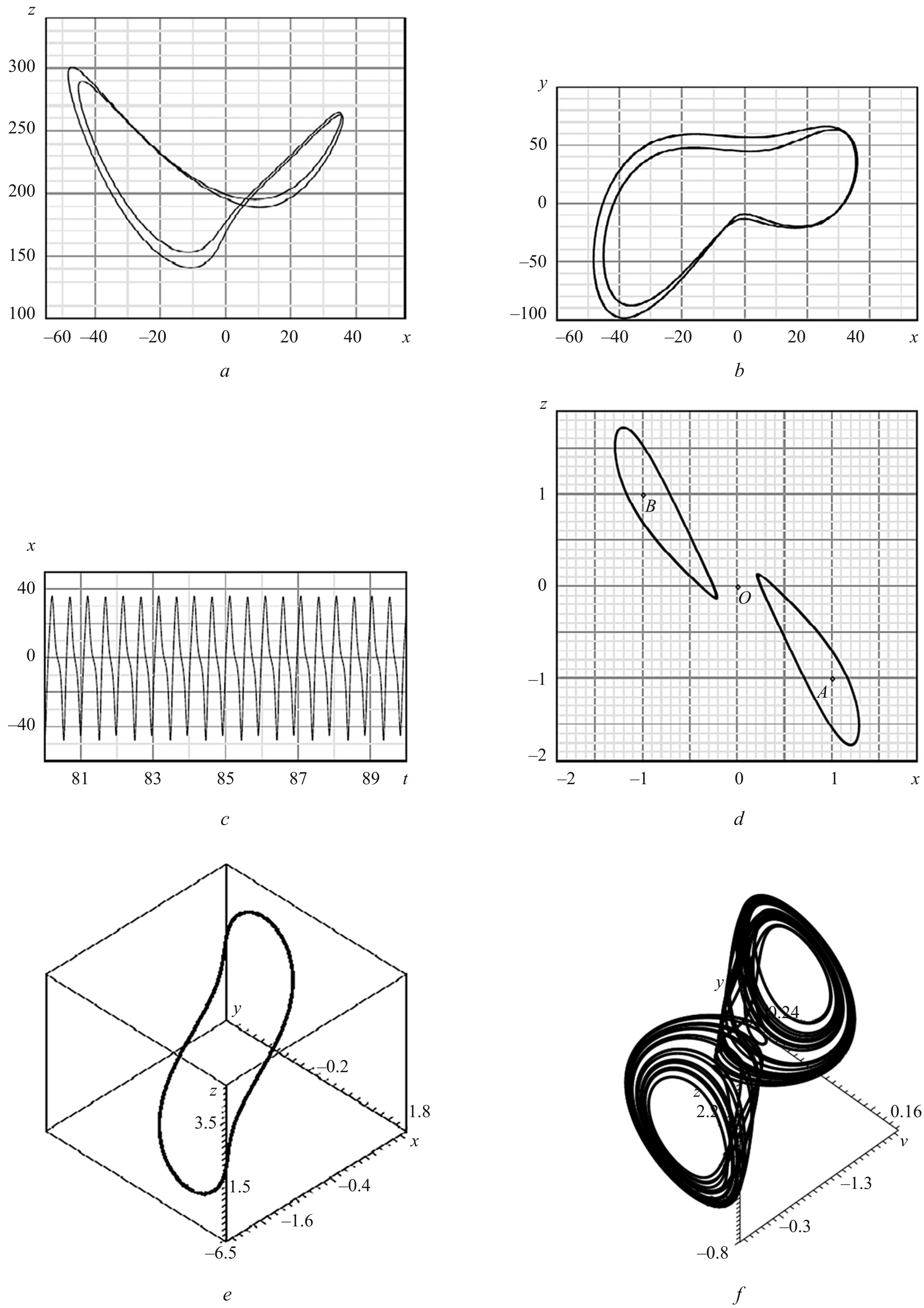
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Fig. 2

states: singular point $O(0, 0, 0)$, singular points $A(x_A = \sqrt{a/b},\ y_A = 0,\ z_A = -\sqrt{a/b})$, $B(x_B = -\sqrt{a/b},\ y_B = 0,\ z_B = \sqrt{a/b})$. Introducing small deviations δx, δy, δz from the partial solutions $\bar{x}$, $\bar{y}$, $\bar{z}$ of (17), we set up variational equations:

$$\frac{d\delta x}{dt} = \alpha(a\delta x - 3b\bar{x}^2\delta x + \delta y), \qquad \frac{d\delta y}{dt} = \delta x - \delta y + \delta z, \qquad \frac{d\delta z}{dt} = -\beta\delta y.$$

The associated characteristic equation is

$$\lambda^3 + \lambda^2(1 + \alpha(-a + 3b\bar{x}^2)) + \lambda(\beta - \alpha(1 + a - 3b\bar{x}^2)) + \alpha\beta(-a + 3b\bar{x}^2) = 0. \tag{18}$$

The bifurcation process in the system is due to the change in the coordinate x because the characteristic equation depends on the partial solution $\bar{x}$ only. The CNs of the point O are determined from the equation $\lambda^3 + \lambda^2(1 - \alpha a) + \lambda(\beta - \alpha(1 + a)) - \alpha\beta a = 0$. Let

$$(a, \alpha) = (1/6,\ 6), \qquad b = a, \qquad \beta \in (7, \ldots, 10, 1). \tag{19}$$

Parameters (19) must be such that the saddle value of the point $O(0, 0, 0)$ is equal to zero. For the initial perturbations to generate a closed curve around the point O in system (17), it is necessary to select the initial conditions as follows:

$$|x(0)| > |x_A|, \qquad |y(0)| \geq 0, \qquad |z(0)| > |z_A|. \tag{20}$$

This choice precludes the influence of the singular points A and B, which, like the singular point O, are responsible for some motion of system (17). The choice of the initial conditions can be refined numerically (within the framework of inequalities (20)). The characteristic equation $\lambda^2(\lambda + 3\alpha b\bar{x}^2) = 0$ of the matrix $M(\bar{x})$ has eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -3\alpha b\bar{x}^2$. The zero roots are not multiple. If the conditions for parameters (19) are satisfied and the initial conditions (20) are chosen, in system (17) there is an attracting periodic orbit (according to Theorem 2).

Note that the saddle value $\sigma = 2\,\mathrm{Re}\,\lambda_{1,2} + \lambda_3$ calculated from Eq. (18) depends on the coordinate x only.

Let us consider the behavior of the solutions of the Chua system determined by the singular points A and B. The points A and B can make the curve close, except for the point O. The points A and B are antisymmetric. Therefore, we will consider the motion around only one singular point. The parameter β can influence the position of the trajectories relative to the points A and B. The major requirement is that the trajectories must not cross. We will use a coordinate system $Avyw$ fixed to the point A and set up the equations of motion in the new coordiantes:

$$\frac{dv}{dt} = \alpha(-2av - bv^2(3\sqrt{a/b} + v) + y), \qquad \frac{dy}{dt} = v - y + w, \qquad \frac{dw}{dt} = -\beta y, \tag{21}$$

where $v = x - \sqrt{a/b}$, $w = z + \sqrt{a/b}$. The saddle value corresponding to this system (21) depends on one variable. Here we can trace the instability of motion which is first balanced (for certain parameter values) by a periodic motion (period multiplication), followed by the occurrence of chaos, when the periodic motion in the system of equations (17) cannot balance instability. In the coordinate system $Avyw$, the singular point O has the following coordinates: $v_O = -\sqrt{a/b}$, $y_O = 0$, $w_O = \sqrt{a/b}$.

When the initial perturbations are

$$|v(0)| < \sqrt{a/b}, \qquad |y(0)| \geq 0, \qquad |w_O(0)| < \sqrt{a/b} \tag{22}$$

the motion induced by the singular point A dominates. The estimates of the initial conditions can also be refined numerically (within the framework of inequalities (22)). When the representative point moves in the neighborhood of the singular point A (or B), all points of trajectory are attractive and saddle-foci with negative saddle value. The Chua system generates chaotic motions around the singular points A and B. This process is associated with the irregularity of bifurcations. The saddle solutions slow down the motion on a circular trajectory. Periodic motions of multiple period first occur to balance the irregularity of bifurcations. Further changes in the parameter values show that period multiplication fails to balance the instability of oscillations.

Figure 2*d*,*e* show a limit cycle in the coordinate system $Oxyz$ and two limit cycles around the singular points A and B. For certain parameter values, the Chua system generates regular attractors. Figure 2*f* shows the chaotic trajectories around the

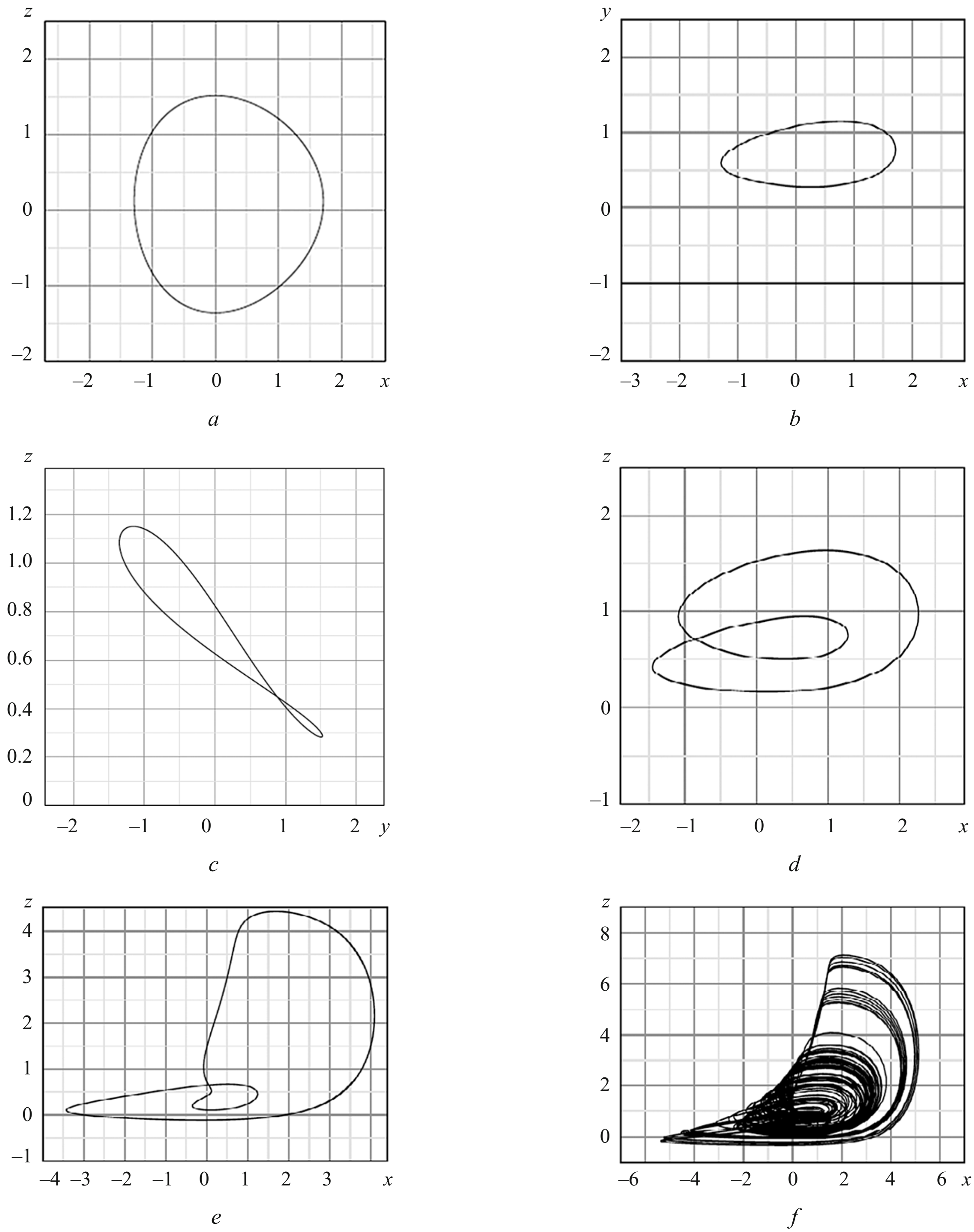


Fig. 3

singular points A and B. Thus, the Chua generator fits the proofs of the theorems above, and the chaos occurs when period multiplication fails to stabilize the motion for some parameter values.

Application of Theorem 3. *Example 3.* Consider a system of three differential nonlinear equations (a generator with exponential inertial nonlinearity [1]):

$$\frac{dx}{dt} = mx - xz + y, \qquad \frac{dy}{dt} = -x, \qquad \frac{dz}{dt} = -b(z - e^x + 1), \tag{23}$$

where m and b are positive parameters. System (23) has one singular point $O(0,0,0)$. The following variational system corresponds to (23):

$$\frac{d\delta x}{dt}=(m-\bar{z})\delta x+\delta y-\bar{x}\delta z, \qquad \frac{d\delta y}{dt}=-\delta x, \qquad \frac{d\delta z}{dt}=-b(\delta z-e^{\bar{x}}\delta x).$$

The characteristic equation of the system of variational equations is

$$(\lambda+b)(\lambda(\lambda-m+\bar{z})+1)+\lambda b\bar{x}e^{\bar{x}}=0.$$

The system $|M(\bar{x})-\lambda E|=0$, where the matrix

$$M(\bar{x},\bar{z})=\begin{pmatrix}\bar{z} & 0 & \bar{x}\\ 0 & 0 & 0\\ -be^{\bar{x}} & 0 & 0\end{pmatrix}$$

has the characteristic equation $\lambda(\lambda^2+\lambda\bar{z}+b\bar{x}e^{\bar{x}})=0$ whose roots

$$\lambda_{1,2}=-\bar{z}/2\pm\sqrt{(\bar{z}/2)^2-b\bar{x}e^{\bar{x}}}, \qquad \lambda_3=0.$$

The saddle value is $\sigma_{\bar{x}}=-\bar{z}<0$.

In this case, it is difficult to estimate $|\bar{z}|$. The roots at the singular point $O(0,0,0)$ $\lambda_{1,2}=m/2\pm\sqrt{(m/2)^2-1}$, $\lambda_3=-b$.

The parameter values for system (23) at which the trajectory closes and forms an almost symmetric projection on the plane xz were found in [10]. The characteristic equation of the variational system does not include the partial solution $\bar{y}$. This suggests that the bifurcation process is not related to the partial solution $\bar{y}$. Separatrices that divide the plane xz into domains with certain quality of points were constructed in [10]. We will use the result of [10] where the parameters of system (23) at which the regular attractor on the plane xz is almost symmetric. The closed trajectory on the plane xz occurs at the following parameter values:

$$(m,b)=(0.7,\ 0.2). \tag{24}$$

The roots λ_1, λ_2, λ_3 form a positive saddle value: $\sigma_O=m-b$, $\sigma_O>0$. At parameter values (24), $|\sigma_{\bar{x}}|$ exceeds σ_O by a small amount. According to Theorem 3, if the algebraic sum $\sigma_O+\sigma_{\bar{x}}<0$, then system (23) generates an attractor. Figures 3*a*,*b*,*c* show the coordinate portraits of the limit cycle of system (23) with parameter values (24).

Let us consider three cases of change in the parameters and the influence of these changes on the quality of the attractor.

1. For $(m,b)=(0.8;\ 0.2)$, the singular point increases the positive value: $\sigma_O=m-b=0.6$. Tending to maintain a status quo, system (23) demonstrates period doubling (Fig. 3*d*). The trajectory acquires the upper branch which increases the negative component $|-\bar{z}|$.

2. For $(m,b)=(1;\ 0.2)$, the singular point even more increases the positive value: $\sigma_O=m-b=0.8$. System (23) not only doubles the period, but also captures very far domains on the plane xz to maintain the condition of Theorem 3 and to generate an attractor (Fig. 3*e*). This attractor is regular yet.

3. For $(m,b)=(1.2;\ 0.2)$, the saddle value $\sigma_O=m-b=1$. System (23) increases the deviations towards an increase in the negative component $|-\bar{z}|$. The attractor is realized as chaos (Fig. 3*f*).

Thus, when there is one singular point, the analysis of bifurcations [10] was used to analyze the occurrence of strange motions. The problem of multistability was considered in [10]. The sign and value of the initial perturbations of system (23) are related to the type of the attractor. The small initial perturbations used in [10] were on the order of 0.01.

In Examples 1 and 2, chaos arose because of it was impossible for the two semiloops to connect into a single loop due to multiple periodic motions (Figs. 1*f* and 2*c*). These systems have three singular points. The semiloop around the point A must connect to the semiloop around the point B to form a closed loop during the multiple periods of oscillations as, for example, in Fig. 1*d*,*e*.

In Example 3, the coordinate portraits look like in Fig. 3*a*,*b*,*c*. The problem of trajectory closing arises. If multiple periodic motions in the system cannot balance the instability of motion, then nonrepeating chaotic motions occur (Fig. 3*f*) and generate a strange attractor in an infinite time. Note that whatever the stabilization of motion is, the main result is obtaining a negative saddle value. This completes both regular and chaotic processes of the generation of an attractor.

The next example is essentially different from the previous three.

Application of Theorem 2. *Example 4.* Consider a system of three differential nonlinear equations (a generator with quadratic nonlinearity [1]):

$$\frac{dx}{dt} = mx - xz + y, \qquad \frac{dy}{dt} = -x, \qquad \frac{dz}{dt} = -b(z - x^2), \tag{25}$$

where m and b are positive parameters. The system has one singular point $O(0, 0, 0)$. Introducing small deviations δx, δy, δz from the partial solutions $\bar{x}$, $\bar{y}$, $\bar{z}$ of (25), we set up variational equations:

$$\frac{d\delta x}{dt} = (m - \bar{z})\delta x + \delta y - \bar{x}\delta z, \qquad \frac{d\delta y}{dt} = -\delta x, \qquad \frac{d\delta z}{dt} = -b(\delta z - 2\bar{x}\delta x).$$

The characteristic equation of this system is

$$\lambda^3 + \lambda^2(b - m + \bar{z}) + \lambda(b(-m + \bar{z} + 2\bar{x}^2) + 1) + b = 0.$$

At the point $O(0, 0, 0)$, the characteristic equation takes the form

$$(\lambda + b)(\lambda^2 - \lambda m + 1) = 0.$$

The CNs of the point $O(0, 0, 0)$ are $\lambda_{1,2} = m/2 \pm \sqrt{(m/2)^2 - 1}$, $\lambda_3 = -b$. Let $(m, b) = (1, 1)$. The point O is a saddle-focus with saddle value $\sigma_o = 0$. The matrix

$$M(\bar{x}, \bar{z}) = \begin{pmatrix} \bar{z} & 0 & \bar{x} \\ 0 & 0 & 0 \\ -2b\bar{x} & 0 & 0 \end{pmatrix}$$

corresponds to the characteristic equation

$$\lambda(\lambda^2 + \lambda\bar{z} + 2b\bar{x}^2) = 0$$

with roots

$$\lambda_{1,2} = -\bar{z}/2 \pm \sqrt{(\bar{z}/2)^2 - 2b\bar{x}^2}, \quad \lambda_3 = 0.$$

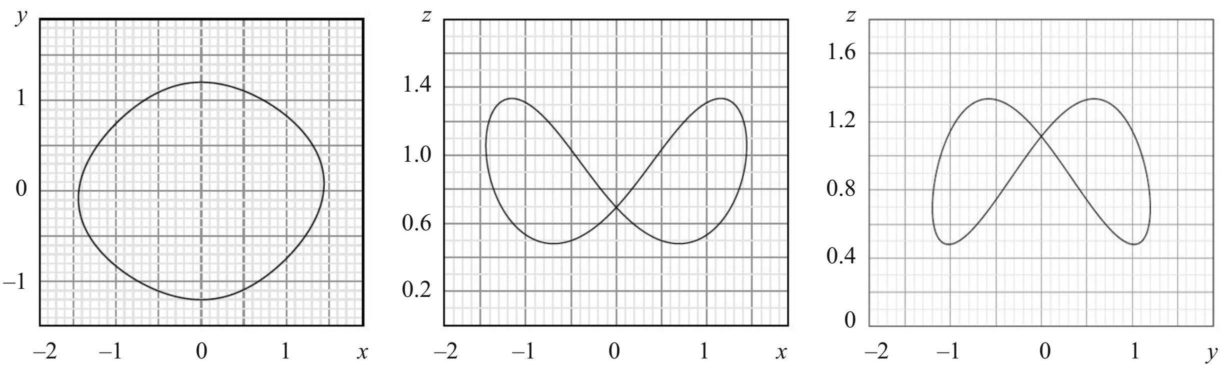
The saddle value is $\sigma_{\bar{x}} = -\bar{z}$. Let us show that the matrix $M(\bar{x}, \bar{z})$ corresponds to a dissipative oscillator. Consider a linear system

$$\frac{dX}{dt} = -\bar{z}X - \bar{x}Z, \qquad \frac{dZ}{dt} = -2b\bar{x}X,$$

which is identical to the dissipative oscillator

$$\frac{d^2 Z}{dt^2} + \frac{dZ}{dt} + 2b\bar{x}^2 Z = 0.$$

In Example 4, the existence of a closed trajectory is established based on Theorem 2. There is one singular point $O(0, 0, 0)$ in the three-dimensional system (25). There is symmetry about the *Ox*- and *Oy*-axes on the plane *xy* and about the



a *b* *c*

Fig. 4

Oz-axis on the planes *xz* and *yz*. If $\sigma < 0$, then the HL forms a closed trajectory, which is a regular attractor. There exists only a regular attractor in system (25).

According to Theorem 2, only a periodic orbit exists in system (25). Figure 4*a*,*b* shows the closed trajectory of system (25) in projections onto the coordinate planes. This result can be obtained using the principle of symmetry for three-dimensional systems.

Conclusions. We have addressed here bifurcation processes in three-dimensional systems. Three theorems related to the formation of attractors from HLs in three-dimensional systems have been formulated. It has been established that an HL also forms closed trajectories with symmetry [7]. The proof of the existence of attractors is related to HLs, and this representation has a wider application for different types of attractors, including chaotic. Chaotic attractors have high information capacity. Therefore, they can be used for data transfer. Theorems that allow establishing the sign of the saddle value of an HL have been formulated. Closing occurs due to the negative saddle value of the HL, including closing of strange attractors in an infinite time. All the three theorems are related to variational equations.

We have examined the qualitative side of the motion in examples of well-known basic models. What was new in analyzing the well-studied the Lorentz system? First of all, at some parameters value, this model fits in systems with symmetry and can have the form of a “butterfly.” Here chaos is due to the topology of the three-dimensional space defined by Eq. (11).

Theorems 1, 2, and 3 have been proved using Shilnikov’s theorem and an approach that allows using expression (3) of the matrix of a system of variational equations. The sign of the saddle value of a loop has been determined by dividing the matrix of the right-hand side of the variational system.

REFERENCES

1. V. S. Anishchenko, *Complex Oscillations in Simple Systems* [in Russian], Nauka, Moscow (1990).
2. T. S. Akhromeeva, S. P. Kurdyumov, G. G. Malinetskii, and A. A. Samarskii, *Structures and Chaos in Nonlinear Media* [in Russian], Fizmatlit, Moscow (2007).
3. T. A. Gurina and I. A. Dorofeev, “Existence of a homoclinic butterfly in the model of stability of a moderate-sized firm,” *Dinam. Sist.*, No. 28, 63–68 (2010).
4. G. A. Leonov, “The Tricomi problem ofn the existence of homoclinic orbits in dissipative systems,” *J. Appl. Math. Mech.*, **77**, No. 3, 296–304 (2013).
5. A. A. Martynyuk and N. V. Nikitina, “On periodic motion and bifurcations in three-dimensional nonlinear systems,” *J. Math. Sci.*, **208**, No. 5, 593–606 (2015).
6. N. V. Nikitina, *Nonlinear Systems with Complex and Chaotic Behavior of Trajectories* [in Russian], Feniks, Kyiv (2012).
7. N. V. Nikitina, “Symmetry principle in three-dimensional systems,” *Dop. NAN Ukrainy*, No. 7, 21–28 (2017).
8. H. Haken, *Synergetics*, Springer-Verlag, Berlin–Heidelberg–New York (1978).

9. G. A. Leonov, *Strange Attractors and Classical Stability Theory*, St. Peterburg University Press, St. Peterburg (2008).
10. A. A. Martynyuk and N. V. Nikitina, "Bifurcations and multistability of the oscillations of a three-dimensional system," *Int. Appl. Mech.*, **51**, No. 2, 223–232 (2015).
11. A. A. Martynyuk and N. V. Nikitina, "Bifurcation and sinchronization of two coupled generators," *Int. Appl. Mech.*, **53**, No. 2, 369–379 (2017).
12. Yu. I. Neimark and P. S. Landa, *Stochastic and Chaotic Oscillations*, Kluwer, Dordrecht (1992).
13. V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equation*, Princeton Univ. Press, Princeton (1960).
14. N. V. Nikitina, "Analyzing the mechanisms of loss of orbital stability in mathematical models of three-dimensional systems," *Int. Appl. Mech.*, **53**, No. 6, 716–726 (2017).
15. N. V. Nikitina, "On existence of attractors in some three-dimensional systems," *Int. Appl. Mech.*, **55**, No. 1, 95–102 (2019).
16. L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, and L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, Part I, World Scientific, Singapore (1998).
17. L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, and L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II, World Scientific, Singapore (2001).