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ON EXISTENCE OF ATTRACTORS IN SOME THREE-DIMENSIONAL SYSTEMS

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Two cases of existence of attractors in the basic models of a three-dimensional system are analyzed.

Keywords: two-disk dynamo model, Rössler model, bifurcation, chaos

Introduction. The methods of the qualitative theory of the nonlinear mechanics of two-dimensional systems came to fruition in the second half of the last century. At the end of the last century, three-dimensional systems attracted interest due to the study of chaotic motion. Problems were related to the variety of mathematical models that generate chaotic attractors, the principles of studying the occurrence of regular attractors, and the mechanisms of occurrence of chaos in three-dimensional systems. Despite some experience of studying nonlinear systems [3–8, 10, 12], the problem of existence of attractors remains open and is associated with several mechanisms of orbital instability.

Let us consider some cases where the principle of symmetry and qualitative analysis are used to prove the theorem of existence of an attractor and to establish the type of orbit instability and change over from a regular attractor to a strange attractor.

1. Attractors whose occurrence can be proved using the symmetry principle have certain symmetry of the closed trajectory on the coordinate planes [3].

2. The following three-dimensional system in which the first two equations are linear and the third equation is nonlinear

$$\frac{dx_1}{dt} = -x_2 - x_3, \quad \frac{dx_2}{dt} = x_1 + ax_2, \quad \frac{dx_3}{dt} = f_3(x_1, x_3) \tag{1}$$

can be represented as two two-dimensional subsystems:

$$\frac{dx_1}{dt} = -x_2, \quad \frac{dx_2}{dt} = x_1 + ax_2, \tag{2}$$

$$\frac{dx_1}{dt} = x_3, \quad \frac{dx_3}{dt} = f_3(x_1, x_3). \tag{3}$$

The system is decomposed into two subsystems so that the linear subsystem (2) describes the process on the plane x_1x_2 and subsystem (3) includes the nonlinear equation of three-dimensional system (1) and describes the process on the plane x_1x_3. The plane x_2x_3 is omitted from consideration because system (1) does not have the corresponding right-hand side. On the planes x_1x_2 and x_1x_3, the subsystems are related by the variable x_1. Subsystem (2) is a dissipative oscillator $\ddot{x}_2 - a\dot{x}_2 + x_2 = 0$ whose trajectory emerges from the zero of the plane x_1x_2.

If subsystem (3) is stable, then a three-dimensional attractor can occur in system (1). These physical assumptions underlie the proof of the theorem of the existence of an attractor in system (1).

3. Three-Dimensional System with a Plane Attractor. The existence of a plane attractor in a three-dimensional system is associated with the type of the singular point and with certain conditions that show that there is a closed trajectory on a plane in

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this three-dimensional system. The qualitative analysis of a plane attractor is related to a certain mechanism of orbital instability. Under certain initial conditions, the orbit of a closed trajectory loses stability and forms a strange attractor.

We will consider the last two cases. Basic mathematical models of processes will be considered in the appendix.

1. Three-Dimensional Systems of the Form (1)–(3). The system of three equations interacts on the two planes (x_1x_2, x_1x_3) of systems (2) and (3). Let us consider the principle of antisymmetry for two-dimensional systems. Let us use the principle of antisymmetry introduced in [2] for nonlinear two-dimensional systems. It is assumed that the nonlinear oscillator has linear and nonlinear dissipation components. The linear one generates an unstable singular point (unstable focus) at zero and the linear component of antisymmetry of the trajectory. The nonlinear component also bounds the domain of the trajectory's leaving the zero of the antisymmetric curve. This curve forms a limit cycle. An example of a nonlinear two-dimensional system is Van der Pol's oscillator:

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -x_1 + \mu(x_2 + x_1^2 x_2), \tag{4}$$

where μ is a parameter. Antisymmetry is associated with two axes of coordinates.

In a nonlinear system of the form (4) (with linear and nonlinear dissipation components)

$$\frac{dx_1}{dt} = F_1(x), \quad \frac{dx_2}{dt} = F_2(x) \tag{5}$$

there exists a closed trajectory if the functions on the right-hand side of system (5) are related as follows:

$$F_1(x_1, -x_2) = -F_1(-x_1, x_2), \quad F_2(x_1, -x_2) = -F_2(-x_1, x_2), \tag{6}$$

$$F_1(-x_1, x_2) = -F_1(x_1, -x_2), \quad F_2(-x_1, x_2) = -F_2(x_1, -x_2). \tag{7}$$

These are antisymmetry conditions. The nonlinear dissipation component bounds the domain and, thus, prevents the leaving of the representative point. The above facts related to the qualitative analysis of two-dimensional systems will be applied to subsystems (2), (3) in proving the existence of an attractor in system (1).

Assumption 1. Let the three-dimensional nonlinear system (1) be reduced to the form (2), (3). The singular point (0, 0, 0) of the three-dimensional system is a saddle-focus. Subsystem (2) forms a linear dissipative oscillator on the coordinate plane x_1x_2; the singular point on the plane is an unstable focus. The linear system on the plane x_1x_2 satisfies conditions (6) and (7). The trajectory on the plane x_1x_2 has focal points. Subsystem (3) on the plane x_1x_3 is nonlinear. The singular point (0, 0) has characteristic exponents (CE) $\lambda_1 < 0, \lambda_2 < 0$. The trajectory of the subsystem on the plane x_1x_3 tends to zero.

Theorem 1. If the three-dimensional system (1) satisfies Assumption 1, then an attractor is born in the neighborhood of the singular point (0, 0, 0).

Proof. Since the singular point of the three-dimensional system at zero is a saddle-focus, the trajectory goes away from zero. In the three-dimensional system, the first subsystem (2) (plane x_1x_2) has an unstable singular point at zero and satisfies conditions (6), (7). However, the subsystem is linear, and no closed curve forms. Subsystems (2) and (3) are related by the variable x_1. The second subsystem is in the mode of attraction to zero on the plane x_1x_3. In this case, the trajectory does not go away on the plane x_1x_2. The attraction process dominates on the plane x_1x_3. The field generated by the variational equations of subsystem (3) is that of a stable system. The trajectory of the three-dimensional system either closes near (0, 0, 0) or undergoes infinite search (due to the saddle-focus solutions), tending to close near zero.

Remark on the Symmetry of the Projections of the Closed Trajectory on the Coordinate Planes. Let the trajectory close at some values of the parameters. It is nearly antisymmetric on the plane x_1x_2. The three-dimensional system (1) has saddle-focus solutions that make the antisymmetry approximate. An increase in the saddle component of the partial solution leads to period multiplying of oscillations or to chaotic oscillations.

Example 1: Attractors in the Rössler Model. The Rössler model arose in the dynamics of chemical reactions in some stirred medium [9]. It is described by the dimensionless system of equations

$$\frac{d\xi}{dt} = -\eta - \zeta, \quad \frac{d\eta}{dt} = \xi + a\eta, \quad \frac{d\zeta}{dt} = b - c\zeta + \xi\zeta, \tag{8}$$

where a, b, c are positive. System (8) was studied by many researchers (see [7, 11] and the references therein). The coordinates of the singular points of system (8) can be found from the equation $a\zeta^2 - c\zeta + b = 0$.

Consider the singular point $A(\xi = az_a, \eta = -z_a, \zeta = z_a)$ that is nearest to zero, where $z_a = c/(2a) - \sqrt{(c/2a)^2 - b/a}$. The other singular point $B(\xi = az_b, \eta = -z_b, \zeta = z_b)$, where $z_b = c/(2a) + \sqrt{(c/2a)^2 - b/a}$.

We will use a coordinate system $Axyz$ fixed to the point A, where $x = \xi - aza_a$, $y = \eta + z_a$, $z = \zeta - z_a$. Then system (8) becomes

$$\frac{dx}{dt} = -y - z, \qquad \frac{dy}{dt} = x + ay, \qquad \frac{dz}{dt} = z_a x - b/z_a z + xz, \tag{9}$$

where $-b/z_a = -c + az_a$. Let us set up the characteristic equation corresponding to system (9):

$$\lambda^3 + \lambda^2 (b/z_a - a) + \lambda(1 + z_a - ab/z_a) + b/z_a - az_a = 0. \tag{10}$$

For certain parameter values, Eq. (10) shows that the singular point of system (9) is a saddle-focus. Let us transform Eqs. (9) to the form (2), (3):

$$\frac{dx}{dt} = -y, \qquad \frac{dy}{dt} = ay + x, \tag{11}$$

$$\frac{dx}{dt} = -z, \qquad \frac{dz}{dt} = -b/z_a z + z_a x + xz. \tag{12}$$

Introducing small deviations δx, δy, δz from the partial solutions $\bar{x}$, $\bar{y}$, $\bar{z}$ in (11), (12) we set up variational equations:

$$\frac{d\delta x}{dt} = -\delta y, \qquad \frac{d\delta y}{dt} = a\delta y + \delta x,$$

$$\frac{d\delta x}{dt} = -\delta z, \qquad \frac{d\delta z}{dt} = (-b/z_a + \bar{x})\delta z + (\bar{z} + z_a)\delta x. \tag{13}$$

The singular point of subsystem (11) at zero has CEs $a/2 \pm \sqrt{(a/2)^2 - 1}$. According to variational subsystem (13), the singular points in subsystem (12) on the plane xz have CEs that depend on the partial solutions:

$$\lambda_{1,2}(\bar{x}, \bar{z}) = \frac{-b/z_a + \bar{x}}{2} \pm \sqrt{\frac{(-b/z_a + \bar{x})^2}{4} - (z_a - \bar{z})}. \tag{14}$$

The CEs show that, according to (14), a stable node is at zero. When $|\bar{x}| < b/z_a$ and $\bar{x} < 0$, the representative point of system (12) describes a trajectory with stable points (according to (14)). The overall pattern that generates an attractor is as follows: an unstable focus ($\mathrm{Re}\lambda_{1,2} > 0$) is at zero on the plane xy (Eqs. (11)). Stable points on the trajectory of the plane xz can be an indication of closing of the three-dimensional system (on the plane xy, the coordinate portrait of system (11), Fig. 1*a*). Subsystem (11) corresponds to linear equations of motion. On the plane xz (subsystem (12)), the trajectory tends to zero. The attractors mentioned above can be either regular or chaotic. The mechanism of occurrence of a chaotic attractor is associated with the excess of saddle solutions on the trajectory. The excess of saddle solutions leads to period doubling (multiple period). Saddle solutions (compared with focal) slow down the motion of the representative point. Therefore, the representative point does the second turn of oscillations. The further development of the periodic process (after change in the parameters) involves a multiple increase in the period, which leads to nonrecurring bifurcations (disappearance of a certain sequence of bifurcations on the periodic trajectory). The oscillatory process becomes chaotic. This type of chaos is frequent.

Introducing small deviations δx, δy, δz from the partial solutions $\bar{x}$, $\bar{y}$, $\bar{z}$ in (9), we set up variational equations:

$$\frac{d\delta x}{dt} = -\delta y - \delta z, \qquad \frac{dy}{dt} = \delta x + a\delta y, \qquad \frac{d\delta z}{dt} = -(b/z_a - \bar{x})\delta z + (z_a + \bar{z})\delta x.$$

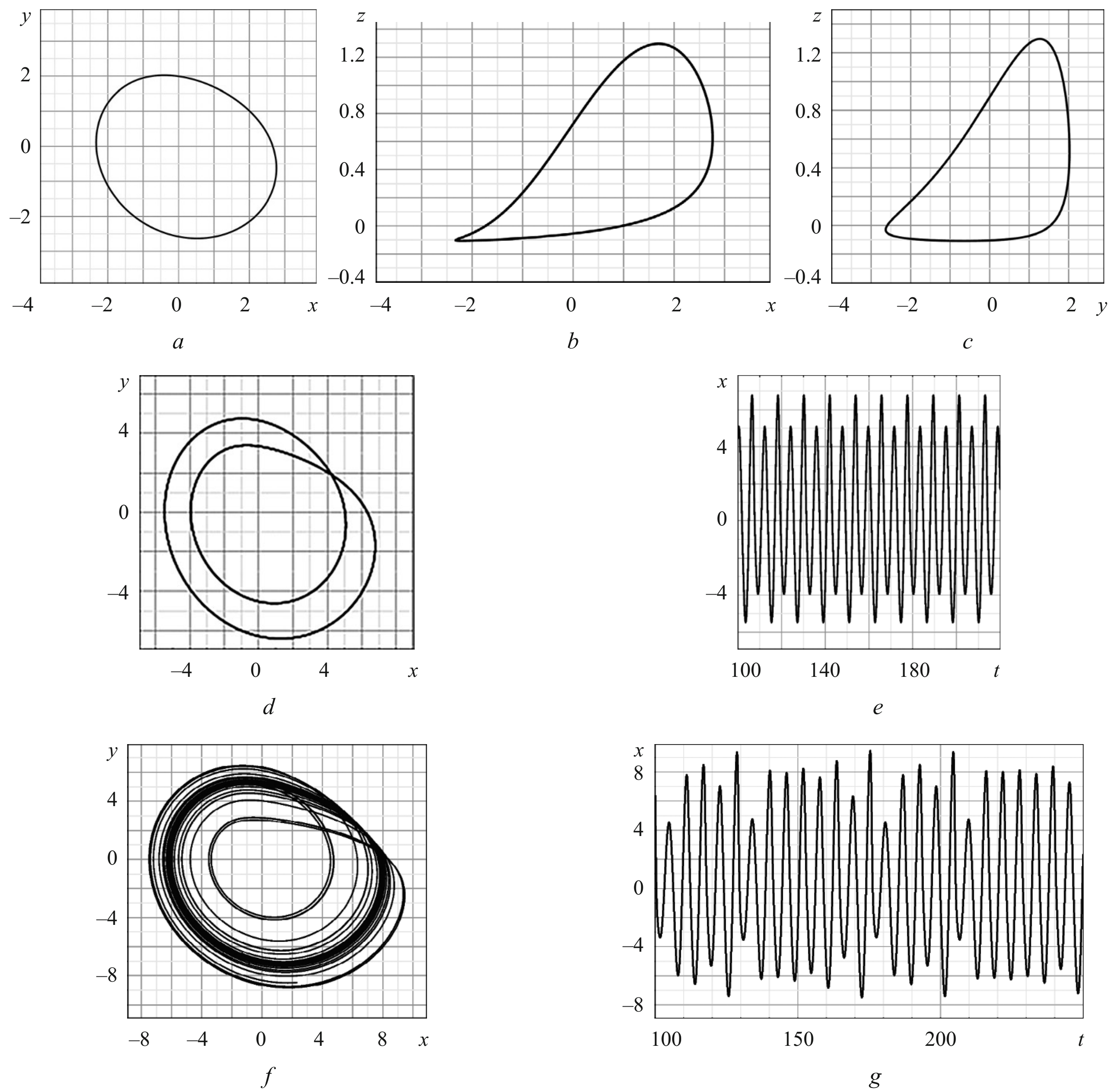

Fig. 1

The characteristic equation of the system of variational equations is

$$\lambda^3+\lambda^2\left(-a+\frac{b}{z_a}-\bar{x}\right)+\lambda\left(1-\frac{ab}{z_a}+z_a-a\bar{x}+\bar{z}\right)+a\left(z_a+\frac{b}{z_a}+\bar{z}\right)-\bar{x}=0. \tag{15}$$

Equation (15) shows that the bifurcation process is associated with the solutions $\bar{x}$ and $\bar{z}$. Equation (15) can be used to analyze how the parameter values affect saddle solutions on the trajectory. Figures 1*a*,*b*,*c* show the coordinate portraits of system (9) for the parameters $(a, b, c) = (0.2, 0.4, 2)$. A variation in b and c leads to a variation in z_a, which affect the excess of saddle solutions (Eq. (15)). At small values of z_a, period doubling (Fig. 1*d*,*e*, $(a, b, c) = (0.2, 0.4, 4)$) and chaos (Fig. 1*g*, $(a, b, c) = (0.2, 0.2, 4.7)$) occur.

Thus, Theorem 1 states the existence of a certain attractor in the three-dimensional system (1).

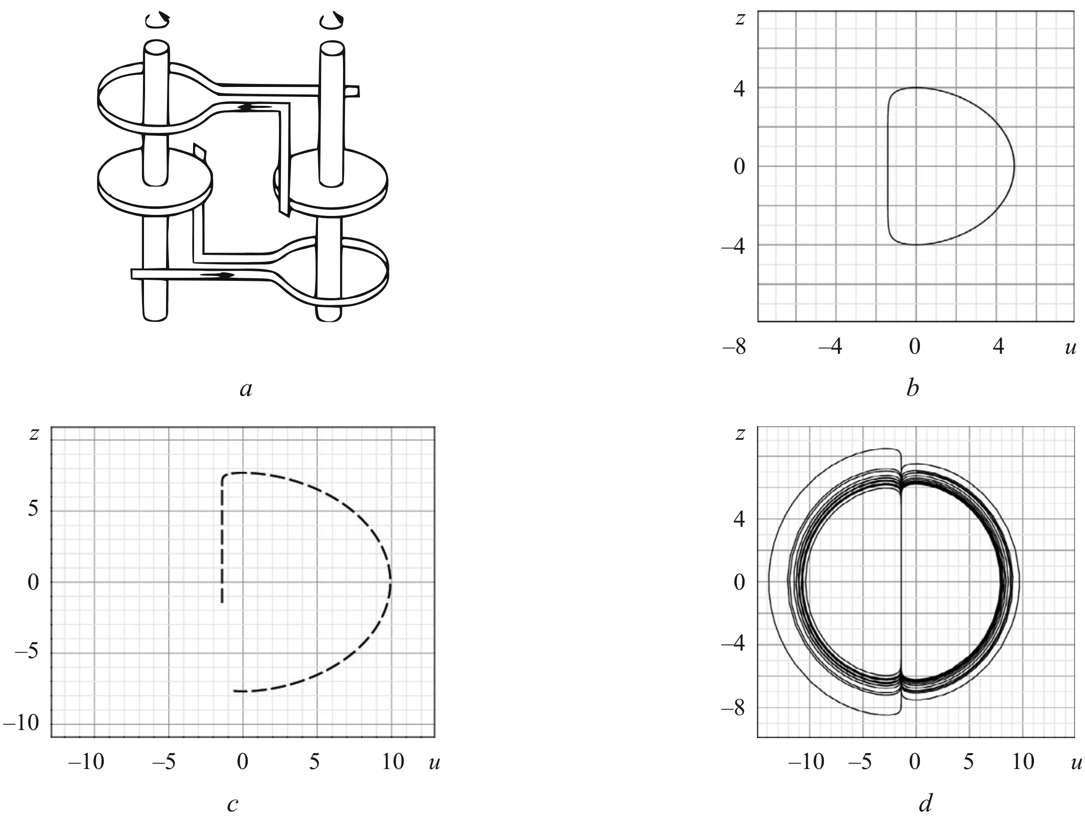

Fig. 2

2. Example 2. Three-Dimensional System with a Plane Attractor. Earth dynamo theory is based on a model that is a three-dimensional system with a plane attractor. The changeover of a plane attractor into a three-dimensional one is the changeover of a regular attractor into a strange one. Let us analyze the mechanism of orbital instability.

Paleomagnetic studies show that the magnetic field of the Earth undergoes reversion of direction (polarity). Over the last hundreds of millions of years, the direction changed irregularly. Besides the detailed magnetohydrodynamic model, there are simple models. Earth dynamo theory is based on the Rikitake model proposed in 1955 [1]. This model describes two dynamo disks coupled as shown in Fig. 2*a*. The disks model the two major vortices in the Earth's core. Let us consider how the fields are excited in the two disks. The current I_1 generates a magnetic field in which the second disk induces the current I_2. This current, in turn, generates a field in which the first disk induces the current I_1. Nonlinearity is associated with the reverse reaction of the moment of electromagnetic force to motion, i.e., to the angular velocities Ω_1 and Ω_2.

The system of equations of mechanics and electrodynamics is

$$L\frac{dI_1}{d\tau} + RI_1 = MI_2\Omega_1, \qquad L\frac{dI_2}{d\tau} + RI_2 = MI_1\Omega_2, \tag{16}$$

where L is the inductance of the disks; R is the resistance of the circuits; M is the mutual inductance between the circuit and the disk. System (16) is called the dynamo equations. The system

$$J\frac{d\Omega_1}{d\tau} = G - MI_1I_2, \qquad J\frac{d\Omega_2}{d\tau} = G - MI_1I_2, \tag{17}$$

where J is the moment of inertia of the disks; G is the moment of the external forces, describes the reverse reaction of the electromagnetic forces to motion. Let us introduce dimensionless variables x_1, x_2, x_3, x_4, t as in [1]:

$$I_1 = x_1\sqrt{\frac{G}{M}}, \quad I_2 = x_2\sqrt{\frac{G}{M}}, \quad \Omega_1 = x_3\sqrt{\frac{GL}{JM}}, \quad \Omega_2 = x_4\sqrt{\frac{GL}{JM}}, \quad \tau = t\sqrt{\frac{JL}{MG}}.$$

System (16), (17) has the following dimensionless form:

$$\frac{dx_1}{dt} = -\mu x_1 + x_2 x_3, \quad \frac{dx_2}{dt} = -\mu x_2 + x_1 x_4, \quad \frac{dx_3}{dt} = 1 - x_1 x_2, \quad \frac{dx_4}{dt} = 1 - x_1 x_2, \tag{18}$$

where $\mu = R\sqrt{J/(GLM)}$ is the coefficient of ohmic dissipation. From system (18) it follows that the difference of angular velocities is constant ($x_3 - x_4 = a$, where $a = \text{const}$). Let $a = 0$.

The coordinates of the singular points of system (18) can be found from the system of equations

$$-\mu x_1 + x_2 x_3 = 0, \quad -\mu x_2 + x_1 x_3 = 0, \quad 1 - x_1 x_2 = 0. \tag{19}$$

The square of the coordinate x_1 is denoted by ν and is found from Eqs. (19):

$$\nu = \pm\sqrt{1}, \quad \nu^2 - 1 = 0.$$

System (18) has two singular points hold: $A(\sqrt{\nu}, 1/\sqrt{\nu}, \mu/\nu)$, $B(-\sqrt{\nu}, -1/\sqrt{\nu}, -\mu/\nu)$.

Let us introduce a coordinate system fixed to the point A ($x = x_1 - \sqrt{\nu}$, $y = x_2 - 1/\sqrt{\nu}$, $z = x_3 - \mu/\nu$) and write the equations of motion:

$$\frac{dx}{dt} = -\mu x + \mu y + \frac{z}{\sqrt{\nu}} + yz,$$

$$\frac{dy}{dt} = \frac{\mu}{\nu} x - \mu y + \sqrt{\nu} z + xz,$$

$$\frac{dz}{dt} = -\frac{x}{\sqrt{\nu}} - \sqrt{\nu} y - xy. \tag{20}$$

Let $\mu = 1$. The characteristic equation corresponding to (20) is

$$\lambda^3 + 2\lambda^2 + 2\lambda + 4 = 0. \tag{21}$$

The singular points of system (20) do not have, according to (21), saddle solutions: $\mathrm{Im}\lambda_1 < 0$, $\mathrm{Im}\lambda_2 > 0$, $\lambda_3 < 0$. These points are node-foci. Let us turn the axes of $Axyz$ by an angle $\gamma = \arctan(1/\nu)$.

$$u = x\cos\gamma + y\sin\gamma, \quad w = -x\sin\gamma + y\cos\gamma.$$

In the new coordinate system $Auwz$, the three-dimensional system becomes

$$\frac{du}{dt} = \sqrt{2}z + uz, \quad \frac{dw}{dt} = -2\mu w - wz, \quad \frac{dz}{dt} = -\sqrt{2}u - u^2/2. \tag{22}$$

Consider system (22) in the general form:

$$\frac{du}{dt} = F_1(u, z), \quad \frac{dw}{dt} = F_2(w, z), \quad \frac{dz}{dt} = F_3(u).$$

Theorem 2. Three-dimensional system (22) forms a plane regular attractor with one axis of symmetry.

Proof. One sign of existence of a plane attractor is the singular three-dimensional point's having only node-focus solution rather than saddle solutions. The parameters of a system that form node-focus solutions at the singular point also assume existence of a regular plane attractor. In this attractor, there are saddle solutions at points that do not affect the regular behavior of the attractor. The right-hand sides of the first and third equations in (22) satisfy the following conditions:

$$F_1(u, -z) = -F_1(u, z), \quad F_3(u, -z) = F_3(x, u, z). \tag{23}$$

Based on the principle of symmetry for two-dimensional systems, it may be concluded that there is a closed trajectory in system (22) if conditions (23) (parity of the function $F_1(u, -z)$ in z and oddness of $F_3(u)$ in z) are satisfied. Here Au is the axis of symmetry, according to (23). Thus, a closed trajectory in system (22) exists on the plane uz. Whether an attractor exists in a three-dimensional system can be determined using the principle of symmetry for two-dimensional systems [8]. A regular attractor appears if $|z_0|<|\pm z^*|$, $u_0=0$. Figure 2*b* shows a trajectory closed about the singular point *A*. Let us specify initial conditions for z in system (22). Let us construct a regular plane limit cycle of system (22) (Fig. 2*b*). On the left boundary ($u=-\sqrt{2}$), the CEs are $\lambda_1>0, \lambda_2<0, \lambda_3=0$. The symmetric cycle of system (24) occurs on the left.

Let us introduce a coordinate system fixed to the point *B* ($x=x_1+\sqrt{\nu},\ y=x_2+1/\sqrt{\nu},\ z=x_3+\mu/\nu$) and write the equations of motion:

$$\frac{dx}{dt}=-\mu x+\mu y-\frac{z}{\sqrt{\nu}}+yz,$$

$$\frac{dy}{dt}=\frac{\mu}{\nu}x-\mu y-\sqrt{\nu}z+xz,$$

$$\frac{dz}{dt}=\frac{x}{\sqrt{\nu}}+\sqrt{\nu}y-xy.$$

In the coordinate system *Buwz*, the three-dimensional system satisfies conditions (23). The trajectory is in the plane uz. Here the axes are fixed to the singular point *B*:

$$\frac{du}{dt}=-\sqrt{2}z+uz,\quad \frac{dw}{dt}=-2\mu w-wz,\quad \frac{dz}{dt}=\sqrt{2}u-u^2/2. \tag{24}$$

For system (24), as well as for (22), Theorem 2 holds.

3. Mechanism of Orbital Instability in a System with a Plane Attractor. Consider the mechanism of orbital instability associated with the passage of the representative point from right to left and back in the half-planes *Auz* and *Buz*. Introducing small deviations δu, δw, δz from the partial solutions $\bar{u}(t)$, $\bar{w}(t)$, $\bar{z}(t)$ in (22), we set up variational equations:

$$\frac{d\delta u}{dt}=\sqrt{2}\delta z+\bar{u}\delta z+\bar{z}\delta u,$$

$$\frac{d\delta w}{dt}=-2\mu\delta w-\bar{w}\delta z-\bar{z}\delta w,$$

$$\frac{d\delta z}{dt}=-\sqrt{2}\delta z-\bar{u}\delta u. \tag{25}$$

The CEs of the points of the field generated by the variational system (25) can be found from the characteristic equation of system (25). The numerical solution corresponds to the partial solutions $\bar{u}(t)$, $\bar{z}(t)$.

Consider Fig. 2*c*. The points forming the vertical section of the trajectory on which the CEs are $\lambda_1<0, \lambda_2<0, \lambda_3=0$, according to the characteristic equation of system (25). According to these CEs, the trajectory rises upward to the point at which $z=0$. Next, the CEs are $\lambda_1>0, \lambda_2<0, \lambda_3=0$, and

$$|\lambda_2|>|\lambda_1|. \tag{26}$$

Condition (26) represents the upward movement of the trajectory where points with three nonzero CEs ($\lambda_1>0, \lambda_2<0$, $\lambda_3>0$) appear in the neighborhood of the point ($u=0.88$, $z=7.62$). Then saddle-nodes transform into saddle-foci. In this case, in Fig. 2*c*, the trajectory turns and the saddle solutions on this section cause the trajectories not to coincide at each turn. This generates a chaotic attractor.

Increasing the initial perturbations in z changes the topology of the CE field so that the trajectory passes from one domain to the other. The passage of the representative point through the vertical boundary between the limit cycles occurs instantaneously; therefore, this does not affect the solution. Figure 2*c* shows a plane stochastic attractor. The CEs of the points are determined from the three-dimensional system (25). The strange attractor remains plane; however, the CEs have the form of saddle-nodes or saddle-foci at the instant of passage. Saddle-focus solutions introduce stochasticity. The attractor becomes strange.

Conclusions. Two cases of existence of attractors in a three-dimensional system have been considered and the causes of chaotization of attractors have been analyzed. The first case is related to Theorem 1. An example of this case is the Rössler attractor that is not symmetric on the coordinate planes (it is approximately antisymmetric on one plane). The mechanism of chaos is associated with saddle solutions that (unlike focal solutions) cause a multiple increase in the period of oscillations. The multiple increase in the period results in chaos.

The other case is related to the existence of plane attractors in a three-dimensional system (a special case of the Rikitake problem). Two node-foci on a plane generate two attractors. For some initial conditions, the attractors have a common boundary and the trajectory does not cross this boundary. For other initial conditions, the trajectory stochastically passes from one domain to the other. The boundary (vertical line) between the attractors is filled by points with the CEs $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0$. The orientation of the saddle CEs of the points at different values of the coordinates z changes so that for $|z_0| < |\pm z^*|$, the attractor is regular (plane) and the trajectory does not pass from domain to domain (Fig. 2*b*); for $|z_0| > |\pm z^*|$, the attractor strange and plane (Fig. 2*d*). The strange attractor remains plane due to the instantaneous passage from one domain to the other.

REFERENCES

1. A. E. Cook and P. H. Roberts, "The Rikitake two-disc dynamo system," *Math. Proc. Cambridge Philos. Soc.*, **68**, No. 2, 547–569 (1970).
2. N. V. Nikitina, *Nonlinear Systems with Complex and Chaotic Behavior of Trajectories* [in Russian], Feniks, Kyiv (2012).
3. N. V. Nikitina, "Symmetry principle in three-dimensional systems," *Dop. NAN Ukrainy*, No. 7, 21–28 (2017).
4. G. A. Leonov, *Strange Attractors and Classical Stability Theory*, St. Peterburg Univ. Press, St. Peterburg (2008).
5. A.A. Martynyuk and N. V. Nikitina, "Stability and bifurcation in a model of the magnetic field of the Earth," *Int. Appl. Mech.*, **50**, No. 6, 721–731 (2014).
6. A. A. Martynyuk and N. V. Nikitina, "Bifurcations and multi-stability of the oscillations of a three-dimensional system," *Int. Appl. Mech.*, **51**, No. 2, 223–232 (2015).
7. A. A. Martynyuk and N. V. Nikitina, "On periodical motions in three-dimensional systems," *Int. Appl. Mech.*, **51**, No. 4, 369–379 (2015).
8. Yu. I. Neimark and P. S. Landa, *Stochastic and Chaotic Oscillations*, Kluwer, Dordrecht (1992).
9. V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equation*, Princeton Univ. Press, Princeton (1960).
10. O. E. Rössler, "Chemical turbulence: chaos in a simple reaction-diffusion system," *Z. Naturforsch*, **31a**, No. 10, 1168–1172 (1976).
11. L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, and L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, Part I, World Scientific, Singapore (1998).
12. L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, and L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II, World Scientific, Singapore (2001).