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VARIANT OF THE NONLINEAR WAVE EQUATIONS DESCRIBING CYLINDRICAL AXISYMMETRICAL WAVES

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New nonlinear wave equations are derived using the Murnaghan five-constant elastic potential. A feature of these equations is the following two assumptions: the elastic deformation process is physically nonlinear (geometric nonlinearity is neglected) and the deformation is geometrically axisymmetric and described by cylindrical coordinates. Therefore, the system of wave equations contains only two coupled equations. Such a statement allows us to use these new equations to analyze surface waves propagating along a circular cylindrical cavity in an elastic medium. Another feature of the nonlinear equations is that every equation includes the classical linear part. The nonlinear terms of the equations are quadratically nonlinear and contain twenty-three types of nonlinearities in the first equation and twenty-two types in the second equation.

Keywords: Murnaghan's potential, physical nonlinearity, cylindrical surface wave, nonlinear wave equations

Introduction. Cylindrical harmonic waves are the subject of fairly long research in the theory of waves [2, 4], which can hardly be considered complete. Such waves have been studied theoretically, experimentally, and appliedly [7–9, 20–22]. The transition from the linear model describing cylindrical waves to different nonlinear models has revealed a number of problems in the analytical description and experimental observations of the waves.

Numerous types of nonlinear wave equations corresponding to the Murnaghan five-constant model were described in [16, 17]. When analyzing a surface wave propagating along a circular cylindrical cavity, we need nonlinear wave equations for cylindrical axisymmetric waves. Four configurations are distinguished in the problem statement on waves described by cylindrical coordinates [13–15, 17]. However, in the case of configuration II describing such surface wave, the system of wave equations can only obtained for one particular case. This is the case where only the geometric nonlinearity is taken into account, and the Murnaghan model is reduced to the simplest nonlinear model (Jon model or neo-Hookean model). Therefore, it is necessary to considering the other cases, including the case where only physical nonlinearity is considered, and which is more typical of materials studied using the Murnaghan model. This model is known to describe weak physical nonlinearity for small strains. The consistent analysis of this case requires some general information and formulas.

1. Problem Statement. Select the initial state (configuration) of a continuum characterized by cylindrical coordinates $\theta^1 = r, \theta^2 = \vartheta, \theta^3 = z$, and restrict ourselves to the case where the state is symmetric about the symmetry axis Oz. Then, only two coordinates and two displacements are necessary to describe this state $(u_r(r, z, t), 0, u_z(r, z, t)) u^1 = u_1 = u_r, u^2 = u_2 = u_\vartheta = 0$, $u^3 = u_3 = u_z$. Denote this configuration by *As*. Using the general expressions for the Christoffel symbols $\Gamma^1_{22} = -r$, $\Gamma^2_{12} = \Gamma^2_{21} = (1/r)$ in the configuration *As* to evaluate the components of the strain tensor $\varepsilon_{ij}(r, z, t)$, the covariant derivatives of the covariant and contravariant components of the displacement vector [1, 3, 5]

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$$\varepsilon_{ij} = (1/2)(\nabla_i u_j + \nabla_j u_i + \nabla_i u_k \nabla_j u^k),$$

$$\nabla_i u^k = (\partial u^k / \partial \theta^i) + u^j \Gamma^k_{ji},$$

$$\nabla_i u_j = (\partial u_j / \partial \theta^i) - u_k \Gamma^k_{ji},$$

we get

$$\nabla_1 u_1 = u_{1,1} - u_1 \cancel{\Gamma^1_{11}} - u_2 \cancel{\Gamma^2_{11}} - u_3 \cancel{\Gamma^3_{11}} = u_{r,r}, \quad \nabla_1 u^1 = u^1_{,1} = u^1 \cancel{\Gamma^1_{11}} + u^2 \cancel{\Gamma^2_{11}} + u^3 \cancel{\Gamma^3_{11}} = u_{r,r},$$

$$\nabla_2 u_2 = -u_1 \Gamma^1_{22} - u_2 \cancel{\Gamma^2_{22}} - u_3 \cancel{\Gamma^3_{22}} = r u_r, \quad \nabla_2 u^2 = u^1 \Gamma^2_{12} + \cancel{u^2 \Gamma^2_{22} + u^3 \Gamma^2_{32}} = (1/r) u_r,$$

$$\nabla_3 u_3 = u_{3,3} - u_1 u^1 \cancel{\Gamma^1_{33}} - u_2 \cancel{\Gamma^2_{33}} - u_3 \cancel{\Gamma^3_{33}} = u_{z,z}, \quad \nabla_3 u^3 = u^3_{,3} + u^1 \cancel{\Gamma^1_{33}} + u^2 \cancel{\Gamma^2_{33}} + u^3 \cancel{\Gamma^3_{33}} = u_{z,z},$$

$$\nabla_3 u_1 = u_{1,3} + u_1 \cancel{\Gamma^1_{31}} + u_2 \cancel{\Gamma^2_{31}} + u_3 \cancel{\Gamma^3_{31}} = u_{r,z}, \quad \nabla_3 u^1 = u^1_{,3} + u^1 \cancel{\Gamma^1_{11}} + u^2 \cancel{\Gamma^2_{11}} + u^3 \cancel{\Gamma^3_{11}} = u_{r,z}, \tag{1}$$

$$\varepsilon_{11} = \nabla_1 u_1 + (1/2) \nabla_1 u_1 \nabla_1 u^1 + (1/2) \nabla_1 u_2 \nabla_1 u^2 + (1/2) \nabla_1 u_3 \nabla_1 u^3 \to$$

$$\to \varepsilon_{rr} = u_{r,r} + (1/2)(u_{r,r})^2 + (1/2)(u_{z,r})^2,$$

$$\varepsilon_{22} = \nabla_2 u_2 + (1/2)(\nabla_2 u_1 \nabla_2 u^1 + \nabla_2 u_2 \nabla_2 u^2 + \nabla_2 u_3 \nabla_2 u^3) \to \varepsilon_{\vartheta\vartheta} = (1/r) u_r + (1/2r^2)(u_r)^2,$$

$$\varepsilon_{33} = \nabla_3 u_3 + (1/2) \nabla_3 u_1 \nabla_3 u^1 + (1/2) \nabla_1 u_3 \nabla_3 u^3 \to \varepsilon_{zz} = u_{z,z} + (1/2)(u_{z,z})^2 + (1/2)(u_{r,z})^2,$$

$$\varepsilon_{13} = (1/2)(\nabla_1 u_3 + \nabla_3 u_1 + \nabla_1 u_1 \nabla_3 u^1 + \nabla_1 u_3 \nabla_3 u^3) \to \varepsilon_{rz} = (1/2)(u_{z,r} + u_{r,z} + u_{r,r} u_{r,z} + u_{z,r} u_{z,z}),$$

$$\varepsilon_{r\vartheta} = \varepsilon_{\vartheta z} = 0. \tag{2}$$

Let the geometric nonlinearity in the description of the strain tensor is not taken into account by means of the Cauchy linear relations:

$$\varepsilon_{rr} = u_{r,r}, \quad \varepsilon_{\vartheta\vartheta} = (1/r) u_r, \quad \varepsilon_{zz} = u_{z,z},$$

$$\varepsilon_{r\vartheta} = 0, \quad \varepsilon_{rz} = (1/2)(u_{z,r} + u_{r,z}), \quad \varepsilon_{r\vartheta} = \varepsilon_{\vartheta z} = 0 \tag{3}$$

and the physical nonlinearity is taken into account by means of the Murnaghan potential [10]:

$$W(I_1, I_2, I_3) = (1/2)\lambda I_1^2 + \mu I_2 + (1/3) A I_3 + B I_1 I_2 + (1/3) C I_1^3. \tag{4}$$

It is necessary to derive the constitutive equations corresponding to (2), (3), (4).

2. Constitutive Equations. The first step is to represent potential (4) in terms of strains. To this end, we need the general expression of the invariants in terms of strains using formulas (3):

$$I_1(\varepsilon_{ik}) = \varepsilon_{ik} g^{ik} = \varepsilon_{11} \cdot 1 + (1/r^2) \varepsilon_{22} \cdot 1 + \varepsilon_{33} \cdot 1,$$

$$I_2(\varepsilon_{ik}) = \varepsilon_{im} \varepsilon_{nk} g^{ik} g^{nm} = (\varepsilon_{11} \cdot 1)^2 + ((1/r^2) \varepsilon_{22} \cdot 1)^2 + (\varepsilon_{33} \cdot 1)^2 + 2 \cdot (\varepsilon_{13})^2,$$

$$I_3(\varepsilon_{ik}) = \varepsilon_{pm} \varepsilon_{in} \varepsilon_{kq} g^{im} g^{pq} g^{kn} = (\varepsilon_{11})^3 + ((1/r^2) \varepsilon_{22})^3 + (\varepsilon_{33})^3 + 3 \cdot (\varepsilon_{13})^2 (\varepsilon_{11} + \varepsilon_{33}) \tag{5}$$

or

$$I_1 = \varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{zz}, \quad I_2 = \varepsilon^2_{rr} + \varepsilon^2_{\vartheta\vartheta} + \varepsilon^2_{zz} + \varepsilon^2_{rz} + \varepsilon^2_{zr},$$

$$I_3 = \varepsilon_{rr}^3 + \varepsilon_{\vartheta\vartheta}^3 + \varepsilon_{zz}^3 + 3(\varepsilon_{rz}^2)\cdot(\varepsilon_{rr} + \varepsilon_{zz}). \tag{6}$$

Substituting invariants (6) into potential (4), we obtain

$$W = (1/2)\lambda(\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{zz})^2 + \mu(\varepsilon_{rr}^2 + \varepsilon_{\vartheta\vartheta}^2 + \varepsilon_{zz}^2 + \varepsilon_{rz}^2 + \varepsilon_{zr}^2) + (1/3)C(\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{zz})^3$$

$$+B(\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{zz})(\varepsilon_{rr}^2 + \varepsilon_{\vartheta\vartheta}^2 + \varepsilon_{zz}^2 + \varepsilon_{rz}^2 + \varepsilon_{zr}^2)$$

$$+(1/3)A(\varepsilon_{rr}^3 + \varepsilon_{\vartheta\vartheta}^3 + \varepsilon_{zz}^3 + \varepsilon_{rr}(\varepsilon_{rz}^2 + \varepsilon_{rz}\varepsilon_{zr} + \varepsilon_{zr}^2) + \varepsilon_{zz}(\varepsilon_{rz}^2 + \varepsilon_{rz}\varepsilon_{zr} + \varepsilon_{zr}^2)). \tag{7}$$

The components of the Lagrange symmetric stress tensor are determined by the formulas $\sigma_{rr} = \partial W / \partial\varepsilon_{rr}, \ldots$, $\sigma_{rz} = \partial W / \partial\varepsilon_{rz}$ under the condition that the stress and strain tensors are represented as if they are asymmetric:

$$\sigma_{rr} = \lambda(\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{zz}) + 2\mu\varepsilon_{rr} + (A + 3B + C)\varepsilon_{rr}^2 + (A + 2B)\varepsilon_{rz}^2$$

$$+(B + C)\left[2\varepsilon_{rr}(\varepsilon_{\vartheta\vartheta} + \varepsilon_{zz}) + \varepsilon_{\vartheta\vartheta}^2 + \varepsilon_{zz}^2\right] + 2C\varepsilon_{\vartheta\vartheta}\varepsilon_{zz},$$

$$\sigma_{\vartheta\vartheta} = \lambda(\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{zz}) + 2\mu\varepsilon_{\vartheta\vartheta} + (A + 3B + C)\varepsilon_{\vartheta\vartheta}^2 + 2B\,\varepsilon_{rz}^2$$

$$+(B + C)(2\varepsilon_{\vartheta\vartheta}(\varepsilon_{rr} + \varepsilon_{zz}) + \varepsilon_{rr}^2 + \varepsilon_{zz}^2) + 2C\varepsilon_{rr}\varepsilon_{zz},$$

$$\sigma_{zz} = \lambda(\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{zz}) + 2\mu\varepsilon_{zz} + (A + 3B + C)\varepsilon_{zz}^2 + (A + 2B)\varepsilon_{rz}^2$$

$$+(B + C)[2\varepsilon_{zz}(\varepsilon_{\vartheta\vartheta} + \varepsilon_{rr}) + \varepsilon_{\vartheta\vartheta}^2 + \varepsilon_{rr}^2] + 2C\,\varepsilon_{\vartheta\vartheta}\varepsilon_{rr},$$

$$\sigma_{rz} = 4\mu\varepsilon_{rz} + 2(A + 2B)\varepsilon_{rz}(\varepsilon_{rr} + \varepsilon_{zz}). \tag{8}$$

The constitutive equations $\sigma \sim \varepsilon$ (8) can be represented as the dependence of the components of the stress tensor on the components of the displacement vector $\sigma \sim u$ (the number of types of nonlinearities is indicated in brackets):

$$\sigma_{rr} = \lambda(u_{r,r} + (1/r)u_r + u_{z,z}) + 2\mu u_{r,r} + (A + 3B + C)(u_{r,r})^2 + (1/4)(A + 2B)(u_{z,r} + u_{r,z})^2$$

$$+(B + C)\left[2u_{r,r}((1/r)u_r + u_{z,z}) + (1/r^2)(u_r)^2 + (u_{z,z})^2\right] + 2C(1/r)u_r u_{z,z},$$

$$\sigma_{\vartheta\vartheta} = (\lambda + 2\mu)(1/r)u_r + \lambda(u_{r,r} + u_{z,z}) + (A + 3B + C)(1/r^2)(u_r)^2$$

$$+(1/2)B(u_{z,r} + u_{r,z})^2 + (B + C)(2(1/r)u_r(u_{r,r} + u_{z,z}) + u_{r,r}^2 + u_{z,z}^2) + 2Cu_{r,r}u_{z,z},$$

$$\sigma_{zz} = (\lambda + 2\mu)u_{z,z} + \lambda(u_{r,r} + (1/r)u_r) + (A + 3B + C)(u_{z,z})^2 + (A + 2B)(1/4)(u_{z,r} + u_{r,z})^2$$

$$+(B + C)\left[2u_{z,z}((1/r)u_r + u_{r,r}) + (1/r^2)(u_r)^2 + (u_{r,r})^2\right] + 2C(1/r)u_r u_{r,r} \tag{9}$$

(9 types for three components from the top)

$$\sigma_{rz} = 2\mu(u_{z,r} + u_{r,z}) + (A + 2B)(u_{r,r} + u_{z,z})(u_{z,r} + u_{r,z}) + 2B(1/r)u_r(u_{z,r} + u_{r,z})$$

(6 types).

3. Equations of Motion and Nonlinear Wave Equations. The general equations of motion without external forces have the following form [1–3]:

$$\nabla_k[\sigma^{ki}(\delta_i^n + \nabla_i u^n)] - \rho\ddot{u}^n = 0. \tag{10}$$

It should be noted that the expression in parentheses is equal to 1 in the linear theory. If the approach is nonlinear, then the equations of motion (10) contain nonlinear components, namely, products of components of the stress tensor, which are nonlinearly dependent on the displacements in nonlinear theory, and the components of the displacement gradient.

Due to the axial symmetry of the configuration, the indices i, k in (10) do not take the value 2, since we use the cylindrical coordinate system $\theta_1 = r, \theta_2 = \vartheta, \theta_3 = z$. Therefore, it is possible to represent (10) as only two equations ($k = 1, 3$):

$$\nabla_1[\sigma^{1i}(\delta_i^1 + \nabla_i u^1)] + \nabla_3[\sigma^{3i}(\delta_i^1 + \nabla_i u^1)] - \rho\ddot{u}^1 = 0,$$

$$\nabla_1[\sigma^{1i}(\delta_i^3 + \nabla_i u^3)] + \nabla_3[\sigma^{3i}(\delta_i^3 + \nabla_i u^3)] - \rho\ddot{u}^3 = 0. \quad (11)$$

Let us further use the following general formula [1–3]:

$$\nabla_k \sigma^{ik} = (\partial\sigma^{ik} / \partial\theta^k) + \sigma^{pk}\Gamma_{pk}^i + \sigma^{ip}\Gamma_{kp}^k,$$

or

$$\nabla_k \sigma^{ik} = (1/\sqrt{g})(\partial(\sqrt{g}\sigma^{ik}) / \partial\theta^k) + \sigma^{pk}\Gamma_{pk}^i. \quad (12)$$

The following relations are valid for the metric tensor in circular cylindrical coordinates $g = g^{11} + g^{22} + g^{33} = 1 + r^2 + 1 = r^2$. Then formula (12) is simplified as applied to Eqs. (11):

$$\nabla_k \sigma^{ik} = (1/\vartheta^1)(\partial(\vartheta^1\sigma^{ik}) / \partial\theta^k) + \sigma^{pk}\Gamma_{pk}^i.$$

Finally, Eqs. (11) take a more specific form for the configuration As:

$$\sigma_{rr,r} + \sigma_{rz,z} + (1/r)(\sigma_{rr} - \sigma_{\vartheta\vartheta}) - \rho\ddot{u}_r = -\sigma_{rr,r}u_{r,r} - \sigma_{zr,r}u_{r,z} - \sigma_{rz,z}u_{r,r} - \sigma_{zz,z}u_{r,z}$$
$$-(1/r)\sigma_{rr}u_{r,r} + (1/r)\sigma_{\vartheta\vartheta}u_{r,r} - (1/r)\sigma_{zr}u_{r,z} - \sigma_{rr}u_{r,rr} - 2\sigma_{rz}u_{r,zr} - \sigma_{zz}u_{r,zz} = 0. \quad (13)$$

$$\sigma_{zr,r} + (1/r)\sigma_{zr} + \sigma_{zz,z} - \rho\ddot{u}^3 = -\sigma_{rr,r}u_{z,r} - \sigma_{zr,r}u_{z,z} - \sigma_{rz,z}u_{z,r} - \sigma_{zz,z}u_{z,z}$$
$$-(1/r)\sigma_{rr}u_{z,r} + (1/r)\sigma_{\vartheta\vartheta}u_{z,r} - (1/r)\sigma_{zr}u_{z,z} - \sigma_{rr}u_{z,rr} - 2\sigma_{rz}u_{3,31} - \sigma_{zz}u_{z,zz} = 0. \quad (14)$$

A feature of Eqs. (13) and (14) is that upon substituting the expressions of stresses in terms of displacements into them, the nonlinearity of the resulting equations will be determined by both the nonlinearity of the left-hand sides (4 terms) and the nonlinearity of the right-hand sides (10 terms in Eqs. (13), (14)). Here the nonlinearity of the left-hand sides is only determined by the nonlinear components of the stress tensor, is of the second order (quadratic nonlinearity), and the coefficients of each type of nonlinearity are linearly dependent only on the Murnaghan constants.

The nonlinearity of the right-hand sides includes the second and third orders due to the quadratic nonlinearity of both right-hand sides and the components of the stress tensor, and the coefficients at every type of quadratic nonlinearity depend linearly only on the Lame elastic constants, and for cubic nonlinearity, they depend only on the Murnaghan elastic constants.

Substituting (9) into Eqs. (13) and (14), we obtain nonlinear wave equations in which the classical linear terms form the left-hand sides, and the quadratic and cubic nonlinear terms form the right-hand sides. We will now show nonlinear wave equations that contain only quadratic nonlinear terms.

$$(\lambda + 2\mu)[u_{r,rr} + (1/r)u_{r,r} - (1/r^2)u_r + u_{z,rz}] + \mu(u_{r,zz} - u_{z,rz}) - \rho u_{r,tt}$$
$$= -2[\lambda + 2\mu + A + 3B + C]u_{r,rr}u_{r,r} - [\lambda + 2\mu + (1/2)A + B]u_{r,zz}u_{z,z}$$
$$-[\lambda + 2\mu + (1/2)A + B]u_{z,zz}u_{r,z} - [(\lambda + 3\mu) + (1/2)A + B]u_{r,zr}u_{r,z}$$
$$-[\mu + (1/4)(A + 2B)]u_{z,rr}u_{r,z} - [\lambda + \mu + (1/2)A + B]u_{r,zz}u_{r,r}$$
$$-[\lambda + \mu + (1/2)A + 3B + 2C]u_{z,rz}u_{r,r} - [2\mu + (3/4)(A + 2B)]u_{r,zr}u_{z,r}$$

$$-[\lambda+2(B+C)]u_{r,rr}u_{z,z}-(1/2)(A+2B)u_{z,rr}u_{z,r}-((1/2)A+3B+2C)u_{z,zr}u_{z,z}$$

$$-[(1/2)A+B]u_{z,zz}u_{z,r}+(A+4B+2C)(1/r^3)u_r^2+2C(1/r^2)u_{z,z}u_r$$

$$-(\lambda+B)(1/r)u_{r,zz}u_r-[\lambda+2(B+C)](1/r)u_{r,rr}u_r-(\lambda+2\mu)(1/r^2)u_{r,r}u_r$$

$$-(B+2C)(1/r)u_{z,zr}u_r-[\lambda+\mu+(1/4)A-B](1/r)\boxed{u_{r,z}^2}$$

$$-2[\lambda+B+C](1/r)\boxed{u_{r,r}u_{z,z}}-[3\lambda+2\mu+A+4B+2C](1/r)\boxed{u_{r,r}^2}-(1/r)[\mu+(1/2)A-B]\boxed{u_{r,z}u_{z,r}}-A(1/4r)\boxed{u_{z,r}^2}=0, \quad (15)$$

$$(\lambda+2\mu)\Big[u_{r,rz}+(1/r)u_{r,z}+u_{z,zz}\Big]-\mu\Big[(1/r)(u_{r,z}-u_{z,r})+(u_{r,rz}-u_{z,rr})\Big]-\rho u_{z,tt}$$

$$=-2(\lambda+2\mu+A+3B+C)u_{z,zz}u_{z,z}-\big[\lambda+2(B+C)\big]u_{z,zz}u_{r,r}-(\lambda+2\mu+(1/2)A+B)u_{z,rr}u_{r,r}$$

$$-(\lambda+\mu+(1/2)A+B)u_{z,rr}u_{z,z}-\big[\lambda+2\mu+(3/2)(A+2B)\big]u_{z,rz}u_{z,r}$$

$$-\big[3\mu+(3/2)(A+2B)\big]u_{z,rz}u_{r,z}-((1/2)A+3B+2C)u_{r,zr}u_{r,r}$$

$$-(\lambda+\mu+(1/2)A+3B+2C)u_{r,zr}u_{z,z}-\big[\lambda+2\mu+(1/2)A+B\big]u_{r,rr}u_{z,r}$$

$$-\big[\mu+(1/2)(A+2B)\big]u_{r,zz}u_{z,r}-((1/2)A+B)u_{r,rr}u_{r,z}-(1/2)u_{r,zz}u_{r,z}$$

$$-2(B+C)(1/r^2)u_{r,z}u_r+(\lambda+2\mu)(1/r^2)u_{z,r}u_r-(\lambda+2B+2C)(1/r)u_{z,zz}u_r$$

$$-(\lambda+B)(1/r)u_{z,rr}u_r-2C(1/r)u_{r,rz}u_r-(1/r)Bu_{r,zr}u_r-\big[\lambda+2\mu+(1/3)A+2B\big](1/r)\boxed{u_{r,r}u_{z,r}}$$

$$-\big((1/3)A+2B+2C\big)(1/r)\boxed{u_{r,z}u_{r,r}}-(\lambda+\mu+3B+2C)(1/r)\boxed{u_{z,z}u_{z,r}}-(\mu+3B+2C)(1/r)\boxed{u_{z,z}u_{z,r}}. \quad (16)$$

In a quadratic nonlinear description of deformation, there may appear many products of 12 functions, i.e., the displacement u_r, its two first derivatives $u_{r,r}, u_{r,z}$, and three second derivatives $u_{r,rr}, u_{r,zr}=u_{r,rz}, u_{r,zz}$, as well as the displacement u_z, its two first derivatives $u_{z,r}, u_{z,z}$ and three second derivatives $u_{z,rr}, u_{z,zr}=u_{z,rz}, u_{z,zz}$.

The total number of products is determined by the number of combinations of six elements taken 2 at a time:

$$\overline{C}_{12}^2=C_{12+2-1}^2=((12+2-1)!/2!(12-1)!)=((13)!/2!\cdot 11!)=78.$$

However, products of displacements and their second derivatives are absent in (15) and (16). Therefore, 23 and 22 types of quadratic nonlinear terms are present in Eqs. (15) and (16), respectively.

This situation looks discouraging. However, it is possible to substantially reduce the number of nonlinearities or simplify the computation of all types of nonlinearities in some cases. First, the wave equations (15), (16) were derived neglecting the products of displacement gradients in the Cauchy relations (4). Therefore, it looks logical to neglect these products on the right-hand side of the wave equations (whose number is 5 in (15) and 4 in (16)). These products are shown by rectangular frames in the equations.

The analysis of harmonic waves within the first two approximations of the method of successive approximations requires computation of all types of nonlinearities in terms of the linear approximation in the form of the first harmonics. Finally, the right-hand (inhomogeneous) side of the equation will have the form of a product of the first two harmonics (second harmonics) and the sum of the coefficients of nonlinearities [6, 17, 18]. This can considerably simplify the description of all nonlinearities in the second approximation.

Conclusions. New nonlinear wave equations based on the Murnaghan five-constant model are obtained. Their feature is two assumptions: the elastic deformation process is physically nonlinear (geometric nonlinearities are neglected) and the deformation is geometrically axisymmetric and is described by circular cylindrical coordinates. Therefore, the system of wave equations contains only two coupled equations. This statement enables the obtaining new equations for the analysis of surface waves propagating along a circular cylindrical cavity in an elastic medium. Another feature of the nonlinear equations is that every equation includes the classical linear part. The nonlinear components are quadratically nonlinear and contain 23 types of nonlinearities in the first equation and 22 types of nonlinearities in the second equation.

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