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ELASTOPLASTIC STATE OF AN ELLIPTICAL CYLINDRICAL SHELL WITH A CIRCULAR HOLE

E. A. Storozhuk*, I. S. Chernyshenko, and O. V. Pigol'*****

Static problems for an elastoplastic elliptical cylindrical shell with a circular hole are formulated and a numerical method for solving it is developed. The basic equations are derived using the Kirchhoff–Love theory of deep shells and the theory of small elastoplastic strains. The method employs the method of additional stresses and the finite-element method. The influence of plastic strains and geometrical parameters of the shell subject to internal pressure on the distributions of stresses, strains, and displacements in the zone of their concentration is studied.

Keywords: cylindrical shell, elliptical cross-section, circular hole, Kirchhoff–Love theory, plasticity, finite-element method, static load

Introduction. Thin cylindrical shells of circular and non-circular cross-sections are widely used as structural elements of machines and devices in various fields of modern engineering. In most cases, such elements have design or service holes. Under high operation loads, zones of higher stresses occur near holes while their material becomes nonlinear.

Basic theoretical results on stress concentration in cylindrical shells with holes under static loads were obtained by using solutions of linear elastic problems for shells with a circular cross-section. They are outlined in [2, 4, 11, 13].

Boundary-value problems where nonlinear factors such as plastic strains and finite deflections are taken into account were solved only for circular cylindrical shells with one or two curved holes [9, 15].

Most studies on noncircular cylindrical shells disregard stress concentrators (holes, notches, etc.). To study the stress–strain state (SSS) [3, 5, 12, 16, 18], stability [8, 10, 17], and vibrations [14, 19, 20] of oval and elliptical cylindrical shells under various loads, analytical, numerical, and experimental methods were used.

A few studies addressed stress concentration in noncircular cylindrical shells using just a linear elastic problem statement. For example, the SSS of an elliptical cylindrical shell with a rectilinear notch under pressure nonuniformly distributed along the directrix was studied in [1]. The effect of cross-sectional ellipticity on the SSS near a circular hole in a cylindrical shell under an axial tensile force was studied in [13].

A review indicates that there are no theoretical studies on the elastoplastic state of noncircular cylindrical shells with curved holes. Therefore, we will formulate elastoplastic static problems for elliptical cylindrical shells with a circular hole, derive basic nonlinear equations, outline a method for numerical solution of problems of this class, and present specific numerical results for a shell under uniform internal pressure.

1. Problem Statement. Basic Equations. Consider a thin elliptical cylindrical shell of thickness h with a circular hole of radius r_0 described in a curvilinear orthogonal coordinate system (x, φ, γ), where x and γ are the longitudinal and normal (to the shell midsurface) coordinates, φ is the angle between the normal to the midsurface and the vertical axis (Fig. 1). Assume that the shell is made of a homogeneous isotropic material and subject to surface forces $\{p\} = \{p_1, p_2, p_3\}^T$, boundary forces and moments $\{m_k\} = \{T_k, S_k, Q_k, M_k\}^T$.

S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterova St., Kyiv, Ukraine 03057, e-mail: *stevan@ukr.net, **prikl@inmech.kiev.ua, ***pigolo@ukr.net. Translated from Prikladnaya Mekhanika, Vol. 53, No. 6, pp. 49–56, November–December, 2017. Original article submitted December 16, 2016.

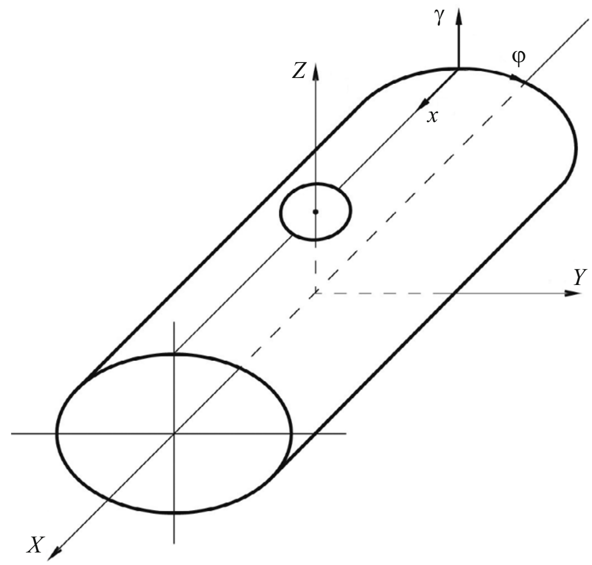

Fig. 1

The midsurface is described in a global coordinate system (X, Y, Z), whose OX-axis is parallel to the generatrix while the OZ-axis passes through the hole center. The shell cross-section lies in the plane (Y, Z) and is described parametrically as follows:

$$Y = \frac{a^2 \sin\varphi}{(a^2 \sin^2\varphi + b^2 \cos^2\varphi)^{1/2}},$$

$$Z = \frac{b^2 \cos\varphi}{(a^2 \sin^2\varphi + b^2 \cos^2\varphi)^{1/2}} \qquad (\varphi_1 \le \varphi \le \varphi_N), \tag{1.1}$$

where a and b are the ellipse semiaxes.

Using the formula for arc length

$$s(\varphi) = \int_{\varphi_1}^{\varphi} \sqrt{\left(\frac{dY(t)}{dt}\right)^2 + \left(\frac{dZ(t)}{dt}\right)^2}\, dt = \int_{\varphi_1}^{\varphi} \frac{a^2 b^2 dt}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}, \tag{1.2}$$

we set up a table function

$$\varphi_i = \varphi(s_i) \quad (i = 1, 2, \ldots, N) \tag{1.3}$$

that describes the dependence of the parameter φ on the arc length s of the ellipse and is used for partitioning the shell midsurface into finite elements (FE).

The coefficients in the expressions of the first quadratic form and curvatures of the shell midsurface take the form

$$A_1 = 1, \quad A_2 = \frac{a^2 b^2}{(a^2 \sin^2\varphi + b^2 \cos^2\varphi)^{3/2}},$$

$$k_1 = 0, \quad k_2 = \frac{(a^2 \sin^2\varphi + b^2 \cos^2\varphi)^{3/2}}{a^2 b^2}. \tag{1.4}$$

Let us represent the kinematic equations in vector form according the theory of deep shells based on the Kirchhoff–Love hypotheses [4, 11]:

$$\varepsilon_{11} = \vec{e}_1 \cdot \frac{\partial \vec{u}}{A_1 \partial x}, \quad \varepsilon_{22} = \vec{e}_2 \cdot \frac{\partial \vec{u}}{A_2 \partial \varphi}, \quad \varepsilon_{12} = \vec{e}_2 \cdot \frac{\partial \vec{u}}{A_1 \partial x} + \vec{e}_1 \cdot \frac{\partial \vec{u}}{A_2 \partial \varphi},$$

$$\mu_{11} = \vec{e}_1 \cdot \frac{\partial \vec{\vartheta}}{A_1 \partial x}, \quad \mu_{22} = \vec{e}_2 \cdot \frac{\partial \vec{\vartheta}}{A_2 \partial \varphi}, \quad 2\mu_{12} = \vec{e}_2 \cdot \frac{\partial \vec{\vartheta}}{A_1 \partial x} + \vec{e}_1 \cdot \frac{\partial \vec{\vartheta}}{A_2 \partial \varphi}, \tag{1.5}$$

where $\vec{u} = u\vec{e}_1 + v\vec{e}_2 + w\vec{n} = u_1\vec{i}_1 + u_2\vec{i}_2 + u_3\vec{i}_3$ is the displacement vector for points of the shell midsurface; $\vec{e}_1$, $\vec{e}_2$, $\vec{n}$ are the unit vectors of the coordinate system (x, φ, γ); $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$ are the unit vectors of the global Cartesian coordinate system (X, Y, Z); $\vec{\vartheta} = \vartheta_x \vec{e}_1 + \vartheta_\varphi \vec{e}_2 = \vartheta_1 \vec{i}_1 + \vartheta_2 \vec{i}_2 + \vartheta_3 \vec{i}_3$ is the vector of angles of rotation of the normal, which are determined, according to the Kirchhoff–Love hypotheses, as

$$\vartheta_x = -\vec{n} \cdot \frac{\partial \vec{u}}{A_1 \partial x},$$

$$\vartheta_\varphi = -\vec{n} \cdot \frac{\partial \vec{u}}{A_2 \partial \varphi}. \tag{1.6}$$

If the loading of the shell is simple, the constitutive equations can be taken from the theory of small elastoplastic deformations [4, 11]:

$$\sigma_{11} = \sigma_{11}^0 + \sigma_{11}^P, \quad \sigma_{12} = \sigma_{12}^0 + \sigma_{12}^P,$$

$$\sigma_{11}^0 = \frac{2G}{1-\nu}(e_{11} + \nu e_{22}), \quad \sigma_{12}^0 = G e_{12},$$

$$\sigma_{11}^P = 2G\left[\left(\frac{1-\omega_i}{1-\nu_i} - \frac{1}{1-\nu}\right)e_{11} + \left(\frac{(1-\omega_i)\nu_i}{1-\nu_i} - \frac{\nu}{1-\nu}\right)e_{22}\right],$$

$$\sigma_{12}^P = -G\omega_i e_{12}, \quad e_{11} = \varepsilon_{11} + \gamma\mu_{11}, \quad e_{12} = \varepsilon_{12} + 2\gamma\mu_{12} \quad (1 \leftrightarrow 2), \tag{1.7}$$

where G and ν are the shear modulus and Poisson's ratio of the shell material; ω_i and ν_i are the plasticity function and the variable coefficient of transverse strain; the superscript "0" refers to the linear shell theory, and the superscript "P" refers to the plastic component. With (1.7), the forces and moments can be represented as sums of linear and nonlinear parts [11]:

$$T_{ij} = T_{ij}^0 + T_{ij}^P, \quad M_{ij} = M_{ij}^0 + M_{ij}^P \quad (i, j = 1, 2).$$

2. Numerical Solution of Physically Nonlinear Problems for an Elliptical Cylindrical Shell with a Hole. To derive the system of governing equations, we will use the virtual displacement principle in combination with the method of additional stresses and finite-element method (FEM). Linearizing the problem, we arrive at the functional

$$\Pi^{LN} = \frac{1}{2}\iint_\Sigma (T_{11}^0\varepsilon_{11} + T_{22}^0\varepsilon_{22} + T_{12}^0\varepsilon_{12} + M_{11}^0\mu_{11} + M_{22}^0\mu_{22} + 2M_{12}^0\mu_{12})\,d\Sigma$$

$$+\iint_\Sigma (T_{11}^P\varepsilon_{11} + T_{22}^P\varepsilon_{22} + T_{12}^P\varepsilon_{12} + M_{11}^P\mu_{11} + M_{22}^P\mu_{22} + 2M_{12}^P\mu_{12})\,d\Sigma - A, \tag{2.1}$$

where A is the work done by the external surface $\{p\}$ and boundary $\{m_k\}$ forces; Σ is the domain occupied by the shell midsurface.

In solving the linear problem, we will apply a FEM that takes into account the features of the deformation of thin noncircular cylindrical shells with holes under static loads. This modification of the FEM is a development of the finite-element approach to numerical solution nonlinear problems for shells of complex geometry with curved holes. This approach is outlined in [9, 11, 15] and has a number of features.

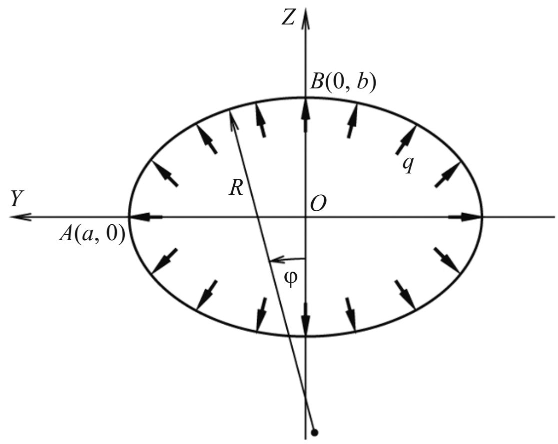

Fig. 2

(i) The strain components of the shell are expressed in vector form (1.5).

(ii) In constructing a finite element of the shell, the displacement vector $\vec{u}$ is approximated, i.e., its projections u_k onto the axes of the global Cartesian coordinate system XYZ:

$$u_k = \sum_{i=1}^{4} u_k^{(i)} L_i(\xi_1, \xi_2) \qquad (k = 1, 2, 3), \tag{2.2}$$

rather than its components (u, v, w) in the coordinate system (x, φ, γ).

In (2.2), i is the local number of a FE node; $L_i(\xi_1, \xi_2)$ are bilinear shape functions of the local coordinates ξ_1 and ξ_2.

(iii) The vector $\vec{\vartheta}$ of angles of rotation of the normal is not determined by (1.6), as in the classical FEM for thin shells, but is approximated by biquadratic serendipity polynomials $K_i(\xi_1, \xi_2)$ with the Kirchhoff–Love hypotheses satisfied only at the FE nodes [6, 9, 11, 15]:

$$\vartheta_k = \sum_{i=1}^{8} \vartheta_k^{(i)} K_i(\xi_1, \xi_2) \qquad (k = 1, 2, 3). \tag{2.3}$$

Thus, the derivatives higher then the first order of the approximating functions are absent in this FEM modification, which considerably simplifies the discretization of the problem.

(iv) To exclude the adverse effect of membrane locking on the convergence of numerical calculations of the tangential strain ε_{ij}, the double approximation method [7] is used.

A curvilinear FE constructed in such a way satisfies the continuity conditions for the displacement vectors and rotation angles, accurately describes the rigid-body translation of the FE, and is free of membrane locking.

From the stationarity condition for the discrete analog of functional (2.1), we obtain a system of governing equations in matrix form for a thin elliptical cylindrical shell with a hole deformed beyond the elastic limit:

$$[K]\{q\} = \{P\} - \{\Phi\}, \tag{2.4}$$

where $[K]$ is the stiffness matrix of a linear elastic shell; $\{q\}$ is the vector of nodal freedom degrees; $\{P\}$ is the load vector; $\{\Phi\}$ is the vector of nonlinearities that allow for plastic strains.

3. Testing of the Numerical Approach. To evaluate the efficiency of the technique developed, we will solve a test problem and compare its solution with the exact solution. Let us solve, as an example, the boundary-value problem of the SSS of a closed infinite long oval cylindrical shell subject to internal pressure of intensity $q = 1$ kPa (Fig. 2). The analytical (exact) solution of the problem was obtained in [16].

It is assumed that the cross-section has two axes of symmetry and is described parametrically as

TABLE 1

FEM without double approximation					FEM with double approximation				
N	$\tilde{w}(0)$	Δ, %	$\tilde{w}(0)$	Δ, %	N	$\tilde{w}(0)$	Δ, %	$\tilde{w}(0)$	Δ, %
250	1.088	9.0	–0.719	8.6	3	1.137	4.9	–0.740	6.0
500	1.167	2.4	–0.769	2.3	5	1.169	2.3	–0.769	2.3
1000	1.189	0.6	–0.783	0.5	10	1.189	0.6	–0.783	0.5
2000	1.194	0.2	–0.786	0.1	20	1.194	0.2	–0.786	0.1
Analytical (exact) solution: $\tilde{w}(0)=1.196$, $\tilde{w}(\pi/2)=-0.787$									

$$Y=R_0\left[\left(1+\frac{\xi}{2}\right)\sin\varphi+\frac{\xi}{6}\sin 3\varphi\right],\qquad Z=R_0\left[\left(1-\frac{\xi}{2}\right)\cos\varphi+\frac{\xi}{6}\cos 3\varphi\right],$$

$$R_0=\frac{a+b}{2},\qquad \xi=3\frac{a-b}{a+b},\qquad -\pi\le\varphi\le\pi, \tag{3.1}$$

where a and b are the major and minor semiaxes of the cross-section. The radius of the oval curvature is calculated by

$$R=R_0\,(1+\xi\cos 2\varphi). \tag{3.2}$$

The input data: $R_0/h=100$, $a/b=1.5$, $E=70\,\text{GPa}$, $\nu=0.3$; h is the thickness, which is constant.

Table 1 summarizes the values of relative deflection $\tilde{w}=w/h$ at two points of the cross-section (at the ends of the minor and major semiaxes). These data were obtained using the technique developed (FEM with double approximation) and the FEM described in [9, 11, 15] (FEM without double approximation). Table 1 also contains the results of analytical (exact) solution and demonstrates how the errors of numerical solutions (Δ) depend on the number of elements along the quarter the length of the cross-section (N).

It can be seen that the use of the FEM without double approximation leads to membrane locking: for the error of the numerical solution to be lower than 1%, it is necessary to partition a quarter of the cross-section into 1000 elements. Contrastingly, the FEM with double approximation needs only 10 elements, which is lower by two orders of magnitude.

Thus, the numerical technique developed to solve boundary-value problems for noncircular cylindrical shells makes it possible to avoid the adverse effect of locking on the convergence of the results, which substantially improves the accuracy of the solution.

4. Analysis of the Numerical Results. Let us analyze the results obtained in analyzing the elastoplastic state of a long elliptical cylindrical shell with a circular hole (Fig. 1). The shell is made of AMg-6 alloy and subject to internal pressure with intensity $q=q^*\cdot 10^5$ Pa.

Input data: $(a+b)/h=100$, $a/b=11/10, 1, 10/11$, $r_0/h=12$, $E=70\,\text{GPa}$, $\nu=0.3$–0.5, $\sigma_n=140\,\text{MPa}$, $\varepsilon_n=0.002$. The hole is closed by a plug that transfers shear forces $Q_k=qr_0/2$ only.

Numerical results were obtained by solving linear (LP) and physically nonlinear (PNP) problems for uniform internal pressure of intensity $q^*=4$.

Table 2 shows the distribution of the relative deflection (w/h), circumferential strain e_θ, and stress σ_θ^* ($\sigma_\theta=\sigma_\theta^*\cdot 10^5$ Pa) along the hole boundary ($0\le\theta\le 90°$, where θ is reckoned from the generatrix) on the outside and inside surfaces of the shell ($\zeta=\gamma/h=\pm 0.5$) for two elliptical ($a=1.1b$ and $b=1.1a$) and one circular ($a=b$) cylinders for both LP and PNP.

TABLE 2

Solution	θ, deg.	ζ	LP			PNP		
			$b = 1.1a$	$a = b$	$a = 1.1b$	$b = 1.1a$	$a = b$	$a = 1.1b$
w/h	0	0	–6.519	–0.057	7.099	–13.73	–0.058	13.88
	45	0	–5.967	0.015	6.751	–12.13	0.015	12.96
	90	0	–5.433	0.087	6.413	–10.86	0.088	12.07
$e_\theta \cdot 10^2$	0	0.5	–0.6666	0.0894	0.8209	–2.4439	0.0891	2.521
		–0.5	0.7231	0.2581	–0.2065	2.4262	0.2610	–0.4738
	45	0.5	–0.2378	0.1497	0.5326	–0.3228	0.1501	1.3108
		–0.5	0.3654	0.1015	–0.1671	0.4525	0.1013	–0.3695
	90	0.5	–0.0364	0.1006	0.2397	–0.0455	0.1006	0.2625
		–0.5	–0.1112	–0.1622	–0.2206	–0.1806	–0.1622	–0.2315
σ_θ^*	0	0.5	–4670	643	5781	–2166	645	2213
		–0.5	5060	1824	–1410	2149	1503	–1592
	45	0.5	–1661	1061	3749	–1485	1062	1873
		–0.5	2563	724	–1151	1641	722	–1282
	90	0.5	–266	698	1676	–334	698	1474
		–0.5	–789	–1143	–1548	–1281	–1143	–1439

TABLE 3

a/b	φ, deg.	LP			PNP		
		w/h	$e_{22} \cdot 10^2$	σ_{22}^*	w/h	$e_{22} \cdot 10^2$	σ_{22}^*
1.1	0	6.523	$\frac{0.3876}{-0.3381}$	$\frac{2980}{-2599}$	10.22	$\frac{0.7513}{-0.6282}$	$\frac{2077}{-2013}$
	45	–0.309	$\frac{-0.0182}{0.0702}$	$\frac{-139}{540}$	–0.402	$\frac{-0.0134}{0.0655}$	$\frac{-103}{504}$
	90	–5.849	$\frac{-0.3533}{0.4078}$	$\frac{-2716}{3135}$	–9.238	$\frac{-0.6803}{0.8235}$	$\frac{-2038}{2104}$
1.0	0	0.013	0.026	200	0.013	0.026	200

Table 3 collects the values of the relative deflection (w/h), strain e_{22}, and stress σ^*_{22} at several points of the cross-section ($0 \le \varphi \le 90°$) on the outside (numerator) and inside (denominator) surfaces of long elliptical and circular cylindrical shells without a hole.

The results show that the most dangerous are the points on the hole boundary in both elliptical and circular cylindrical shells in the section $\theta = 0°$, where the stresses and strains are maximum.

The plastic strains equalize the stresses across the shell thickness and on the hole boundary and decrease the maximum stresses by 57% at $b = 1.1a$, by 18% at $a = b$, and by 62% at $a = 1.1b$, compared with the linear elastic solution. Moreover, the maximum strains and deflections in the PNP are greater than those in the LP by 238% and 111% at $b = 1.1a$, by 1% and 1% at $a = b$, and by 207% and 96% at $a = 1.1b$.

An analysis of the results shows that the stress concentration factor k_θ for the elliptical shell is considerably less than that for the circular shell: this factor is equal to 1.61 for $b = 1.1a$, 9.12 for $a = b$, and 1.94 for $a = 1.1b$ in the linear elastic problem and to 1.03 for $b = 1.1a$, 7.52 for $a = b$, and 1.07 for $a = 1.1b$ in the physically nonlinear problem.

Conclusions. Two-dimensional elastoplastic problems for thin elliptical cylindrical shells with a circular hole have been formulated and a numerical technique for solving them has been developed. The technique is based on the method of additional stresses and finite-element method with double approximation of strains. With this technique, we have analyzed the elastoplastic state of elliptical and circular cylindrical shells with a hole under uniform internal pressure. Numerical results have been presented in tables for several values of the aspect ratio. Of interest is to solve nonlinear boundary-value problems for thin cylindrical shells of arbitrary cross-section with curved holes under arbitrary loads.

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