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SOLUTION OF BOUNDARY-VALUE PROBLEMS OF THE THEORY OF PLATES WITH VARIABLE PARAMETERS USING PERIODICAL B-SPLINES

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An approach to solving static problems for ring plates with parameters varying in two coordinate directions is proposed. The system of equations and boundary conditions are formulated for displacements, forces, and moments. The two-dimensional boundary-value problem is reduced to one-dimensional one using the spline-collocation method. This problem is solved with the stable numerical method of discrete orthogonalization. The numerical results presented as a table are analyzed.

Keywords: bending of ring plates, spline-collocation method, periodical B-splines, method of discrete orthogonalization

Introduction. Analysis of the stress–strain state of shells and plates subjected to various loads under different boundary conditions leads to formulating and solving boundary-value problems for systems of partial differential equations with variable coefficients. The complexity of solving these problems is due not only to the high order of the systems and variability of the coefficients but also to the necessity to satisfy exactly the boundary conditions. The application of one method or another to solving the problem with sufficient degree of accuracy depends, to a large measure, on geometrical and mechanical parameters characterizing some features of the problem and on the type of boundary conditions. This fact sometimes restricts the capabilities of solving important applied problems for shells and plates with variable stiffness and different boundary conditions. Moreover, local and boundary effects observed in shell problems impose certain stiffness conditions associated with computational instability in solving boundary-value problems [1, 2, 7, 9].

Spline functions have been widely used to solving problems of computational mathematics, mathematical physics, and mechanics [1, 4, 5, 8] because spline approximations demonstrate a number of advantages over other ones. These advantages are the following: stability of splines with respect to local disturbances (behavior of the spline in the vicinity of a point does not affect the spline behavior at a whole, as it takes place, for example, in the case of polynomial approximation), high convergence of spline-interpolation as distinct from polynomial one, simplicity and convenience in implementation of algorithms of construction and computer-aided calculation of splines.

Due to the features indicated above, spline functions are used in solving problems of the theory of deformable elastic bodies. Particularly, they are efficient in the case of two-dimensional shell and plate problems as well as in the case of spatial elasticity problems [3, 4, 11]. Herewith, boundary-value problems are stated only for displacements while the approaches to solving them are based on the method of spline-collocation in one coordinate direction and numerical solution in the other direction.

In the case of thin closed shells of revolution, the displacement formulation essentially narrows the class of problems being solved.

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In what follows, we will consider an approach based on the use of the spline-collocation method [1, 6, 10], when the governing system of equations and boundary conditions are formulated for displacements, forces, and moments, i.e., in mixed form. As an example, the bending problem for ring plates will be solved.

1. Problem Statement. Consider an isotropic ring plate whose thickness $h = h(r,\theta)$ varies in two coordinate directions. It is under a surface load $q_z = q_z(r,\theta)$. Let the coordinate plane be the middle plane described by polar coordinates r, θ. The coordinate z is reckoned along the normal to the mid-plane.

Since the plate is circumferentially closed, we use w, ϑ_r, Q_r, M_r as unknown functions, where w is the deflection, ϑ_r is the angle of rotation, Q_r is the shearing force, M_r is the bending moment.

The pure bending of the plate is described as follows [2]:

$$\frac{\partial w}{\partial r} = -\vartheta_r, \qquad \frac{\partial \vartheta_r}{\partial r} = \frac{\nu}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2} - \frac{\nu}{r}\vartheta_r + \frac{12(1-\nu^2)}{h^2}M_r,$$

$$\frac{\partial Q_r}{\partial r} = -\frac{h^2}{2(1+\nu)r^4}\frac{\partial h}{\partial \theta}\frac{\partial w}{\partial \theta} + \frac{1}{r^4}\left(\frac{1}{4}\frac{\partial}{\partial \theta}\left(h^2\frac{\partial h}{\partial \theta}\right) - \frac{h^3}{6(1+\nu)}\right)\frac{\partial^2 w}{\partial \theta^2}$$

$$+\frac{h^2}{2r^4}\frac{\partial h}{\partial \theta}\frac{\partial^3 w}{\partial \theta^3} + \frac{h^3}{12r^4}\frac{\partial^4 w}{\partial \theta^4} - \frac{1}{4r^3}\frac{\partial}{\partial \theta}\left(h^2\frac{\partial h}{\partial \theta}\right)\vartheta_r - \frac{(2+\nu)h^2}{2(1+\nu)r^3}\frac{\partial h}{\partial \theta}$$

$$-\frac{(3+\nu)h^3}{12(1+\nu)r^3}\frac{\partial^2 \vartheta_r}{\partial \theta^2} - \frac{1}{r}Q_r - \frac{\nu}{r^2}\frac{\partial^2 M_r}{\partial \theta^2} - q_z,$$

$$\frac{\partial M_r}{\partial r} = -\frac{h^2}{2(1+\nu)r^3}\frac{\partial h}{\partial \theta}\frac{\partial w}{\partial \theta} - \frac{(3+\nu)}{12(1+\nu)}\frac{h^3}{r^3}\frac{\partial^2 w}{\partial \theta^2} + \frac{h^3}{12r^2}\vartheta_r$$

$$-\frac{h^2}{2(1+\nu)r^2}\frac{\partial h}{\partial \theta}\frac{\partial \vartheta_r}{\partial \theta} - \frac{h^3}{6(1+\nu)r^2}\frac{\partial^2 \vartheta_2}{\partial \theta^2} + Q_r - \frac{1-\nu}{r}M_r \quad (r_0 \le r \le r_1). \tag{1.1}$$

In formulating the boundary-value problem for the governing system of equations (1.1), loading or boundary conditions should be specified on the plate boundaries $r = r_0$, $r = r_1$. Let us consider some boundary conditions on the boundaries $r = r_0$, $r = r_1$:

(i) the boundary is clamped:

$$w = 0, \quad \vartheta_r = 0, \tag{1.2}$$

(ii) the boundary is free:

$$Q_r = 0, \quad M_r = 0; \tag{1.3}$$

(iii) the boundary is loaded by a transverse force:

$$\vartheta_r = 0, \quad Q_r = Q_0. \tag{1.4}$$

2. Problem-Solving Method. Let the solution of the boundary-value problem for the governing system (1.1) be the following (variant 1):

$$\{w(r,\theta),\ \vartheta_r(r,\theta),\ Q_r(r,\theta),\ M_r(r,\theta)\} = \sum_{i=0}^{N-1}\{w_i(r),\ \vartheta_{ri}(r),\ Q_{ri}(r),\ M_{ri}(r)\}\psi_i(\theta), \tag{2.1}$$

where $w_i(r)$, $\vartheta_{ri}(r)$, $Q_{ri}(r)$, $M_{ri}(r)$ $(i = \overline{0, N-1})$ are the unknown functions, while $\psi_i(\theta)$ $(i = \overline{0, N-1}, N > 6)$ are linear combinations of quintic B-splines on a uniform mesh of points $\theta_0 < \theta_1 < \theta_2 < \ldots < \theta_N$, which satisfy periodicity conditions for the solution. In this case, the functions $\psi_i(\theta)$ take the form

$$\Psi_0(\theta)=2B_5^0(\theta), \quad \Psi_1(\theta)=2B_5^1(\theta), \quad \Psi_2(\theta)=2B_5^2(\theta),$$

$$\Psi_i(\theta)=B_5^i(\theta) \quad (i=\overline{3,N-3}),$$

$$\Psi_{N-2}(\theta)=2B_5^{N-2}(\theta), \quad \Psi_{N-1}(\theta)=2B_5^{N-1}(\theta). \tag{2.2}$$

Applying the collocation method, we arrive at the following system of ordinary differential Cauchy-form equations that contains $4\times N$ equations of the first order:

$$\frac{d\overline{Y}}{dr}=B\overline{Y}+\bar{f} \tag{2.3}$$

$$\Big(\overline{Y}=\{\overline{Y_1^T}(r),\overline{Y_2^T}(r),\overline{Y_3^T}(r),\overline{Y_4^T}(r)\}^T;$$

$$\overline{Y_1^T}(r)=\{w_1,\, w_2,\, w_3,\, w_4\}; \quad \overline{Y_2^T}(r)=\{\vartheta_{r1},\, \vartheta_{r2},\, \vartheta_{r3},\, \vartheta_{r4}\};$$

$$\overline{Y_3^T}(r)=\{Q_{r1},\, Q_{r2},\, Q_{r3},\, Q_{r4}\}, \quad \overline{Y_4^T}(r)=\{M_{r1},\, M_{r2},\, M_{r3},\, M_{r4}\}\Big).$$

Let us specify arbitrary correct boundary conditions on the boundary $r=\text{const}$:

$$B_1\overline{Y}(r_0)=\overline{b_1}, \quad B_2\overline{Y}(r_1)=\overline{b_2}. \tag{2.4}$$

To solve the boundary-value problem (2.3), (2.4), we will use the stable numerical method of discrete orthogonalization [1, 2].

Since Eqs. (1.1) contain major second partial derivatives of the functions ϑ_r, Q_r, M_r with respect to θ, we will consider (along with search of a solution in the form (2.1) when all the unknown functions w, ϑ_r, Q_r, M_r are expressed in terms of quintic B-splines) a variant where cubic and quintic B-splines are used and the solution has the following form (variant 2):

$$w(r,\theta)=\sum_{i=0}^{N-1} w_i(r)\psi_i(\theta),$$

$$\{\vartheta_r(r,\theta),Q_r(r,\theta),M_r(r,\theta)\}=\sum_{i=0}^{N-1}\{\vartheta_{ri}(r),Q_{ri}(r),M_{ri}(r)\}\,\varphi_i(\theta). \tag{2.5}$$

Here $\varphi(\theta)$ $(i=\overline{0,N-1},\ N>6)$ are linear combinations of cubic B-splines on the uniform mesh $\theta_0<\theta_1<\theta_2<\ldots<\theta_N$ that satisfy periodicity conditions for the solution. Then the functions $\varphi_i(\theta)$ become

$$\varphi_0(\theta)=2B_3^0(\theta), \quad \varphi_1(\theta)=2B_3^1(\theta), \quad \varphi_{N-1}(\theta)=2B_3^{N-1}(\theta),$$

$$\varphi_i(\theta)=B_3^i(\theta) \quad (i=\overline{2,N-2}). \tag{2.6}$$

After solving problem (2.3), (2.4), we substitute the unknown functions into solutions (2.1) and (2.5), respectively, and determine the stress–strain state (SSS) of the plates.

3. Analysis of the Numerical Results. To illustrate the above approach, we will consider a number of problems.

(i) Let us analyze the deformation of a ring plate with circumferentially varying thickness $h=h_0(1+\gamma\cos\theta)$. The outer boundary $r=r_1$ of the plate is clamped, while the inner one $r=r_0$ is subject to a normal force Q_0 and is fixed so that only a normal displacement is only possible, i.e., the boundary conditions are

$$\vartheta_r=0, \quad Q_r=Q_0 \quad \text{at} \quad r=r_0;$$

$$w=\vartheta_r=0 \quad \text{at} \quad r=r_1.$$

TABLE 1

Function	θ/π					
	0	0.2	0.4	0.6	0.8	1.0
w/h_0	0.7085	0.7379	0.8221	0.9428	1.057	1.105
	0.7084	0.7376	0.8220	0.9428	1.056	1.105
	0.7085	0.7379	0.8222	0.9429	1.057	1.105
	0.7182	0.7460	0.8243	0.9341	1.035	1.077
$M_r/10E_0$	0.2024	0.2000	0.1936	0.1853	0.1783	0.1755
	0.2023	0.2001	0.1935	0.1852	0.1782	0.1754
	0.2024	0.2001	0.1937	0.1852	0.1783	0.1756
	0.2040	0.2013	0.1939	0.1842	0.1761	0.1729

The parameters have the following values: $r_0 = 0.4$, $r_1 = 1$, $E = 1$, $\nu = 0.3$, $\gamma = 0.1$, $Q_0 = -8$, $h_0 = 1$.

Tables 1 and 2 collect the results calculated for each function: the first row contains results for solution (2.1) with quintic B-splines (variant 1); the second row contains results for solution (2.5) with cubic and quintic B-splines (variant 2); the third row contains results from [3], where the boundary-value problem is formulated for displacements, using cubic and quintic B-splines; the fourth row contains results from [4] obtained with the method of lines.

Table 1 summarizes distributions of the deflection w and bending moment M_r over the inner boundary $r = r_0$. As is seen, the results differ by no greater than 2%, which justifies the efficiency and accuracy of the approach proposed.

(ii) Let us analyze the bending of a ring plate with thickness varying in two coordinate directions $h = h_0(1+\gamma\cos\theta)(1-2r/3)$. The plate is under a distributed normal load $q_z = q_0$. The inner boundary of the plate is clamped, while the outer one is free. The boundary conditions are:

$$w = \vartheta_r = 0 \quad \text{at} \quad r = r_0,$$

$$Q_r = M_r = 0 \quad \text{at} \quad r = r_1.$$

The input data: $r_0 = 0.5$, $r_1 = 1$, $E = 1$, $\nu = 0.3$, $\gamma = 0.1$, $q_0 = 2.5$, $h_0 = 1$.

Table 2 summarizes distributions of the deflection w over the outer boundary $r = r_1$ and bending moments M_r over the inner boundary $r = r_0$.

The moment peaks in the section $\theta = 0$ while the deflection in $\theta = \pi$. As is seen, the results differ by no greater than 1%, which justifies the efficiency and accuracy of the approach proposed.

Conclusions. The results obtained show that the mixed statement of a boundary-value problem, i.e., for displacements, forces, and moments, and the use of periodical B-splines expand the class of problems with arbitrary correct boundary and loading conditions, simplifies the algorithm, and allows determining the SSS of the plate without additional calculations. Such an approach can be used to study, using varies problem statements, the SSS of various shell elements with a coordinate surface in the form of circumferentially closed bodies of revolution.

TABLE 2

Function	θ/π					
	0	0.2	0.4	0.6	0.8	1.0
$\frac{w}{10h_0}$	0.0917	0.0964	0.1104	0.1318	0.1533	0.1627
	0.0917	0.0964	0.1104	0.1318	0.1533	0.1630
	0.0914	0.0961	0.1103	0.1319	0.1538	0.1634
	0.0917	0.0964	0.1104	0.1318	0.1533	0.1627
M_r / E_0	–0.4569	–0.4561	–0.4538	–0.4501	–0.4462	–0.4445
	–0.4569	–0.4560	–0.4539	–0.4502	–0.4463	–0.4446
	–0.4558	–0.4552	–0.4533	–0.4504	–0.4472	–0.4458
	–0.4560	–0.4561	–0.4537	–0.4500	–0.4462	–0.4446

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