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FRACTURE PROCESS ZONE AT THE TIP OF A MODE I CRACK IN A NONLINEAR ELASTIC ORTHOTROPIC MATERIAL

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A body with a fracture process zone at the crack front is considered. The constitutive equations relating the components of the stress vectors at points on the opposite boundaries of the fracture process zone and the components of the vector of relative displacements of these points are derived. A local fracture criterion is formulated. A boundary-value problem for a plate made of a nonlinear elastic orthotropic material with a mode I crack is stated in terms of the components of the displacement vector. By solving the problem numerically, it is revealed how the fracture process zone develops under loading. Features of the deformation field at the end of the fracture process zone are established. The critical load on the plate that causes the crack to grow is determined.

Keywords: nonlinear elastic orthotropic material, mode I crack, fracture process zone, constitutive equations, local fracture criterion

Introduction. Experimental studies show that a fracture process zone in the form of a narrow region containing microcracks, pores, and delaminations origins at the crack front [9]. The presence of this zone should be allowed for in formulating boundary-value problems on the equilibrium (including ultimate one) of cracked bodies. However, this meets severe difficulties. Usually they may be evaded by applying models of the fracture process zone. For example, in [6], the fracture process zone is modeled by an open slit whose surfaces aligned with the boundaries of the fracture process zone are acted upon by opposite stress vectors. In conformity with up-to-date tendency in fracture mechanics, it is necessary to employ the constitutive equations that relate the components of stress and displacement vectors at points on the opposite boundaries of the fracture process zone [7, 8].

Recently, much attention has been paid to the derivation of these equations [12]. The constitutive equations were mainly derived based on different assumptions and hypotheses for the fracture process zone at the front of mode I and II cracks. Despite a number of important results obtained, the issue has not been resolved completely.

We believe that the components of the stress vectors at points on the opposite boundaries of the fracture process zone must depend on the distance between them. However, just a few constitutive equations satisfy this condition.

The constitutive equations derived in [14] are best known. In these equations, the normal and reduced tangential (with respect to the crack) components of the vector of relative displacements of points on the opposite boundaries of the fracture process zone appear as arguments. These arguments have a scalar multiplier that is a function of the square root of a certain quadratic invariant composed of the components of the displacement vector. Thus, the Tvergaard–Hutchinson constitutive equations [14] satisfy the above condition only for a mode I crack.

Unlike the Tvergaard–Hutchinson equations [14], the normal and tangential components of the displacement vector appear in the equations derived in [5, 11] as arguments, while the scalar multiplier is a function of the square root of the second invariant composed of these above components. Thus, according to the Needleman–Banks-Sills equations [5, 11], the

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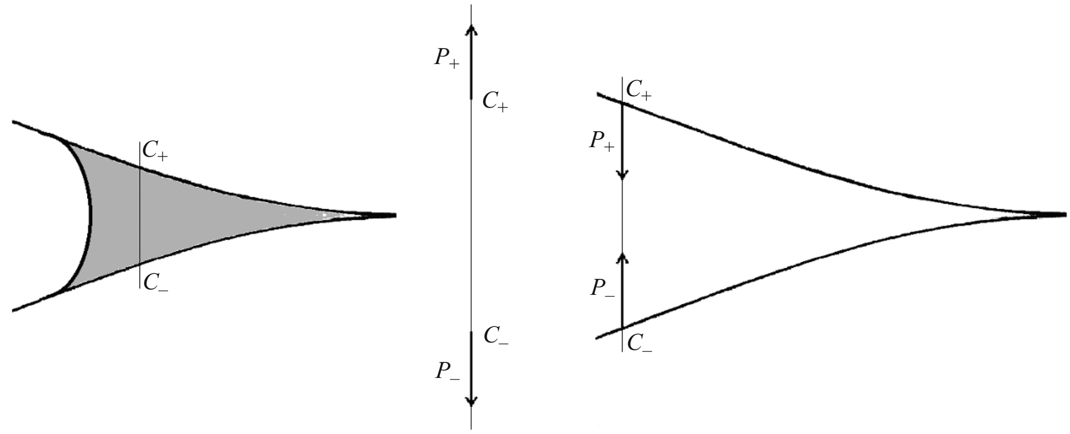


a *b*

Fig. 1

components of the stress vectors at points on the opposite boundaries of the fracture process zone depend on the distance between these points.

To allow for the crack mode (either mode I or mode II) in [5, 11], a special parameter was introduced into the constitutive equations, without any substantiation of how this is done.

In the present paper, we will study the fracture process zone at the tip of a mode I crack in a nonlinear elastic orthotropic material in a plane stress state. Using the model proposed in [6], we will proceed as follows:

(i) The internal stresses acting on the boundaries of the fracture process zone are considered external ones by representing the boundaries of the fracture process zone as external surfaces.

(ii) In solving the boundary-value problem, it is required that the stresses at the end of the fracture process zone be continuous and the strength criterion be satisfied.

Suppose that the components of the stress vectors at points on the boundaries of the fracture process zone depend on the components of the vector of mutual displacements of these points. We will derive the constitutive equations analytically using general principles and formulate a local fracture criterion.

Using the tensor linear constitutive equations relating the components of the stress and strain tensors and the constitutive equations derived, we will formulate (in terms of the components of the displacement vector) a boundary-value problem on the equilibrium of a plate made of a nonlinear elastic orthotropic material and having a mode I crack. Solving the problem numerically, we obtain the dependence of the crack tip opening displacement on the load applied to the plate. The deformation field near the end of the fracture process zone will also be analyzed. Special attention will be paid to the ultimate equilibrium state of the plate in which the crack tip opening displacement is critical.

1. Derivation of the Constitutive Equations. Consider a cracked body in which a narrow fracture process zone origins at the crack front under loading. Assume that the fracture process zone consists of straight elements attached to its boundaries [1]. Let us select a point C at the crack front that transforms into points C_+ and C_- on the boundaries of the fracture process zone during deformation (Fig. 1*a*). Consider a straight element attached to the boundaries of the fracture process zone at the points C_+ and C_-. Next, we apply stress vectors $\boldsymbol{P}_+$ and $\boldsymbol{P}_-$ to the ends of the element (Fig. 1*b*) and stress vectors $-\boldsymbol{P}_+$ and $-\boldsymbol{P}_-$ to the boundaries of the fracture process zone (Fig. 1*c*). These vectors are opposite and lie on the straight line passing through the points C_+ and C_-.

Assume that the vectors $\overrightarrow{CC_+} \equiv \boldsymbol{u}^+$ and $\overrightarrow{CC_-} \equiv \boldsymbol{u}^-$ describing the displacements of the points C_+ and C_- relative to the point C are known. Let us form the vector $\boldsymbol{v}^{+-}$ ($\boldsymbol{v}^{+-} = \boldsymbol{u}^+ - \boldsymbol{u}^-$) describing the displacements of the point C_+ relative to the point C_-, and the vector $\boldsymbol{v}^{-+}$ ($\boldsymbol{v}^{-+} = \boldsymbol{u}^- - \boldsymbol{u}^+$) describing the displacements of the point C_- relative to the point C_+.

To describe the state of the element, we will select either $\boldsymbol{P}_+$ and $\boldsymbol{v}^{+-}$ or $\boldsymbol{P}_-$ and $\boldsymbol{v}^{-+}$. For simplicity, we will denote these vectors by $\boldsymbol{P}$ and $\boldsymbol{v}$. Thus, these vectors are collinear and, moreover, codirectional.

Let the body be described by non-orthogonal curvilinear coordinates x^1, x^2, x^3 characterized by a covariant metric tensor with components $g_{\alpha\beta}$ and a contravariant metric tensor with components $g^{\alpha\beta}$.

Let there be reciprocal bases represented by local basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ and $\boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3$:

$$\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta = g_{\alpha\beta}, \quad \boldsymbol{e}^\alpha \cdot \boldsymbol{e}^\beta = g^{\alpha\beta} \tag{1.1}$$

with

$$\boldsymbol{e}_\alpha \cdot \boldsymbol{e}^\beta = \delta^\beta_\alpha, \tag{1.2}$$

where δ^β_α is the Kronecker delta,

$$\delta^\beta_\alpha = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases} \tag{1.3}$$

In what follows, we will replace dummy indices without any explanation.

The vector $\boldsymbol{P}$ can be expressed in terms of its contravariant components:

$$\boldsymbol{P} = \boldsymbol{P}^\gamma \boldsymbol{e}_\gamma. \tag{1.4}$$

The modulus $|\boldsymbol{P}| \equiv P$ of the vector $\boldsymbol{P}$ is

$$P = \sqrt{\boldsymbol{P} \cdot \boldsymbol{P}}. \tag{1.5}$$

According to formula (1.4) and the first expression in (1.1), the scalar product $\boldsymbol{P} \cdot \boldsymbol{P}$ is

$$\boldsymbol{P} \cdot \boldsymbol{P} = g_{\alpha\beta} P^\alpha P^\beta. \tag{1.6}$$

The vector $\boldsymbol{v}$ is expressed in terms of its contravariant components:

$$\boldsymbol{v} = \mathrm{v}_\gamma \boldsymbol{e}^\gamma. \tag{1.7}$$

The modulus $|\boldsymbol{v}| \equiv v$ of the vector $\boldsymbol{v}$ is

$$v = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}. \tag{1.8}$$

According to formula (1.7) and the second expression in (1.1), the scalar product $\boldsymbol{v} \cdot \boldsymbol{v}$ is

$$\boldsymbol{v} \cdot \boldsymbol{v} = g^{\alpha\beta} \mathrm{v}_\alpha \mathrm{v}_\beta. \tag{1.9}$$

Let us derive equations relating the contravariant components of the vector $\boldsymbol{P}$ and the contravariant components of the vector $\boldsymbol{v}$.

Consider a unit vector $\boldsymbol{i}$ codirectional with the vector $\boldsymbol{P}$,

$$\boldsymbol{P} = P\boldsymbol{i}. \tag{1.10}$$

Multiplying both sides of formula (1.10) by $\boldsymbol{e}^\alpha$, we get

$$\boldsymbol{P} \cdot \boldsymbol{e}^\alpha = P\boldsymbol{i} \cdot \boldsymbol{e}^\alpha. \tag{1.11}$$

According to formulas (1.4), (1.2), and (1.3), the scalar products $\boldsymbol{P} \cdot \boldsymbol{e}^\alpha$ are

$$\boldsymbol{P} \cdot \boldsymbol{e}^\alpha = P^\alpha. \tag{1.12}$$

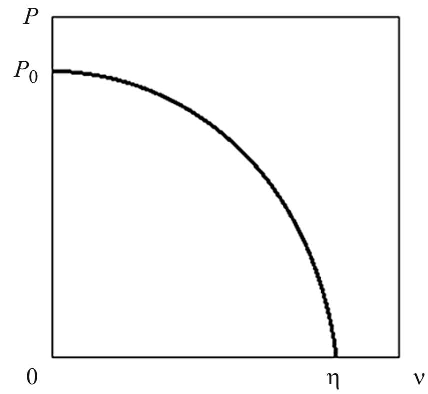


Fig. 2

Since the vectors $\boldsymbol{P}$ and $\boldsymbol{v}$ are codirectional, then

$$\boldsymbol{i} = \boldsymbol{v} / v. \tag{1.13}$$

According to formula (1.7) and the second expression in (1.1), the scalar products $\boldsymbol{v} \cdot \boldsymbol{e}^{\alpha}$ are

$$\boldsymbol{v} \cdot \boldsymbol{e}^{\alpha} = g^{\alpha\beta} v_{\beta}. \tag{1.14}$$

According to (1.13) and (1.14), the scalar products $\boldsymbol{i} \cdot \boldsymbol{e}^{\alpha}$ are

$$\boldsymbol{i} \cdot \boldsymbol{e}^{\alpha} = \frac{g^{\alpha\beta} v_{\beta}}{v}. \tag{1.15}$$

Substituting (1.12) and (1.15) into (1.11), we obtain

$$P^{\alpha} = P \frac{g^{\alpha\beta} v_{\beta}}{v}. \tag{1.16}$$

Thus, we have derived the necessary equations.

Relationship (1.16) between the contravariant components of the vector $\boldsymbol{P}$ and the contravariant components of the vector $\boldsymbol{v}$ is known if so is the functional dependence of the modulus P on the modulus v (Fig. 2). Following [16], we suppose that this dependence is

$$P = P_o f(v), \tag{1.17}$$

where $f(v)$ is a function decreasing in the range (o, η).

Note that the value η of the modulus v of the vector $\boldsymbol{v}$ must depend on the orientation of the element relative to the crack.

Let the function $f(v)$ be defined by

$$f(v)\Big|_{v=o} = 1, \quad \frac{d}{dv} f(v)\Big|_{v=o} = 0,$$

$$f(v)\Big|_{v=\eta} = 0, \quad \frac{d}{dv} f(v)\Big|_{v=\eta} = m. \tag{1.18}$$

For many applications, it can be approximated by

$$f(v) = 1 - b_{k_1} v^{k_1} - b_{k_2} v^{k_2}, \tag{1.19}$$

where k_1 and k_2 are integers ($1 < k_1 < k_2$).

Differentiating (1.19), we obtain

$$\frac{d}{dv} f(v) = -k_1 b_{k_1} v^{k_1 - 1} - k_2 b_{k_2} v^{k_2 - 1}. \tag{1.20}$$

It is clear that the first and second conditions in (1.18) become identities due to (1.19) and (1.20).

Considering the third and fourth conditions in (1.18), we find the coefficients b_{k_1} and b_{k_2} using (1.19) and (1.20):

$$b_{k_1} = \frac{k_2 + m\eta}{(k_2 - k_1)\eta^{k_1}}, \quad b_{k_2} = \frac{k_1 + m\eta}{(k_1 - k_2)\eta^{k_2}}. \tag{1.21}$$

Let us represent the function $f(v)$ as follows:

$$f(v) = [1 - \tilde{f}(v)], \tag{1.22}$$

where the function $\tilde{f}(v)$ increases in the range (o, η).

Comparing formulas (1.19) and (1.22), we see that

$$\tilde{f}(v) = b_{k_1} v^{k_1} + b_{k_2} v^{k_2}. \tag{1.23}$$

With (1.17) and (1.22), Eqs. (1.16) become:

$$P^{\alpha} = P_o [1 - \tilde{f}(v)] \frac{g^{\alpha\beta} v_{\beta}}{v}. \tag{1.24}$$

Using formulas (1.21), (1.23), and (1.24), we calculate the contravariant components of the vectors ***P*** and −***P***.

At the instant of failure of the element, the modulus P of the vector ***P*** becomes equal to zero. According to formula (1.17) and the third condition in (1.18), this occurs when the modulus v of the vector ***v*** becomes equal to η. Thus, we have the local fracture criterion

$$v = \eta. \tag{1.25}$$

This criterion is a generalization of the critical opening displacement criterion [13, 15].

2. Statement of the Boundary-Value Problem. In what follows, we will restrict our consideration to small strains. Let us employ tensor-linear constitutive equations relating the contravariant components of the stress tensor ***S*** and the covariant components of the strain tensor ***D*** [3]:

$$S^{\alpha\beta} = G^{\alpha\beta\gamma\delta} D_{\gamma\delta} - \tilde{\varphi}(\Omega)\left(G^{\alpha\beta\gamma\delta} D_{\gamma\delta} - \frac{\mathrm{E}}{\mathrm{Z}} g^{\alpha\beta} \right), \tag{2.1}$$

where

$$\Omega = \sqrt{\Xi - \frac{\mathrm{E}^2}{\mathrm{Z}}} \tag{2.2}$$

is the argument of the function $\tilde{\varphi}(\Omega)$. The invariants Z, E, and Ξ are defined by

$$\mathrm{Z} = F_{\alpha\beta\gamma\delta} g^{\alpha\beta} g^{\gamma\delta}, \quad \mathrm{E} = g^{\alpha\beta} D_{\alpha\beta}, \quad \Xi = G^{\alpha\beta\gamma\delta} D_{\alpha\beta} D_{\gamma\delta}. \tag{2.3}$$

The reciprocal fourth-rank tensors ***F*** and ***G*** characterize anisotropy and possess high symmetry. In other words, we can exchange either the indices within any pair of indices or pairs of indices themselves in the components of these tensors.

Let the coordinate system x^1, x^2, x^3 be Cartesian. Then

$$g^{\varepsilon\zeta} = \begin{cases} 1, & \varepsilon = \zeta, \\ 0, & \varepsilon \neq \zeta. \end{cases} \tag{2.4}$$

Let us derive basic equations for the components of the displacement vector $\boldsymbol{u}$. We will use the Cauchy relations [10]:

$$D_{\varepsilon\zeta} = \frac{\partial u_\varepsilon}{\partial x^\zeta} \, (\varepsilon, \zeta), \tag{2.5}$$

where symmetrization over the indices ε and ζ is assumed. With (2.5), Eqs. (2.1) become

$$S^{\alpha\beta} = G^{\alpha\beta\gamma\delta} \frac{\partial u_\gamma}{\partial x^\delta} - \tilde{\varphi}(\Omega) \left(G^{\alpha\beta\gamma\delta} \frac{\partial u_\gamma}{\partial x^\delta} - \frac{\mathrm{E}}{\mathrm{Z}} g^{\alpha\beta} \right) \tag{2.6}$$

and the second and third invariants in (2.3) become

$$\mathrm{E} = g^{\alpha\beta} \frac{\partial u_\alpha}{\partial x^\beta}, \quad \Xi = G^{\alpha\beta\gamma\delta} \frac{\partial u_\alpha}{\partial x^\beta} \frac{\partial u_\gamma}{\partial x^\delta}. \tag{2.7}$$

Assume that the plate material is orthotropic and its principal axes are aligned with the axes x^1, x^2, x^3.

Let the consider the case of plane stress state assuming that

$$S^{\alpha\beta} = S^{\alpha\beta}(x^1, x^2) \quad (\alpha = 1, 2, \ \beta = 1, 2), \tag{2.8}$$

$$S^{\alpha\beta} = 0 \quad (\alpha = 1, 2, \beta = 3, \ \alpha = 3, \beta = 1, 2, \ \alpha = 3, \beta = 3). \tag{2.9}$$

According to (2.4), the first invariant in (2.7) becomes

$$\mathrm{E} = \frac{\partial u_1}{\partial x^1} + \frac{\partial u_2}{\partial x^2} + \frac{\partial u_3}{\partial x^3}. \tag{2.10}$$

Since $\tilde{\varphi}(\Omega) \neq 1$, Eqs. (2.6) lead to the following in view of (2.9) and (2.4):

$$\frac{\partial u_\gamma}{\partial x^\delta} + \frac{\partial u_\delta}{\partial x^\gamma} = 0 \quad (\gamma = 1, 2, \ \delta = 3, \ \gamma = 3, \ \delta = 1, 2). \tag{2.11}$$

Let

$$G^{1111} \equiv \mu_{AA}, \quad G^{1212} \equiv \mu_{BB}, \quad G^{1122} \equiv \mu_{AD}, \quad G^{2222} \equiv \mu_{DD},$$
$$G^{1133} \equiv \mu_{AF}, \quad G^{2233} \equiv \mu_{DF}, \quad G^{3333} \equiv \mu_{FF}. \tag{2.12}$$

With (2.11) and (2.12), the second invariant in (2.7) becomes

$$\Xi = \mu_{AA} \frac{\partial u_1}{\partial x^1} \frac{\partial u_1}{\partial x^1} + 2\mu_{AD} \frac{\partial u_1}{\partial x^1} \frac{\partial u_2}{\partial x^2} + 2\mu_{AF} \frac{\partial u_1}{\partial x^1} \frac{\partial u_3}{\partial x^3}$$
$$+ \mu_{BB} \left(\frac{\partial u_1}{\partial x^2} \frac{\partial u_1}{\partial x^2} + 2 \frac{\partial u_1}{\partial x^2} \frac{\partial u_2}{\partial x^1} + \frac{\partial u_2}{\partial x^1} \frac{\partial u_2}{\partial x^1} \right) + \mu_{DD} \frac{\partial u_2}{\partial x^2} \frac{\partial u_2}{\partial x^2} + 2\mu_{DF} \frac{\partial u_2}{\partial x^2} \frac{\partial u_3}{\partial x^3} + \mu_{FF} \frac{\partial u_3}{\partial x^3} \frac{\partial u_3}{\partial x^3}. \tag{2.13}$$

Taking into account (2.9), (2.4), and (2.6), we arrive at

$$\frac{\partial u_3}{\partial x^3} = \frac{1}{G^{3333}} \left[\tilde{\varphi}(\Omega) \left(G^{3311} \frac{\partial u_1}{\partial x^1} + G^{3322} \frac{\partial u_2}{\partial x^2} + G^{3333} \frac{\partial u_3}{\partial x^3} - \frac{\mathrm{E}}{\mathrm{Z}} \right) - G^{3311} \frac{\partial u_1}{\partial x^1} - G^{3322} \frac{\partial u_2}{\partial x^2} \right]. \tag{2.14}$$

With (2.4) and (2.14), Eqs. (2.6) take the form

$$S^{\alpha\beta}=\left(G^{\alpha\beta 11}-\frac{G^{\alpha\beta 33}}{G^{3333}}G^{3311}\right)\frac{\partial u_1}{\partial x^1}+\left(G^{\alpha\beta 22}-\frac{G^{\alpha\beta 33}}{G^{3333}}G^{3322}\right)\frac{\partial u_2}{\partial x^2}$$

$$-\tilde{\varphi}(\Omega)\left[\left(G^{\alpha\beta 11}-\frac{G^{\alpha\beta 33}}{G^{3333}}G^{3311}\right)\frac{\partial u_1}{\partial x^1}+\left(G^{\alpha\beta 22}-\frac{G^{\alpha\beta 33}}{G^{3333}}G^{3322}\right)\frac{\partial u_2}{\partial x^2}-\left(1-\frac{G^{\alpha\beta 33}}{G^{3333}}\right)\frac{\mathrm{E}}{\mathrm{Z}}\right]$$

$$(\alpha,\beta=1,2,\ \alpha=\beta), \tag{2.15}$$

$$S^{\alpha\beta}=G^{\alpha\beta 12}\frac{\partial u_1}{\partial x^2}+G^{\alpha\beta 21}\frac{\partial u_2}{\partial x^1}-\tilde{\varphi}(\Omega)\left(G^{\alpha\beta 12}\frac{\partial u_1}{\partial x^2}+G^{\alpha\beta 21}\frac{\partial u_2}{\partial x^1}\right)$$

$$(\alpha,\beta=1,2,\ \alpha\neq\beta). \tag{2.16}$$

Denote

$$\frac{G^{1133}}{G^{3333}}\equiv\xi_{AF},\qquad \frac{G^{2233}}{G^{3333}}\equiv\xi_{DF}, \tag{2.17}$$

$$G^{1111}-\frac{G^{1133}}{G^{3333}}G^{3311}\equiv\breve{\mu}_{AA},\qquad G^{1122}-\frac{G^{1133}}{G^{3333}}G^{3322}\equiv\breve{\mu}_{AD},$$

$$G^{2211}-\frac{G^{2233}}{G^{3333}}G^{3311}\equiv\breve{\mu}_{DA},\qquad G^{2222}-\frac{G^{2233}}{G^{3333}}G^{3322}\equiv\breve{\mu}_{DD}. \tag{2.18}$$

Let us use Navier's equations [10]:

$$\frac{\partial S^{\alpha\beta}}{\partial x^\beta}=0. \tag{2.19}$$

Suppose that the material of the plate is homogeneous. Using formulas (2.8), (2.9), (2.15), (2.16), the second notation in (2.12), notation (2.17) and (2.18), and Eqs. (2.19), we obtain

$$\breve{\mu}_{AA}\frac{\partial^2 u_1}{\partial x^1\partial x^1}+(\breve{\mu}_{AD}+\mu_{BB})\frac{\partial^2 u_2}{\partial x^1\partial x^2}+\mu_{BB}\frac{\partial^2 u_1}{\partial x^2\partial x^2}=Q^1,$$

$$\mu_{BB}\frac{\partial^2 u_2}{\partial x^1\partial x^1}+(\mu_{BB}+\breve{\mu}_{DA})\frac{\partial^2 u_1}{\partial x^1\partial x^2}+\breve{\mu}_{DD}\frac{\partial^2 u_2}{\partial x^2\partial x^2}=Q^2, \tag{2.20}$$

where

$$Q^1=\frac{\partial\tilde{\varphi}(\Omega)}{\partial x^1}\left(\breve{\mu}_{AA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{AD}\frac{\partial u_2}{\partial x^2}-\frac{1-\xi_{AF}}{\mathrm{Z}}\mathrm{E}\right)+\frac{\partial\tilde{\varphi}(\Omega)}{\partial x^2}\mu_{BB}\left(\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}\right)$$

$$+\tilde{\varphi}(\Omega)\left[\breve{\mu}_{AA}\frac{\partial^2 u_1}{\partial x^1\partial x^1}+(\breve{\mu}_{AD}+\mu_{BB})\frac{\partial^2 u_2}{\partial x^1\partial x^2}+\mu_{BB}\frac{\partial^2 u_1}{\partial x^2\partial x^2}-\frac{1-\xi_{AF}}{\mathrm{Z}}\frac{\partial\mathrm{E}}{\partial x^1}\right],$$

$$Q^2=\frac{\partial\tilde{\varphi}(\Omega)}{\partial x^1}\mu_{BB}\left(\frac{\partial u_2}{\partial x^1}+\frac{\partial u_1}{\partial x^2}\right)+\frac{\partial\tilde{\varphi}(\Omega)}{\partial x^2}\left(\breve{\mu}_{DA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{DD}\frac{\partial u_2}{\partial x^2}-\frac{1-\xi_{DF}}{\mathrm{Z}}\mathrm{E}\right)$$

$$+\tilde{\varphi}(\Omega)\left[\mu_{BB}\frac{\partial^2 u_2}{\partial x^1 \partial x^1}+(\mu_{BB}+\breve{\mu}_{DA})\frac{\partial^2 u_1}{\partial x^1 \partial x^2}+\breve{\mu}_{DD}\frac{\partial^2 u_2}{\partial x^2 \partial x^2}-\frac{1-\xi_{DF}}{Z}\frac{\partial \mathrm{E}}{\partial x^2}\right]. \tag{2.21}$$

Let a stress vector $\boldsymbol{P}$ with components P^α act on the plate boundaries, crack faces, and boundaries of the fracture process zone. Let us use the following boundary conditions [10]:

$$S^{\alpha\beta} n_\beta = P^\alpha, \tag{2.22}$$

where n_β are the components of the outward normal unit vector $\boldsymbol{n}$.

Using formulas (2.9), (2.15), and (2.16), the second notation in (2.12), notation (2.17) and (2.18), and condition (2.22), we obtain

$$\left(\breve{\mu}_{AA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{AD}\frac{\partial u_2}{\partial x^2}\right)n_1+\mu_{BB}\left(\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}\right)n_2 = P^1+R^1,$$

$$\mu_{BB}\left(\frac{\partial u_2}{\partial x^1}+\frac{\partial u_1}{\partial x^2}\right)n_1+\left(\breve{\mu}_{DA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{DD}\frac{\partial u_2}{\partial x^2}\right)n_2 = P^2+R^2, \tag{2.23}$$

where

$$R^1=\tilde{\varphi}(\Omega)\left[\left(\breve{\mu}_{AA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{AD}\frac{\partial u_2}{\partial x^2}-\frac{1-\xi_{AF}}{Z}\mathrm{E}\right)n_1+\mu_{BB}\left(\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}\right)n_2\right],$$

$$R^2=\tilde{\varphi}(\Omega)\left[\mu_{BB}\left(\frac{\partial u_2}{\partial x^1}+\frac{\partial u_1}{\partial x^2}\right)n_1+\left(\breve{\mu}_{DA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{DD}\frac{\partial u_2}{\partial x^2}-\frac{1-\xi_{DF}}{Z}\mathrm{E}\right)n_2\right]. \tag{2.24}$$

Let us consider a rectangular small-thickness plate with a central crack. The symmetry axes of the plate are aligned with the x^1- and x^2-axes. A load symmetric about the axes is applied to the plate. We can restrict ourselves to consideration of only one quarter of the plate, say, that in the first quadrant (Fig. 3). In this case, Eqs. (2.23) for the upper boundary of the plate ($n_1 = 1$, $n_2 = 0$) take the form

$$\breve{\mu}_{AA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{AD}\frac{\partial u_2}{\partial x^2}=P^1+R^1,\quad \mu_{BB}\left(\frac{\partial u_2}{\partial x^1}+\frac{\partial u_1}{\partial x^2}\right)=P^2+R^2, \tag{2.25}$$

while formulas (2.24) become

$$R^1=\tilde{\varphi}(\Omega)\left(\breve{\mu}_{AA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{AD}\frac{\partial u_2}{\partial x^2}-\frac{1-\xi_{AF}}{Z}E\right),\quad R^2=\tilde{\varphi}(\Omega)\mu_{BB}\left(\frac{\partial u_2}{\partial x^1}+\frac{\partial u_1}{\partial x^2}\right). \tag{2.26}$$

Equations (2.23) for the lateral boundary of the plate ($n_1 = 0, n_2 = 1$) take the form

$$\mu_{BB}\left(\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}\right)=P^1+R^1,\quad \breve{\mu}_{DA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{DD}\frac{\partial u_2}{\partial x^2}=P^2+R^2, \tag{2.27}$$

while formulas (2.24) become

$$R^1=\tilde{\varphi}(\Omega)\mu_{BB}\left(\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}\right),\quad R^2=\tilde{\varphi}(\Omega)\left(\breve{\mu}_{DA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{DD}\frac{\partial u_2}{\partial x^2}-\frac{1-\xi_{DF}}{Z}E\right). \tag{2.28}$$

Equations (2.23) for the upper face of the crack and for the upper boundary of the fracture process zone ($-n_1 = 1, n_2 = 0$) become

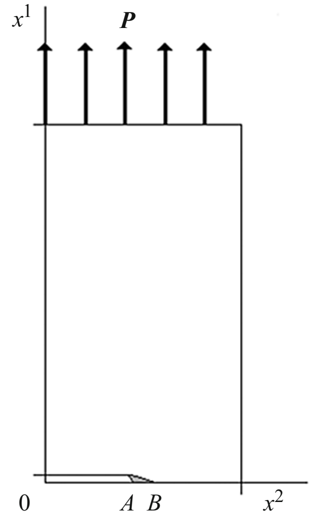


Fig. 3

$$-\left(\breve{\mu}_{AA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{AD}\frac{\partial u_2}{\partial x^2}\right)=P^1+R^1, \quad -\mu_{BB}\left(\frac{\partial u_2}{\partial x^1}+\frac{\partial u_1}{\partial x^2}\right)=P^2+R^2, \tag{2.29}$$

while formulas (2.24) take the form

$$-R^1=\tilde{\varphi}(\Omega)\left(\breve{\mu}_{AA}\frac{\partial u_1}{\partial x^1}+\breve{\mu}_{AD}\frac{\partial u_2}{\partial x^2}-\frac{1-\xi_{AF}}{Z}E\right), \quad -R^2=\tilde{\varphi}(\Omega)\mu_{BB}\left(\frac{\partial u_2}{\partial x^1}+\frac{\partial u_1}{\partial x^2}\right). \tag{2.30}$$

Since the fracture process zone at the tip of a mode I crack is being considered, only the component P^1 of the vector $\boldsymbol{P}$ is nonzero (Fig. 3). The components of the stress vector at points on the lateral boundary of the plate and at points on the upper face of the crack are supposed equal to zero.

Using the constitutive equations, we express the components of the stress vector at points on the upper boundary of the fracture process zone in terms of the components of the vector $\boldsymbol{v}$. It should be taken into account that the components of the stress vector appearing in the boundary conditions are the components of opposite vectors. By the vector $\boldsymbol{v}$ is meant the vector representing the displacements of points on the upper boundary of the fracture process zone relative to points on the lower boundary of the fracture process zone.

It is obvious that

$$v_1>0, \tag{2.31}$$

and

$$v_2=v_3=0. \tag{2.32}$$

In view of (2.4) and (2.32), we have

$$\boldsymbol{v}\cdot\boldsymbol{v}=v_1v_1. \tag{2.33}$$

With (2.4), (2.31), (2.32), (1.8), and (2.33), the first equation in (1.24) becomes

$$P^1=P_o[1-\tilde{f}(v)], \tag{2.34}$$

while the second and third equations in (1.24) yield

$$P^2=P^3=0. \tag{2.35}$$

Note that

$$v_1 = 2u_1. \tag{2.36}$$

Considering (2.31), (1.8), and (2.33), we obtain

$$v = v_1. \tag{2.37}$$

To solve the boundary-value problem, we need the equations for the components u_1 and u_2. These equations follow from the condition of symmetry about the x^1- and x^2-axes:

$$u_1(x^1, -x^2) - u_1(x^1, +x^2) = 0, \quad u_2(x^1, -x^2) + u_2(x^1, +x^2) = 0,$$

$$u_1(-x^1, x^2) + u_1(+x^1, x^2) = 0, \quad u_2(-x^1, x^2) - u_2(+x^1, x^2) = 0. \tag{2.38}$$

From the symmetry about the x^2-axis it follows that

$$u_1 = 0 \tag{2.39}$$

at the end of the fracture process zone.

Let us derive the equation for the component u_2. We select a point with the coordinates a^1, a^2 at the end of the fracture process zone and assume that $u_2(x^1, x^2)$ is a real function whose partial derivatives (up to the second order) are continuous in the vicinity D of the point (a^1, a^2).

Let us expand the function (x^1, x^2) into a Taylor series about the point (a^1, a^2):

$$u_2(x^1, x^2) = u_2(a^1, a^2) + \sum_{\beta=1}^{2} \left.\frac{\partial u_2}{\partial x^\beta}\right|_{(a^1,\, a^2)} (x^\beta - a^\beta)$$

$$+ \frac{1}{2} \sum_{\beta=1}^{2} \sum_{\gamma=1}^{2} \left.\frac{\partial^2 u_2}{\partial x^\beta \partial x^\gamma}\right|_{(a^1,\, a^2)} (x^\beta - a^\beta)(x^\gamma - a^\gamma) \quad ((x^1, x^2) \in D). \tag{2.40}$$

Using (2.40) and denoting the coordinates of the end of the fracture process zone by $a^1 + \varepsilon^1$ and $a^2 + \varepsilon^2$, we obtain

$$-u_2(a^1 + \varepsilon^1, a^2 + \varepsilon^2) + u_2(a^1, a^2) + \left.\frac{\partial u_2}{\partial x^1}\right|_{(a^1,\, a^2)} \varepsilon^1 + \left.\frac{\partial u_2}{\partial x^2}\right|_{(a^1,\, a^2)} \varepsilon^2$$

$$+ \frac{1}{2}\left(\left.\frac{\partial^2 u_2}{\partial x^1 \partial x^1}\right|_{(a^1,\, a^2)} \varepsilon^1 \varepsilon^1 + 2 \left.\frac{\partial^2 u_2}{\partial x^1 \partial x^2}\right|_{(a^1,\, a^2)} \varepsilon^1 \varepsilon^2 + \left.\frac{\partial^2 u_2}{\partial x^2 \partial x^2}\right|_{(a^1,\, a^2)} \varepsilon^2 \varepsilon^2 \right) = 0. \tag{2.41}$$

Due to the symmetry about the x^2-axis, only the component P^1 of the vector $\boldsymbol{P}$ at points on the upper boundary of the fracture process zone is nonzero.

3. Addendum. For the components of the tensors ***F*** and ***G*** appearing in (2.1) as components of reciprocal tensors of the fourth rank, we have

$$F_{\alpha\beta\gamma\delta} G^{\gamma\delta\varepsilon\zeta} = \delta_\alpha^\varepsilon \delta_\beta^\zeta (\varepsilon, \zeta), \tag{3.1}$$

where the symmetrization over the indices ε and ζ is assumed.

The function $\tilde{\varphi}(\Omega)$ appearing in (2.1) has the following form [3]:

$$\tilde{\varphi}(\Omega)=\begin{cases}0, & \Omega\in[o,\upsilon],\\ \dfrac{\Omega-\upsilon-a\ln\left(\dfrac{\Omega-\upsilon}{a}+1\right)}{\Omega}, & \Omega\in[\upsilon,\psi],\end{cases} \tag{3.2}$$

where the constants υ and ψ and the coefficient a are determined experimentally.

To solve the boundary-value problem, we will define nonlinearity and failure criteria.

According to (3.2), the relation of the contravariant components of the stress tensor and the covariant components of the strain tensor expressed by (2.1) become nonlinear when Ω becomes greater than υ and the function $\tilde{\varphi}(\Omega)$ begins to increase. The nonlinearity criterion is

$$\Omega=\upsilon. \tag{3.3}$$

The fracture of the plate occurs when Ω becomes equal to ψ, and the function $\tilde{\varphi}(\Omega)$ peaks. Thus, the fracture criterion is

$$\Omega=\psi. \tag{3.4}$$

The quantity Ω can be interpreted from a physical point of view.

Note that the invariant Ε is the relative variation in the volume of the body element. Then E^2/Z is the double energy of deformation needed to change the volume of a body element. If the contravariant components of the stress tensor and the covariant components of the strain tensor are related linearly, the invariant Ξ is the double energy of deformation of the body element.

Thus, according to (2.2), Ω is the square root of the double energy needed to deform a body element without changing its volume. This interpretation of Ω coincides with Hencky's interpretation of strain intensity [4].

Features of Ω can be seen by assuming that the covariant components of the strain tensor are described by the equality

$$D_{\varepsilon\zeta}=F_{\varepsilon\zeta\iota\kappa}g^{\iota\kappa}\chi, \tag{3.5}$$

where χ is a variable.

In view of (3.5) and the first invariant in (2.3), the second invariant in (2.3) becomes

$$\mathrm{E}=\mathrm{Z}\chi. \tag{3.6}$$

With (3.5), (3.1), and the first invariant in (2.3), the third invariant in (2.3) takes the form

$$\Xi=\mathrm{Z}\chi^2. \tag{3.7}$$

Taking (3.6) and (3.7) into account, we see that $\Xi-\mathrm{E}^2/\mathrm{Z}=0$ and $\Omega=0$, according to (2.2). Then the covariant components of the strain tensor defined by (3.5) do not satisfy criteria (3.3) and (3.4). In this case, the body can neither change over into the nonlinear state nor fracture.

As established in [3], during fracturing, when Ω becomes equal to the constant ψ, the density of the energy spent to deform a body element without change in its volume is defined by the formula

$$\Psi=\upsilon\left(\psi-\frac{\upsilon}{2}\right)+a^2\left\{\left(\frac{\psi-\upsilon}{a}+1\right)\left[\ln\left(\frac{\psi-\upsilon}{a}+1\right)-1\right]+1\right\}. \tag{3.8}$$

4. Numerical Example. To solve the boundary-value problem on the fracture process zone at the tip of a mode I crack in a nonlinear elastic orthotropic material, we use the data for D16 alloy borrowed from [2].

The components of the tensor $\boldsymbol{F}$ are the following:

$$F_{1111}=0.193\cdot10^{-10}\ \mathrm{Pa}^{-1},\quad -F_{1122}=0.045\cdot10^{-10}\ \mathrm{Pa}^{-1},\quad -F_{1133}=0.049\cdot10^{-10}\ \mathrm{Pa}^{-1},$$

$$F_{1212}=0.107\cdot10^{-10}\ \mathrm{Pa}^{-1},\quad F_{1313}=0.121\cdot10^{-10}\ \mathrm{Pa}^{-1},\quad F_{2222}=0.142\cdot10^{-10}\ \mathrm{Pa}^{-1},$$

$$-F_{2233} = 0.045 \cdot 10^{-10}\ \text{Pa}^{-1}, \quad F_{2323} = 0.107 \cdot 10^{-10}\ \text{Pa}^{-1}, \quad F_{3333} = 0.193 \cdot 10^{-10}\ \text{Pa}^{-1}.$$

Using the values of the components of the tensor $\boldsymbol{F}$ and Eqs. (1.3), we calculate, by formulas (3.1), the components of the tensor $\boldsymbol{G}$:

$$G^{1111} = 6.395 \cdot 10^{10}\ \text{Pa}, \quad G^{1122} = 2.744 \cdot 10^{10}\ \text{Pa}, \quad G^{1133} = 2.263 \cdot 10^{10}\ \text{Pa},$$

$$G^{1212} = 2.336 \cdot 10^{10}\ \text{Pa}, \quad G^{1313} = 2.066 \cdot 10^{10}\ \text{Pa}, \quad G^{2222} = 8.781 \cdot 10^{10}\ \text{Pa},$$

$$G^{2233} = 2.744 \cdot 10^{10}\ \text{Pa}, \quad G^{2323} = 2.336 \cdot 10^{10}\ \text{Pa}, \quad G^{3333} = 6.395 \cdot 10^{10}\ \text{Pa}.$$

The constants υ and ψ as well as the coefficient a appearing in (3.2) and (3.8) are:

$$\upsilon = 3.25 \cdot 10^{2}\ \text{Pa}^{1/2}, \quad \psi = 93.50 \cdot 10^{2}\ \text{Pa}^{1/2}, \quad a = 1.1112866 \cdot 10^{2}\ \text{Pa}^{1/2}.$$

Using these values and formula (3.8), we find $\Psi = 645.97 \cdot 10^{4}$ Pa

Let $k_1 = 2$, $k_2 = 3$, $m = -0.2 \cdot 10^{5}\ \text{m}^{-1}$, $\eta = 50$ μm. Using these values and formula (1.21), we get:

$$b_{k_1} = 0.08 \cdot 10^{10}\ \text{m}^{-2}, \quad b_{k_2} = -0.008 \cdot 10^{15}\ \text{m}^{-3}.$$

If $v = 50$ μm, then $\tilde{f}(v) = 1$ by formula (1.23). According to (2.34), $P^1 = 0$, i.e., the state of ultimate equilibrium is achieved.

Figure 3 shows the coordinates of the origin of the fracture process zone (point A) and the end of the zone (point B) are denoted by x_f^2 and x_g^2, respectively.

Let

$$x_f^2 = 1.50\ \text{cm}, \quad x_g^2 = 1.58,\ 1.60,\ 1.62\ \text{cm}, \quad -\varepsilon^1 = \varepsilon^2 = 0.2\ \text{mm}.$$

The lengths of the crack, l_R, and fracture process zone, l_S, are given by

$$l_R = x_f^2, \quad l_S = x_g^2 - x_f^2. \tag{4.1}$$

Thus, the crack length is constant and equal to 1.5 cm, while the length of the fracture process zone, according to the second formula in (4.1), takes the values 0.8, 1.0, 1.2 mm.

The component P^1 of the vector $\boldsymbol{P}$ at points on the upper boundary of the plate segment being considered is expressed in terms of the load parameter as $P^1 = w$.

Considering that criterion (3.4) is satisfied at the point B and $\Psi = 645.97 \times 10^4$ Pa, we determine the parameter w. In solving the boundary-value problem, we considered three cases differing by the length of the fracture process zone. In every case, the parameter w was varied.

In solving the boundary-value problem (for each value of w), we supposed that the component S^{11} of the tensor $\boldsymbol{S}$ at the point B satisfies the equality

$$S_B^{11} = P_o. \tag{4.2}$$

The value of P_o was determined in several iterations. Its initial value was 1.90×10^8 Pa.

Considering Eq. (2.34) and the first equation in (2.35), the component P^1 of the vector $\boldsymbol{P}$ at points on the upper boundary of the fracture process zone was expressed in terms of P_o and $\tilde{f}(v)$. Next, using Eqs. (2.20), (2.25), (2.27), (2.29), (2.38), (2.39), and (2.41), we determined the components u_1 and u_2 by representing the partial derivatives by finite differences using a step-by-step method generalizing Il'yushin's one. In the first approximation, $\tilde{\varphi}(\Omega) = 0$. Note that according to (2.21), (2.26), (2.28), and (2.30), $Q^1, Q^2 = 0$ and $R^1, R^2 = 0$. Moreover, it was assumed that $\tilde{f}(v) = 0$ in the first approximation. In each

TABLE 1

l_S, cm	$w \cdot 10^{-7}$, Pa
8	5.846375
10	5.903761
12	5.909860

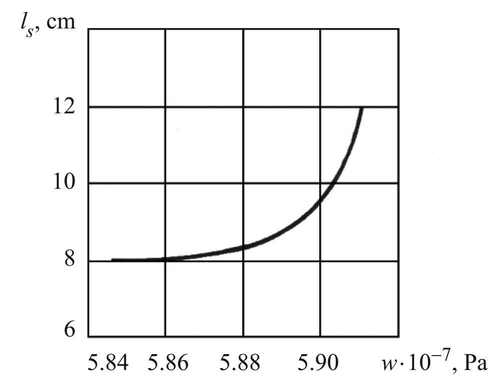


Fig. 4

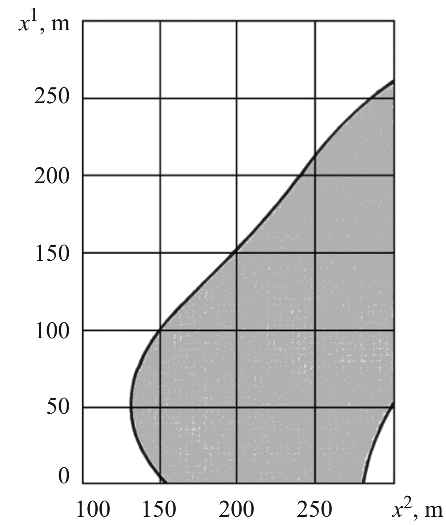


Fig. 5

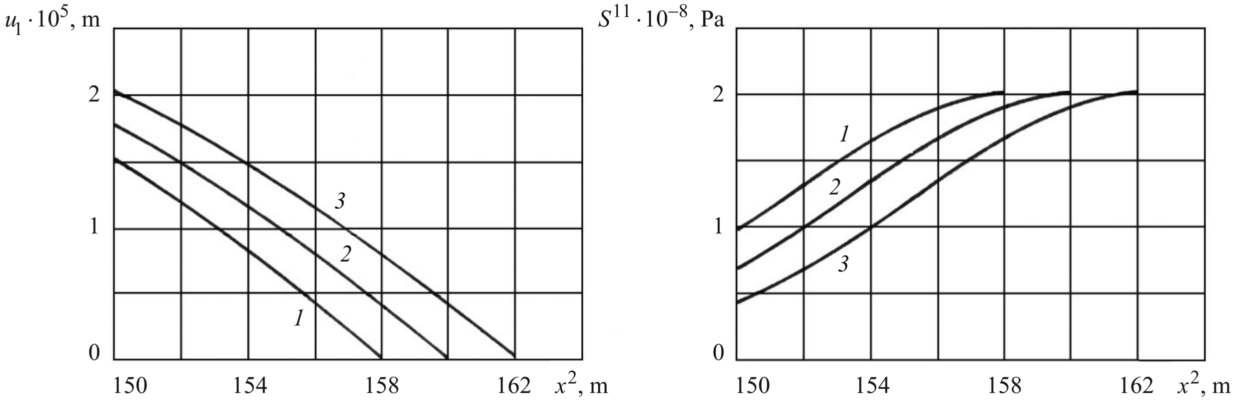
successive approximation, out of 89 approximations, the values of $\tilde{\varphi}(\Omega)$, $\tilde{f}(v)$, Q^1, Q^2, and R^1, R^2 were found from the values of u_1 and u_2 obtained in the previous approximation using formulas (3.2) and (2.2), the first invariant in (2.3), invariants (2.10) and (2.13), and formulas (2.14), (2.21), (2.26), (2.28), (2.30), (1.23), (2.37), and (2.36). Next, using the first equation in (2.15), we calculated the component S^{11} of the tensor $\boldsymbol{S}$ at the point B. If it did not satisfied Eq. (4.2), the value of P_o was corrected, and the whole procedure was repeated.

Using formula (2.2), the first invariant in (2.3), and invariants (2.10) and (2.13), we tested criterion (3.4) at the point B. If it was not met, the parameter w was changed.

5. Analysis of the Results. By solving the boundary-value problem, we determined the load on the plate (parameter w) for different lengths of the fracture process zone. The values of w are presented in Table 1, from which we can see that the length of the fracture process zone and the crack opening displacement at the tip (point A) increase with the load. In other words, the plate tends to the state of ultimate equilibrium in which the modulus v of the vector $\boldsymbol{v}$ takes the value η, while the modulus P of the vector $\boldsymbol{P}$ becomes equal to zero (at the same point A). As the load increases, the length of the fracture process zone varies more intensively. Figure 4 demonstrates it convincingly. The crack opening displacement at the tip varies in the same manner.

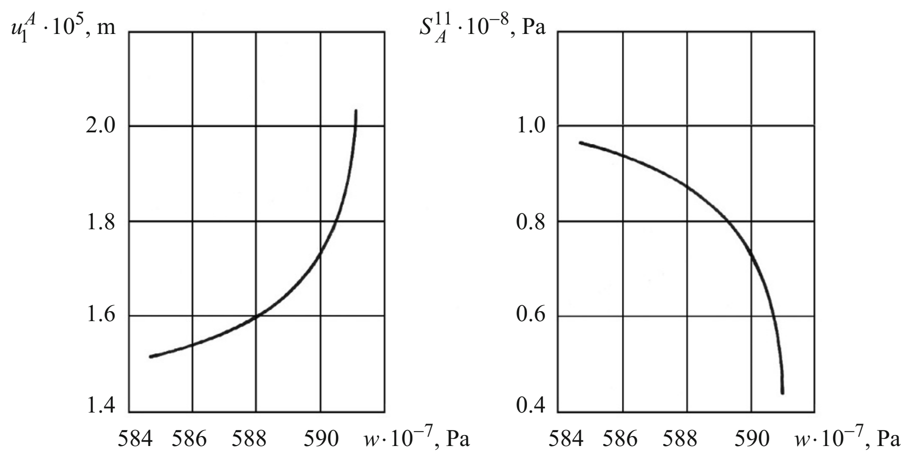
Considering formula (2.2), invariants (2.10) and (2.13), and expression (2.14), we identified the points at which criterion (3.3) is satisfied for every value of the load. The lines drawn through these points represent the boundaries of the nonlinear zone.

Figure 5 shows, as an example, the nonlinear zone (grayed) for a load of $5.9098 \cdot 10^7$ Pa. The nonlinear zone formed at the crack tip with $x^1 = 0$ and $x^2 = 1.5$ cm reached the lateral boundary of the plate part that is as a distance of 3.0 cm from the x^1-axis. This was not so only on the small segment of the boundary opposite to the crack.



a *b*

Fig. 6



a *b*

Fig. 7

As the load is increased, the nonlinear zone expands. Indeed, it extends along the lateral boundary of the plate segment from $x^1 = 0.56$ cm to $x^1 = 2.54$ cm at $w = 5.8463 \cdot 10^7$ Pa and from $x^1 = 0.5$ cm to $x^1 = 2.64$ cm at $w = 5.9098 \cdot 10^7$ Pa. The moderate extension of the nonlinear zone is natural because the load is increased insignificantly.

Figure 6 demonstrates the evolution of the fracture process zone during the loading of the plate. Curves *1*, *2*, *3* correspond to the following lengths of the fracture process zone, respectively: 8, 10, 12 cm. Figure 6*a* shows how the displacement (along the x^1-axis) of the upper boundary of the fracture process zone increases with its length.

Interestingly, the component S^{11} of the tensor $\boldsymbol{S}$ at the point *B* weakly depends on the length of the fracture process zone. Indeed, curves *1*, *2*, *3* in Fig. 6*b* correspond to $2.0026 \cdot 10^8$, $2.0046 \cdot 10^8$, $2.0087 \cdot 10^8$ Pa, respectively. In accordance with the results obtained, S_B^{11} can be considered equal to $2.00 \cdot 10^8$ Pa, irrespectively of the length of the fracture process zone. In view of (4.2), $P_o = 2.00 \cdot 10^8$ Pa.

Figures 7*a* and 7*b* demonstrate the dependence of the component u_1 of the vector $\boldsymbol{u}$ and of the component S^{11} of the tensor $\boldsymbol{S}$, respectively, at the point *A* on the parameter *w*. As can be seen, u_1^A increases sharply while S_A^{11} sharply decreases with increasing *w*.

For each length of the fracture process zone, we calculated the components D_{11}, D_{22}, D_{33} of the tensor $\boldsymbol{D}$ at points near the point *B* using formulas (2.5) and (2.14). Of especial interest are the values of the components at the point *B* (Table 2). Note that the increase in the length of the fracture process zone from 8 to 12 cm causes D_{11}^B to increase from $2.0235 \cdot 10^{-2}$ to

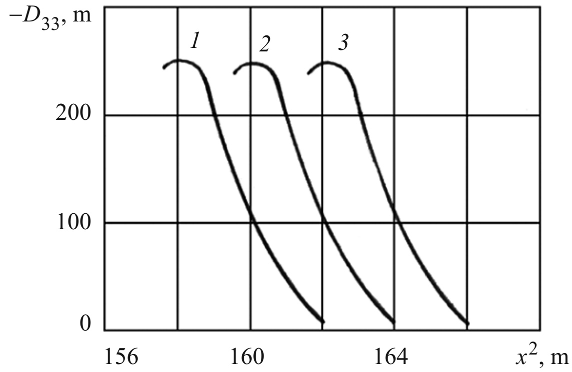


Fig. 8

TABLE 2

l_S, cm	$D_{11}^B \cdot 10^2$	$D_{22}^B \cdot 10^2$	$D_{33}^B \cdot 10^2$
8	2.0235	1.8232	–3.5356
10	2.0281	1.8190	–3.5359
12	2.0311	1.8170	–3.5364

$2.0311 \cdot 10^{-2}$ and D_{22}^B to decrease from $1.8232 \cdot 10^{-2}$ to $1.8170 \cdot 10^{-2}$. In addition, the value of D_{33}^B changed less decreasing from $-3.5356 \cdot 10^{-2}$ to $-3.5364 \cdot 10^{-3}$. It is interesting that the D_{33}^B did not change noticeably despite the substantial increase in the length of the fracture process zone.

The component D_{33} of the tensor $\boldsymbol{D}$ at the point B has the minimum value (Fig. 8). Curves *1*, *2*, *3* correspond to the following lengths of the fracture process zone, respectively: 8, 10, 12 cm. Thus, having determined experimentally the component D_{33} of the tensor $\boldsymbol{D}$ at various points around the crack tip, we can find the point B and, as a result, determine the length of the fracture process zone.

It is very important to establish how the crack opening displacement at the tip depends on the load ($v^A = v^A(w)$). The values of v^A for different values of w are collected in Table 3.

Extrapolating the dependence $v^A = v^A(w)$, we can find the critical value of w at which the plate changes over to the state of ultimate equilibrium.

Let the dependence $v^A = v^A(w)$ be exponential:

$$v^A = b + c\exp(\alpha w) \tag{5.1}$$

and

$$v^A \Big|_{w=w_o, w_p, w_q} = v_o^A,\ v_p^A,\ v_q^A. \tag{5.2}$$

According to the first condition in (5.2), formula (5.1) becomes

$$v_o^A = b + c\exp(\alpha w_o)$$

whence the coefficient c follows:

TABLE 3

$w \cdot 10^{-7}$, Pa	$v^A \cdot 10^5$, m
5.846375	3.043642
5.903761	3.590334
5.909860	4.062284

$$c = \frac{v_o^A - b}{\exp(\alpha w_o)}. \tag{5.3}$$

Substituting (5.3) into (5.1), we obtain

$$v^A = b + (v_o^A - b)\exp[\alpha(w - w_o)]. \tag{5.4}$$

Let us represent (5.4) as follows:

$$\alpha(w - w_o) = \ln\frac{v^A - b}{v_o^A - b}. \tag{5.5}$$

With (5.5), we have

$$\alpha = \frac{1}{w - w_o}\ln\frac{v^A - b}{v_o^A - b}. \tag{5.6}$$

Considering the second and third conditions from (5.2) and using (5.6), we derive the equation for the constant b:

$$\frac{1}{w_p - w_o}\ln\frac{v_p^A - b}{v_o^A - b} = \frac{1}{w_q - w_o}\ln\frac{v_q^A - b}{v_o^A - b}. \tag{5.7}$$

Solving it, we get $b = 3.042052 \cdot 10^{-5}$ m.

Setting $w = w_p$ and $v^A = v_p^A$, we calculate the coefficient α by formula (5.6): $\alpha = 101.8201 \cdot 10^{-7}$ Pa^{-1}. Then, from (5.6) we have

$$w = w_o + \frac{1}{\alpha}\ln\frac{v^A - b}{v_o^A - b}. \tag{5.8}$$

Since $v^A = 50$ μm for the state of ultimate equilibrium according to criterion (1.25), we calculate the critical value of w by formula (5.8). It is equal to $5.916262 \cdot 10^7$ Pa, which slightly exceeds the last value of w in Table 3.

It can be seen from Fig. 9 plotted from the results obtained, the crack tip opening displacement sharply increases with the load.

Let us analyze the derivative dv^A / dw of the function $v^A = v^A(w)$. Differentiating (5.4), we get

$$\frac{dv^A}{dw} = (v_o^A - b)\alpha\exp[\alpha(w - w_o)]. \tag{5.9}$$

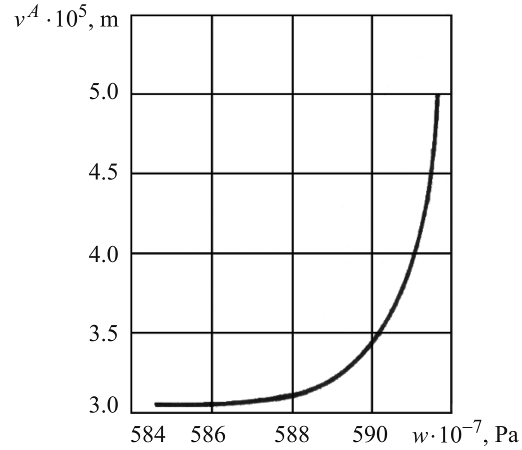


Fig. 9

Formula (5.9) shows that the derivative dv^A / dw sharply increases with w. Indeed, it is equal to $0.16 \cdot 10^{-12}$ m/Pa at $w = 5.846375 \cdot 10^7$ Pa, to $55.82 \cdot 10^{-12}$ m/Pa at $w = 5.903761 \cdot 10^7$ Pa, and to $103.87 \cdot 10^{-12}$ m/Pa at $w = 5.909860 \cdot 10^7$ Pa. At the critical value $5.916262 \cdot 10^7$ Pa, however, the derivative dv^A / dw is equal to $199.35 \cdot 10^{-12}$ m/Pa.

Conclusions. A body with a crack with a fracture process zone at its front has been studied. The constitutive equations relating the components of the stress vectors at points on the opposite boundaries of the fracture process zone and the components of the displacement vector of these points relative to each other have been derived. The local fracture criterion has been formulated. The boundary-value equilibrium problem for a plate made of a nonlinear elastic orthotropic material and having a mode I crack has been stated for the components of the displacement vector. The numerical solution of the problem shows how the fracture process zone develops in the plate under loading. Features of the deformation field near the end of the fracture process zone have been established and the critical load initiating crack growth has been determined.

REFERENCES

1. O. S. Bogdanova, A. A. Kaminsky, and E. E. Kurchakov, "Fracture process zone at the front of a crack in a solid," *Dop. NAN Ukrainy*, No. 5, 25–33 (2017).
2. E. E. Kurchakov, "Stress–strain relations for nonlinear anisotropic medium," *Int. Appl. Mech.*, **15**, No. 9, 803–807 (1979).
3. E. E. Kurchakov, "Thermodynamic validation of the constitutive equations for a nonlinear anisotropic body," *Dop. NAN Ukrainy*, No. 5, 46–53 (2015).
4. H. Hencky, "Development and modern state of plasticity theory," *Prikl. Mat. Mekh.*, **4**, No. 3, 31–36 (1940).
5. L. Banks-Sills, N. Travitzky, D. Ashkenazi, and R. Eliasi, "A methodology for measuring interface fracture properties of composite materials," *Int. J. Fract.*, **99**, No. 3, 143–160 (1999).
6. A. A. Kaminsky and O. S. Bogdanova, "Long-term crack-resistance of orthotropic viscoelastic plate under biaxial loading," *Int. Appl. Mech.*, **31**, No. 9, 747–753 (1995).
7. A. A. Kaminsky and E. E. Kurchakov, "Influence of tension along a mode I crack in an elastic body on the formation of a nonlinear zone," *Int. Appl. Mech.*, **51**, No. 2, 130–148 (2015).
8. A. A. Kaminsky and M. F. Selivanov, "On modeling of subcritical crack growth in viscoelastic body under point forces," *Int. Appl. Mech.*, **53**, No. 5, 538–544 (2017).
9. E. E. Kurchakov, "Experimental study of the plastic zone at the front of a mode I crack," *Int. Appl. Mech.*, **54**, No. 2, 213–219 (2018).
10. A. Love, *Treatise on the Mathematical Theory of Elasticity*, The University Press, Cambridge (1927).

11. A. Needleman, “A continuum model for void nucleation by inclusion debonding,” *J. Appl. Mech.*, **54**, No. 3, 525–531 (1987).
12. K. Park and G. H. Paulino, “Cohesive zone models: A critical review of traction-separation relationship across fracture surfaces,” *Appl. Mech. Rev.*, **64**, No. 11, 1–20 (2011).
13. G. N. Savin, and A. A. Kaminsky, “The growth of cracks during the failure of hard polymers,” *Sov. Appl. Mech.*, **3**, No. 9, 22–25 (1967).
14. V. Tvergaard and J. W. Hutchinson, “The influence of plasticity on mixed mode interface toughness,” *J. Mech. Phys. Solids*, **41**, No. 6, 1119–1135 (1993).
15. A. A. Wells, “Critical tip opening displacement as fracture criterion,” in: *Proc. Crack. Propagation Symp.*, 1, Granfield (1961), pp. 210–221.
16. F. H. Wittmann, K. Rokugo, E. Bruehwiler, H. Mihashi, and P. Simonin, “Fracture energy and strain softening of concrete as determined by means of compact tension specimens,” *Mater. Struct.*, **21**, No. 1, 21–32 (1988).