DOI 10.1007/s10778-018-0862-6
International Applied Mechanics, Vol. 54, No. 1, January, 2018

FORCED VIBRATIONS AND DISSIPATIVE HEATING OF HINGED FLEXIBLE VISCOELASTIC RECTANGULAR PLATES WITH ACTUATORS UNDER SHEAR DEFORMATION

V. G. Karnaukhov[1], V. N. Kozlov[1], and T. V. Karnaukhova[2]

The forced resonant vibrations and vibrational heating of viscoelastic plates with actuators are modeled considering geometrical nonlinearity and transverse shear. An approximate analytical solution of the problem is obtained for a hinged rectangular plate by the Bubnov–Galerkin method. The effect of geometrical nonlinearity and shear deformations on the efficiency of active damping of vibrations with piezoelectric actuators is analyzed.

Keywords: resonant vibrations, geometrical nonlinearity, transverse shear, piezoelectric actuators, active damping

Introduction. Thin plates made of polymer-based composite materials are widely used in various areas of modern engineering. Forced harmonic vibrations, including resonant ones, are one of the most frequent modes of operation of such elements [2, 4, 18–20]. During vibrations, all materials show hysteresis losses, and mechanical or electromechanical energy is converted into heat. Hysteresis losses are much greater in viscoelastic materials. This effect is widely used in developing passive methods of damping of forced resonant vibrations of thin-walled elements to reduce their dynamical stress when components with high hysteresis losses are incorporated into the structure of an element with low hysteresis, which results in a reduction of amplitudes of vibrations. The extensive literature in the form of encyclopedias, monographs, and articles deals with the study of passive damping of vibrations [6, 7, 9, 11, 16, 17]. However, an increase of hysteresis losses can be accompanied by a significant increase in temperature, which is further called dissipative heating temperature (DHT). DHT can affect all aspects of the mechanical and thermal state of the structure, in particular, the distribution of strains and strains in the deformed body, the dynamic characteristics of resonant vibrations (amplitude-frequency, temperature-frequency characteristics, frequency dependence of the damping factor), dynamic and static stability of thin-walled elements, their mechanical and heat fracture, creep [1, 3–5, 14]. Moreover, every material has a definite temperature, so-called degradation temperature at which mechanical and electromechanical properties become considerably worse. As a rule transition of a material from one aggregate state to another corresponds to the point of degradation. For a passive material, the point of degradation is melting temperature, and for piezo-electric material, this is the Curie point, on reaching which the piezo-effect is lost. Here a specific type of thermal damage takes place, when the structure element is not split into parts, but ceases to fulfill its function. Therefore, it is necessary to take into account DHT in studying the forced vibrations of structural elements made of inelastic materials, since neglecting this phenomenon can result in unreliable results. In spite of this evident fact, the overwhelming majority of works on forced resonant vibrations of inelastic thin-walled elements neglect it.

In recent years, active methods based on piezoactive components incorporate into the structure of a passive thin-walled element made of metallic, polymeric, or composite material have been effectively used for damping the vibrations of thin-walled

[1]S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterova St., Kyiv, Ukraine 03057, e-mail: karn@inmech.kiev.ua. [2]National Technical University “KPI,” 37 Pobedy Av., Kiev, Ukraine 03056, e-mail: karn@inmech.kiev.ua. Translated from Prikladnaya Mekhanika, Vol. 54, No. 1, pp. 101–110, January–February, 2017. Original article submitted March 30, 2017.

elements. In most cases, piezoelectric components act as active elements. Survey of foreign publications on active control of stationary and non-stationary vibrations of structure elements is presented in [11–13, 15, 27–29]. One of the main methods of active damping is the method based on the use of piezo-actuators, to which the potential difference is applied to compensate the action of the mechanical loading, as the result of which the amplitude of vibrations reduces considerably. Here the main problem is to determine the difference of potentials considering the size of the actuator, its location, etc. Temperature, including the DHT has a noticeable effect on the efficiency of actuators. As the vibrational heating temperature reaches the Curie point of the piezo-material, the actuator ceases to fulfill its function due to the loss of the piezoelectric effect by the active material [3–5, 14, 21–25]. For anisotropic materials and for large amplitudes of harmonic loading, it is necessary to consider shear strains and geometric nonlinearity.

Here we state the problem of the forced vibrations and dissipative heating of viscoelastic plates with actuators taking into account geometrical nonlinearity and transverse shear strains. An approximate analytical solution for a hinged rectangular plate is found using the Bubnov–Galerkin method. The dissipative function is determined. The analytical solution of the energy equation is obtained. The effect of geometrical nonlinearity, shear strains, and DHT on the efficiency of actuators is analyzed.

1. Constitutive Equations for Viscoelastic Piezoelectric Plates with Shear Strains. Consider a three-layer plate composed of passive (without piezoeffect) middle orthotropic viscoelastic layer of thickness h_0 and two face layers of thickness h_1 made of transversely isotropic viscoelastic piezoelectric materials polarized across the thickness. The properties of the piezo-layers are the same; they have opposite polarization. The plate is loaded by uniform surface harmonic pressure with frequency close to the resonant frequency of the plate. A Cartesian coordinate system (x, y, z) is used, where the axis Oz is oriented along the plate thickness. The middle plane of the inner layer is chosen the reference one. We accept two hypotheses for modeling the electromagnetic vibrations of the plate, namely,

(i) the normal component of the stress tensor $\sigma_{zz} = 0.2$;

(ii) the normal component of the induction vector D_z is considerably greater than the tangential components, $D_z \neq 0$, $D_x = 0$, $D_y = 0.3$;

(iii) all the components of the vector of electric field strength are not equal to zero, $E_x \neq 0, E_y \neq 0, E_z \neq 0$.

Under these hypotheses, three-dimensional constitutive equations of electromagnetic elasticity take the form

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}) - \gamma_{11}E_3, \quad \sigma_{yy} = \frac{E}{1-\nu^2}(\nu\varepsilon_{xx} + \varepsilon_{yy}) - \gamma_{11}E_3,$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\varepsilon_{xy}, \quad \sigma_{xz} = E_{44}\varepsilon_{xz}, \quad \sigma_{yz} = E_{44}\varepsilon_{yz},$$

$$D_z = \gamma_{33}E_z + \gamma_{11}(\varepsilon_{xx} + \varepsilon_{yy}) \tag{1.1}$$

$$\Big(\nu = -S_{12}/S_{11}; \quad E = 1/S_{11}; \quad E_{44} = 1/\tilde{S}_{44}; \quad \tilde{S}_{44} = S_{44} - d_{15}^2/e_{11};$$

$$\gamma^{11} = \frac{Ed_{31}}{1-\nu}; \quad \gamma_{33} = e_{33}\left(1-k_p^2\right); \quad k_p^2 = \frac{2d_{31}^2}{S_{11}^E e_{33}(1-\nu)}\Big), \tag{1.2}$$

S_{ij}, d_{ij}, e_{ij} are compliances, piezo-modulus, and dielectric constant of the piezoelectric layer [2, 4, 27–29].

For passive elastic materials, the simplified constitutive equations under the Timoshenko hypotheses are given, for example, in [26]. From them, using the correspondence principle [10], we obtain the constitutive equations for viscoelastic materials.

For the refined Timoshenko model, the components of the displacement vector are approximated by the linear law

$$w = w(s,\theta), \quad u(s,\theta,z) = u_0(s,\theta) + z\varphi_x(s,\theta), \quad v = v_0(s,\theta) + z\varphi_y(s,\theta), \tag{1.3}$$

where φ_x, φ_y characterize independent turn of the normal about the plate. Expressions for the plate strains $\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$, $\kappa_1, \kappa_2, \kappa_{12}$ in terms of $u_0, v_0, \varphi_x, \varphi_y, w$ are given, for example, in [26], and, if only squares of angles are retained, have the form

$$\varepsilon_1 = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_2 = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \varepsilon_{12} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y},$$

$$\varepsilon_{13} = \frac{\partial w}{\partial x} + \varphi_x, \quad \varepsilon_{23} = \frac{\partial w}{\partial y} + \varphi_y, \quad \kappa_1 = \frac{\partial \varphi_x}{\partial x}, \quad \kappa_2 = \frac{\partial \varphi_y}{\partial y}, \quad \kappa_{12} = \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}.$$

Substituting these expressions into (1.1) and integrating over thickness, we obtain the constitutive equations for forces and moments in the piezo-layers with shear strains:

$$N_{xx}^{(n)} = \frac{2Eh_1}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2) + \frac{2E^2 d_{31}^2 h_1}{(1-\nu)^2 \tilde{\varepsilon}_{33}}(\varepsilon_1 + \varepsilon_2),$$

$$N_{yy}^{(n)} = \frac{2Eh_1}{1-\nu^2}(\nu\varepsilon_1 + \varepsilon_2) + \frac{2E^2 d_{31}^2 h_1}{(1-\nu)^2 \tilde{\varepsilon}_{33}}(\varepsilon_1 + \varepsilon_2), \quad N_{xy}^{(n)} = \frac{Eh_1}{(1+\nu)}\varepsilon_{12},$$

$$M_{xx}^{(n)} = \frac{2Em_1}{1-\nu^2}(\kappa_1 + \nu\kappa_2) - \frac{2E^2 d_{31}^2 m_1}{(1-\nu)^2 \tilde{\varepsilon}_{33}}(\kappa_1 + \kappa_2) - M_0, \quad m_1 = \frac{1}{2}h_0^2 h_1 + h_0 h_1^2 + \frac{1}{3}h_1^3,$$

$$M_{yy}^{(n)} = \frac{2Em_1}{1-\nu^2}(\nu\kappa_1 + \kappa_2) - \frac{2E^2 d_{31}^2 m_1}{(1-\nu)^2 \tilde{\varepsilon}_{33}}(\kappa_1 + \kappa_2) - M_0, \quad M_{xy}^{(n)} = \frac{Em_1}{2(1+\nu)}\kappa_{12}, \quad M_0 = m_0 V_0,$$

$$Q_x^{(n)} = 2E_{44}h_1\varepsilon_{13}, \quad Q_y^{(n)} = 2E_{44}h_1\varepsilon_{23}, \quad E_{44} = 1/(S_{44} - d_{15}^2/\tilde{\varepsilon}_{33}^T), \; m_0 = \frac{E}{1+\nu}|d_{31}|(h_0 + h_1), \tag{1.4}$$

where $V_0/2$ is the potential difference applied to the active layers deposited on the surface of the passive layer.

The constitutive equations for the passive layer are derived similarly. For elastic orthotropic material, they are given, for example, in [26]. From them, using the correspondence principle [10], we obtain the constitutive equations for the passive viscoelastic layer:

$$N_{xx} = \tilde{A}_{11} * \varepsilon_{xx} + \tilde{A}_{12} * \varepsilon_{yy}, \ldots,$$

$$M_{xx} = \tilde{D}_{11} * \kappa_{xx} + \tilde{D}_{12} * \kappa_{yy}, \ldots,$$

$$Q_x = K_s \tilde{A}_{55} * \varepsilon_{xz}, \quad Q_y = K_s \tilde{A}_{44} * \varepsilon_{yz}. \tag{1.5}$$

Here asterisk "*" denotes the Volterra operator [10] and will be omitted hereafter.

The quantities $\tilde{A}_{ij}, \tilde{D}_{ij}$ for elastic passive orthotropic material are given in [26] and have the form

$$\tilde{A}_{11} = \frac{E_1 h_0}{1-\nu_{12}\nu_{21}}, \quad \tilde{A}_{22} = \frac{E_2 h_0}{1-\nu_{12}\nu_{21}}, \quad \tilde{A}_{12} = \frac{\nu_{12}E_2 h_0}{1-\nu_{12}\nu_{21}}, \quad \tilde{A}_{66} = G_{12}h_0,$$

$$\tilde{A}_{55} = G_{13}h_0, \quad \tilde{A}_{44} = G_{23}h_0, \quad \tilde{B}_{13} = G_{13}h_0, \quad \tilde{B}_{23} = G_{23}h_0,$$

$$\tilde{D}_{11} = \frac{E_1 h_0^3}{12(1-\nu_{12}\nu_{21})}, \quad \tilde{D}_{22} = \frac{E_2 h_0^3}{12(1-\nu_{12}\nu_{21})}, \quad \tilde{D}_{12} = \frac{\nu_{12}E_2 h_0^3}{12(1-\nu_{12}\nu_{21})}, \quad \tilde{D}_{66} = \frac{G_{12}h_0^3}{12}.$$

If we consider the active material as elastic, then m_0 is constant. We obtain the constitutive equations for the layered plate by summing the stiffness characteristics of the active and passive layers. After introduction of the correction coefficient K_s, they take the form

$$N_{xx} = A_{11}\varepsilon_{xx} + A_{12}\varepsilon_{yy}, \ldots,$$

$$M_{xx} = D_{11}\kappa_{xx} + D_{12}\kappa_{yy} - M_0, \ldots,$$

$$Q_x = K_s A_{55}\varepsilon_{xz}, \quad Q_x = K_s A_{44}\varepsilon_{yz}, \tag{1.6}$$

where, for example, the stiffness characteristics A_{11}, D_{11} are defined by

$$A_{11} = \widetilde{A}_{11} + \frac{2Eh_1}{1-\nu^2} + \frac{2E^2 d_{31}^2 h_1}{(1-\nu)^2 \widetilde{\varepsilon}_{33}}, \ldots, D_{11} = \widetilde{D}_{11} + \frac{2Em_1}{1-\nu^2} - \frac{2E^2 d_{31}^2 m_1}{(1-\nu)^2 \widetilde{\varepsilon}_{33}}, \ldots. \tag{1.7}$$

The remaining formulas for the stiffness characteristics are obtained from (1.4), (1.5) by summing the expressions multiplying the corresponding strain components.

2. Forced Vibrations of Viscoelastic Plates with Actuators with Shear Strains and Geometric Nonlinearity. Problem Statement. The universal equations of the refined theory of plates (motion equations for forces and moments, kinematic relations, boundary and initial conditions) have the same form as in the pure mechanical theory of plates [25]. The behavior of the material is described by the above constitutive equations. The constitutive equations (1.6) for plates formally coincide with the constitutive equations of thermoelasticity where the thermal terms are replaced by M_0 depending on the potential difference applied to the electrodes. Therefore, many of the results obtained in the field of thermoelasticity can be applied to electroelasticity. However, the physical meaning of these problems is fundamentally different. For example, it is practically impossible to realize a high-frequency temperature effect on the plate. At the same time, this is easily done by applying a harmonic potential difference to the electrodes on the plate.

Using the universal equations of electroelasticity and the above constitutive equations, it is possible to obtain the equations for displacements and angles. Considering the above-mentioned analogy, we can derive equations from those given in monographs on thermoelasticity of thin-walled elements. Retaining the only normal forces of inertia, we can show that the equations for displacements and angles of rotation coincide with Eqs. (10.1.31)–(10.1.35) from [26], where it is only necessary to modify the strength characteristics and differently interpret the thermal terms. Represent these equations as

$$L_1(u,v,w) = 0, \quad L_2(u,v,w) = 0, \quad L_3(u,v,w,u_1,v_1) + q_0 = I_0 \ddot{w},$$

$$L_4(u_1,v_1,w) - \frac{\partial M_0}{\partial x} = 0, \quad L_5(u_1,v_1,w) - \frac{\partial M_0}{\partial y} = 0, \tag{2.1}$$

where the expressions for $L_i\,(i = 1-5)$ are given in [26]. According to the correspondence principle, it is necessary to replace all elastic characteristics by the Volterra integral operators using operator algebra [10]. For example, the operator L_1 coincides with operator (10.1.31), where it is necessary to replace the elastic constants by the Volterra operators:

$$L_1(u,v,w) = A_{11} * \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} \right) + A_{12} * \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} \right)$$

$$+ A_{66} * \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} \right). \tag{2.2}$$

The operator $L_2(u,v,w)$ derives from (2.2) after the following substitutions: $u \to v, v \to u, x \to y, y \to x, A_{11} \to A_{22}$. The operator L_4 is linear and has the form

$$L_4 = D_{11}\frac{\partial^2 \varphi_x}{\partial x^2} + D_{12}\frac{\partial^2 \varphi_y}{\partial x \partial y} + D_{66}\left(\frac{\partial^2 \varphi_x}{\partial y^2} + \frac{\partial^2 \varphi_y}{\partial x \partial y} \right) - K_S A_{55}\left(\frac{\partial w}{\partial x} + \varphi \right)_x. \tag{2.3}$$

We obtain from (2.3) the operator L_5 after the following substitutions: $\varphi_x \to \varphi_y, \varphi_y \to \varphi_x, x \to y, y \to x, D_{11} \to D_{12}$, $A_{55} \to A_{44}$. The operator L_3 has the form

$$L_3 = K_S A_{55} * \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \varphi_x}{\partial x} \right) + K_S A_{44} * \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \varphi_y}{\partial y} \right) + N(u, v, w) + q(x, y, t) - I_0 \frac{\partial^2 w}{\partial t^2}. \quad (2.4)$$

Retaining only the transverse forces of inertia, we get

$$N = N_{xx} \frac{\partial^2 w}{\partial x^2} + N_{yy} \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}. \quad (2.5)$$

3. Analytical Solution of the Problem of Electroviscoelasticity for a Hinged Plate. As mentioned above, the operators L_4, L_5 are linear in φ_x, φ_y, w. Consider the resonant vibrations in the vicinity of a certain (say, first) resonance frequency of a rectangular plate with sizes $0 \le x \le a$, $0 \le y \le b$. Represent the solution of the problem as

$$w = W_{mn} \sin(k_m x) \sin(p_n y), \quad \varphi_x = U_{mn} \cos(k_m x) \sin(p_n y),$$

$$\varphi_y = V_{mn} \sin(k_m x) \cos(p_n y) \quad (k_m = m\pi / a,\ p_n = n\pi / b), \quad (3.1)$$

satisfying the hinged boundary conditions. The mechanical and electric loads are represented similarly:

$$M_0 = M_{mn} \sin(k_m x) \sin(p_m y), \quad q_0 = q_{mn} \sin(k_m x) \sin(p_n y).$$

Hereafer we omit the indices m, n. Substituting these expressions into the operators L_4, L_5 we obtain

$$U = x_{11} M + w_1 W, \quad V = y_{11} M + w_2 W. \quad (3.2)$$

Here the coefficients multiplying M and W are Volterra-type operators, which are expressed in terms of electric and mechanical characteristics of the plate:

$$x_{11} = \frac{p\Delta_{12} - k\Delta_{22}}{\Delta}, \quad w_1 = \frac{K_S pA_{44}\Delta_{12} - K_S kA_{55}\Delta_{22}}{\Delta},$$

$$y_{11} = \frac{k\Delta_{21} - p\Delta_{11}}{\Delta}, \quad w_2 = \frac{K_S kA_{55}\Delta_{21} - K_S pA_{44}\Delta_{11}}{\Delta}$$

$$(\Delta_{11} = k^2 D_{11} + p^2 D_{66} + K_S A_{55}, \quad \Delta_{22} = p^2 D_{22} + k^2 D_{66} + K_S A_{44},$$

$$\Delta_{12} = \Delta_{21} = kp(D_{12} + D_{66}), \quad \Delta = \Delta_{11}\Delta_{122} - \Delta_{12}^2).$$

Substituting expressions (3.2) into the operators L_1, L_2, we arrive at a linear system of integral-differential equations for u and v:

$$A_{11} \frac{\partial^2 u}{\partial x^2} + A_{12} \frac{\partial^2 v}{\partial x \partial y} + A_{66} \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \right) = [C \sin(2kx)\cos(2py) + C_1 \sin(kx)] W^2,$$

$$A_{22} \frac{\partial^2 v}{\partial x^2} + A_{12} \frac{\partial^2 u}{\partial x \partial y} + A_{66} \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right) = [D \cos(2kx)\sin(2py) + D_1 \sin(py)] W^2, \quad (3.3)$$

where asterisk is omitted, and C, C_1, D, and D_1 are given by

$$C = -\frac{1}{4} A_{11} k^3 - \frac{1}{4} A_{12} kp^2 - \frac{1}{2} A_{66} kp^2, \quad C_1 = \frac{1}{4} A_{11} k^3 - \frac{1}{4} A_{12} kp^2,$$

$$D = -\frac{1}{4} A_{22} p^3 - \frac{1}{4} A_{12} k^2 p - \frac{1}{2} A_{66} k^2 p, \quad D_1 = \frac{1}{4} A_{22} p^3 - \frac{1}{4} A_{12} k^2 p. \quad (3.4)$$

The boundary conditions for u and v are

$$u = 0, \quad \frac{\partial v}{\partial x} = 0 \quad (x = 0, a),$$

$$v = 0, \quad \frac{\partial u}{\partial y} = 0 \quad (y = 0, b). \tag{3.5}$$

For these boundary conditions, the solution of system (3.3) has the form

$$u = W^2 [A\cos(2py) + A_1]\sin(2kx),$$

$$v = W^2 [B\cos(2kx) + B_1]\sin 2py. \tag{3.6}$$

The operator expressions for A, A_1, B, B_1 are

$$A_1 = -C_1 / (4A_{11}k^2), \quad B_1 = -D_1 / (4A_{22}p^2), \quad A = \frac{a_{12}D - a_{22}C}{a_{11}a_{22} - a_{12}^2}, \quad B = \frac{a_{12}C - a_{11}D}{a_{11}a_{22} - a_{12}^2},$$

$$a_{11} = 4(k^2 A_{11} + A_{66}p^2), \quad a_{22} = 4(p^2 A_{22} + A_{66}k^2), \quad a_{12} = 4kp(A_{12} + A_{66}).$$

Substituting expressions (3.1), (3.2), (3.6) into the operator equation (2.4), applying the Bubnov–Galerkin method, and performing very awkward computations, we get a nonlinear integral-differential equation for $W(t)$:

$$I_0 \ddot{W} + L^{(1)} * W + L^{(2)} * W^3 = q - Q * M. \tag{3.7}$$

Here $L^{(1)}, L^{(2)}, Q$ are Volterra operator expressions,

$$L^{(1)} = k_S k^2 A_{55} + k_S kA_{55}w_1 + k_S p^2 A_{44} + k_S pA_{44}w_2 + A_{11}\left(\frac{1}{2}k^3 A - k^3 A_1\right) + A_{22}\left(\frac{1}{2}p^3 B - k^3 B\right)_1$$

$$+ \frac{1}{2}A_{12}[k^2 p(B - 2B_1) + kp^2(A - 2A_1)],$$

$$L^{(2)} = \frac{1}{32}k^4 A_{11} + \frac{1}{32}p^4 A_{22} + \frac{3}{16}k^2 p^2 A_{12} - \frac{1}{8}k^2 p^2 A_{66}, \quad Q = K_S kA_{55}x_{11} + K_S pA_{44}y_{11}.$$

Let the viscoelastic properties of the passive material be described by the same kernel, so that, for example, $\widetilde{A}_{ij} * W = A_{ij}[W - \int_{-\infty}^{t} K(t-\tau)W(\tau)d\tau], \ldots$. Therefore, w_1, w_2, A, A_1, B, B_1 in the operator $L^{(1)}$ should be considered constants rather than operators, so that only A_{ij} remain to be Volterra operators. Similarly, the operator Q should also be considered constant. If we do not accept this assumption, it is necessary to make use of the algebra of Volterra operators. For example, using Rabotnov kernels for describing the viscoelastic properties of a material, product, quotient, sum, and difference of these kernels lead to the same operators with modified parameters. Therefore, it is possible to represent the operator $L^{(1)}$ as a certain operator with kernel depending on all parameters entering the expression of $L^{(1)}$. As a result, we get

$$L^{(1)} * W = L_0[W - \int_{-\infty}^{t} L_1(t-\tau)W(\tau)d\tau],$$

where the constant L_0 and the kernel L_1 depend on all constants and parameters of kernels in $L^{(1)}$.

If the effects of geometrical nonlinearity and viscosity are of same order of smallness, then it is possible to consider the operator $L^{(2)}$ to be a constant. For solving the integral-differential equation, we can use methods of nonlinear mechanics [8], as well as the Bubnov–Galerkin method. In using asymptotic methods [8], any operator $L^{(1)} * w$ is replaced by the expression $Lw + \mu\dot{w}$. Here the integral-differential equation (3.7) is represented as

$$\ddot{W} + 2\tilde{\mu}_1 \dot{W} + \omega_0^2 W + K_1 W^3 = q_1, \quad (3.8)$$

where

$$q_1 = (q - QM) / I_0. \quad (3.9)$$

If the piezo-layers are elastic, then Q is a constant proportional to the difference of potentials applied to the actuator $Q = \overline{Q} V_0$.

Solution of Eq. (3.8) is discussed in detail in [1]. For harmonic loading, it has the form

$$W = W' \cos \omega t - W'' \sin \omega t.$$

An algebraic cubic equation for squared amplitude was obtained in [1] $X = |W|^2 = (W')^2 + (W'')^2$:

$$b_3 X^3 + b_2 X^2 + b_1 X - b_0 = 0. \quad (3.10)$$

The coefficients b_i $(i = 0, 1, 2, 3)$ are given in [1] where also amplitude–frequency curve of hard type is given and so is an expression for the maximal amplitude $|W|_{max}$.

After determination of W, we find u_1 and u_2 from (3.10). Then u and v are determined from (3.6). Knowing w, u_1, v_1, u, v, we can determine the strains, and the forces and moments from the constitutive equations.

4. Determination of Dissipative Heating Temperature. Assuming that it is constant across the plate thickness, the dissipative heating temperature is determined by solving the stationary energy equation

$$\overline{\lambda}(\theta_{xx} + \theta_{yy}) - (2\delta / h)\theta + D / h = 0 \quad (4.1)$$

subject to the appropriate boundary and initial conditions for temperature.

The dissipative function D is determined by the formula

$$D = \frac{\omega}{2}\left[A''_{11} |\varepsilon_{xx}|^2 + 2A''_{12} |\varepsilon_{xx}\varepsilon_{yy}|^2 + A''_{22} |\varepsilon_{yy}|^2 + 2A''_{66} \left|\varepsilon_{xy}\right|^2 + D''_{11} \left|\frac{\partial \varphi_x}{\partial x}\right|^2 + 2D''_{12} \left|\frac{\partial \varphi_x}{\partial x}\frac{\partial \varphi_y}{\partial y}\right|^2 \right.$$

$$\left. + D''_{22} \left|\frac{\partial \varphi_y}{\partial y}\right|^2 + 2D''_{66} \left|\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right|^2 + K_s A''_{44} \left|\frac{\partial w}{\partial y} + \varphi_y\right|^2 + K_s A''_{55} \left|\frac{\partial w}{\partial x} + \varphi_x\right|^2 \right], \quad (4.2)$$

where $|ab|^2 = a'b' + a''b''$.

Assuming that geometrical nonlinearity and viscosity are of the same order of smallness, it is possible to neglect the nonlinear terms $|\varepsilon_{xx}|^2, |\varepsilon_{yy}|^2, |\varepsilon_{xy}|^2, |\varepsilon_{xx}\varepsilon_{yy}|^2$ in the kinematic equations for determining the dissipative function.

If the ends of the plate are heat-insulated, we determine the stationary solution of the energy equation from the solution of the electromechanical problem in the form

$$\theta = \theta_0 + \theta_1 \cos(2kx) + \theta_2 \cos(2py) + \theta_3 \cos(2kx)\cos(2py). \quad (4.3)$$

We easily determine the constants θ_i $(i = 0 - 3)$ by substituting (4.3) into Eq. (4.1) and equating the coefficients of $1, \cos(2kx), \cos(2py), \cos(2kx)\cos(2py)$. Due to their awkwardness, we omit the expressions for θ_i.

5. Analysis of the Solution. When actuators are used for active damping of the vibrations of plates, the main problem is to determine the potential difference to be applied to the actuator to compensate the mechanical load. As can be seen from (3.8)–(3.9), when the potential difference determined from $M = q / Q$ is applied to the actuator, the load on the plate is cancelled and the amplitude of forced transverse vibrations becomes zero. Here geometrical nonlinearity does not affect the potential difference applied to the electrodes. Therefore, to calculate this potential difference, it is possible to use a simpler linear theory detailed in, for example, [2, 3]. For a transversely isotropic active material, this potential difference is determined by the formula

$$V_a^{yt} = V_a^{kl}\left\{1+\frac{2}{1-\nu}\left(\frac{G}{G'}\right)\left(\frac{h}{a}\right)^2\left[m^2+\left(\frac{a}{b}\right)^2 n^2\right]\right\}. \tag{5.1}$$

For the main mode $m = n = 1$, it follows from formula (5.1) that

$$V_a^{yt} = V_a^{kl}\left\{1+\frac{2}{1-\nu}\left(\frac{G}{G'}\right)\left(\frac{h}{a}\right)^2\left[1+\left(\frac{a}{b}\right)^2\right]\right\}. \tag{5.2}$$

In this case, the correction to the classical result obtained based on the Kirchhoff–Love hypotheses depends on the ratio of the shear moduli (G/G') and the ratio of the plate thickness to the size *a*. Depending on their values, the correction can be very large.

It follows from expression (5.2) and physical considerations that when vibrations occur in the first node, the temperature is maximum in the plate center $x = a/2,\ y = b/2$, and it is equal to

$$\theta_{\max} = |\theta_0| + |\theta_1| + |\theta_2| + |\theta_3|. \tag{5.3}$$

Equating the maximum temperature to the point of degradation θ_k, we obtain the critical mechanical or electric load under which the plate loses its function. For example, if the Curie point is the degradation point of a piezo-material, then applying a harmonic potential difference to the electrodes does not generate vibrations of the plate because of depolarization.

For an electric load exceeding the critical value, it is necessary to determine the critical time at which the maximal temperature reaches the Curie point by solving the non-stationary problem of heat conduction under a load exceeding the critical value. Solving this problem for different amplitudes of electric load, we obtain a Weller curve that asymptotically tends to the critical load from above [5, 21, 23].

Conclusion. A model of forced resonant vibrations of flexible viscoelastic rectangular plates with piezo-actuators has been developed based on the refined Timoshenko hypotheses supplemented by corresponding hypotheses on the distribution of electric field quantities. To allow for geometrical nonlinearities in the kinematic equations, we retained squared angles of rotation. The Bubnov–Galerkin method has been used to reduce the problem to a nonlinear integral-differential equation with cubic nonlinearity in transverse bending. Methods of nonlinear mechanics were used to solve it. A cubic algebraic equation for squared amplitude of transverse vibrations has been derived. We have determined the potential difference that should be applied to the actuator to compensate the mechanical load, reducing the amplitudes of forced resonant vibrations to zero. The solution of the problem of electromechanics has been used to determine the dissipative function and represent an approximate solution of the heat conduction equation with a known heat source. Thermal failure of the plate upon temperature's reaching the Curie point and depolarization of the active material have been discussed.

REFERENCES

1. A. F. Bulat, V. I. Dyrda, V. G. Karnaukhov, E. L. Zvyagil'skii, and A. S. Kobets, *Forced Vibrations and Dissipative Heating of Elastic Bodies*, Vol. 3 of the four-volume series *Applied Mechanics of Elastic-Hereditary Media* [in Russian], Naukova Dumka, Kiev (2014).
2. V. T. Grinchenko, A, F, Ulitko, and N. A. Shul'ga, *Electroelasticity*, Vol. 5 of the five-volume series *Mechanics of Coupled Fields in Structural Elements* [in Russian], Naukova Dumka, Kiev (1989).
3. V. G. Karnaukhov and I. F. Kirichok, *Electrothermoviscoelasticity*, Vol. 4 of the five-volume series *Mechanics of Coupled Fields in Structural Elements* [in Russian], Naukova Dumka, Kiev (1988).
4. V. G. Karnaukhov and V. V. Mikhailenko, *Nonlinear Thermal Mechanics of Piezoelectric Inelastic Bodies under Monoharmonic Loading* [in Russian], ZhGTU, Shitomir (2005).
5. V. G. Karnaukhov, V. I. Kozlov, and T. V. Karnaukhova, "Thermal failure of inelastic hinged rectangular plate with piezoelectric sensors and actuators under forced resonant bending vibrations," *Byul. Dnepropetrovsk Univ.*, *Ser. Mekh.*, **2,** No. 5 (15), 68–75 (2011).
6. V. V. Matveev *Damping of Vibrations of Deformable Bodies* [in Russian], Naukova Dumka, Kiev (1985).

7. A. N. Guz (ed.), *Mechanics of Composites* [in Russian], in 12 vols., A.S.K., Kiev (1992).
8. Yu. A. Mitropolsky, *Nonlinear Mechanics*, *Single-Frequency Vibrations* [in Russian], Inst. Mat. NAS Ukrainy, Kiev (1997).
9. A. Nashif, J. Jones, and J. Henderson, *Vibrations Damping*, New York, Wiley (1985).
10. Yu. N. Rabotnov, *Elements of Hereditary Solid Mechanics* [in Russian], Nauka, Moscow (1977).
11. G. S. Agnes, "Development of a modal model for simultaneous active and passive piezoelectric vibration suppression," *J. Intell. Mater. Syst. Struct.*, **6**, No. 4, 482–487 (1995).
12. G. S. Bae, M. K. Kwak, and D. J. Inman, "Vibration suppression of a cantilever beam using eddy current of damper," *J. S. V.*, **284**, No. 3–5, 805–814 (2005).
13. U. Gabbert and H. S. Tzou, *Smart Structures and Structronic Systems*, Kluwer Academic Pub., Dordrecht (2001).
14. I. A. Guz, Y. A. Zhuk, and M. Kashtalyan, "Dissipative heating and thermal fatigue life prediction for structures containing piezoactive layers," *Techn. Mech.*, **32**, No. 2–5, 238–250 (2012).
15. Mel Schwartz (ed.), *Encyclopedia of Smart Materials*, 1–2, Wiley & Sons, New York (2002).
16. D. I. Jones, *Handbook of Viscoelastic Vibration Damping*, Wiley & Sons, New York (2001).
17. J. F. He and D. A. Ma, "Analysis of flexural vibration of viscoellastically damped sandwich plates," *J. Sound Vibr.*, **126**, No. 1, 37–47 (1988).
18. V. L. Karlash, "Influence of electric loading conditions on the vibrations of piezoceramic resonators," *Int. Appl. Mech.*, **53**, No. 2, 220–227 (2017).
19. V. L. Karlash, "Phase–frequency characteristics of the longitudinal and transverse vibrations of planar piezoceramic transformers," *Int. App. Mech.*, **53**, No. 3, 349–355 (2017).
20. V. L. Karlash, "Effect of split or partial electrodes on the forced vibrations of bear-type piezoceramic transducers", *Int. Appl. Mech.*, **52**, No. 5, 535–546 (2016).
21. T. V. Karnaukhova, "Thermal depolarization of a piezoelectric layer under harmonic quasistatic electric loading," *Int. Appl. Mech.*, **34**, No. 4, 373–376 (1998).
22. V. G. Karnaukhov, "Thermomechanics of coupled fields in passive and piezoactive inelastic bodies under harmonic deformations (review)," *J. Therm. Stress.*, **28,** No. 6–7, 783–815 (2005).
23. V. G. Karnaukhov, T. V. Karnaukhova, and O. McGillicaddy, "Thermal failure of flexible rectangular viscoelastic plates with distributed sensors and actuators," *J. Eng. Math.*, **78**, No. 1, 199–212 (2013).
24. V. G. Karnaukhov, I. F. Kirichok, and V. I. Kozlov, "Thermomechanics of inelastic thin-walled structural members with piezoelectric sensors and actuators under harmonic loading (review)," *Int. Appl. Mech.*, **53**, No. 1, 6–59 (2017).
25. I. F. Kirichok, "Damping the radial vibrations and self-heating of viscoelastic shell elements with piezoelectric sensor and actuator," *Int. Appl. Mech*., **52**, No. 4, 354–358 (2016).
26. J. N. Reddy, *Theory and Analysis of Elastic Plates and Shells*, CRC, Boca Raton (2007).
27. J. Tani, T. Takagi, and J. Qui, "Intelligent material systems: Application of functional materials," *Appl. Mech. Rev.*, **51**, No. 8, 505–521 (1998).
28. H. S. Tzou, *Piezoelectric Shells* (*Distributed Sensing and Control of Continua*), Kluwer Academic Publishers, Boston (1993).
29. H. S. Tzou and L. A. Bergman, *Dynamics and Control of Distributed Systems*, Cambridge University Press, Cambridge (1998).