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SOLUTION OF STRESS-STRAIN PROBLEMS FOR COMPLEX-SHAPED PLATES IN A REFINED FORMULATION

A. Ya. Grigorenko, S. A. Pankrat'ev, and S. N. Yaremchenko

A numerical-analytical approach to solving problems on the stress–strain state of quadrangular plates of complex shape is proposed. The governing system of equations is presented in new orthogonal coordinates using transformations that take into account the plate geometry. A two-dimensional boundary-value problem, which is described by a system of partial differential equations derived with the spline-collocation method, is reduced to a one-dimensional one that is solved by the stable numerical discrete-orthogonalization method. The numerical results obtained for plates in the form of a trapezium and parallelogram are compared with the data obtained by other methods. The approach makes it possible to calculate deflections of quadrangular plates of complex shape made of anisotropic materials

Keywords: stress-strain state, plates of different shape, coordinate transformation, spline-collocation method, discrete-orthogonalization method

Introduction. Plates being considered are widely used as structural members in such fields as mechanical engineering, instrument-making and construction. Due to the wide use of composite materials and diversity of shapes of structural members, the set of appropriate problems to be solved becomes extremely large.

The methods and approaches to the practical analysis of the stress–strain state (SSS) of plates of different shapes have been considered yet as far back as the last century. As a result, analytical solutions, including those in the form of expansions into series, have been obtained for different fixation conditions, loadings, and such rather simple geometrical shapes as a circle or square [1–3]. In other cases, the appropriate numerical methods taking into account the symmetry or possibility to reduce a complex domain to a more simple one by the way of parametrization were developed in [6, 10]. Some questions relating to the transformation of coordinates for static analysis of complex-shaped plates are considered in [7–9].

Schemes of the analysis based on the finite-element method are implemented in many specialized program packages. They make it possible to numerically analyze the real subjects of complex shape but, however, have high requirements to computing resources and leave many open questions about the adequacy of the models and the choice of their parameters.

The present paper proposes an approach that widens applicability of the discrete-orthogonalization and spline-collocation methods [4] for the analysis of the SSS of quadrangular plates of complex shape. The approach is based on a refined theory involving the straight-line hypothesis.

1. Problem Statement. Basic Equations. Let us analyze the SSS of a quadrangular plate ($0 \le x_1 \le a, 0 \le x_2 \le b$) of thickness *h*. The equilibrium equations of the refined plate theory are expressed as follows [2]:

$$Q_{1,1} + Q_{2,2} + q = 0, \quad M_{1,1} + M_{12,2} - Q_1 = 0, \quad M_{2,2} + M_{12,1} - Q_2 = 0,$$

where Q_1 and Q_2 are the shearing forces; M_1, M_2, and M_{12} are the bending and twisting moments; *q* is the surface load.

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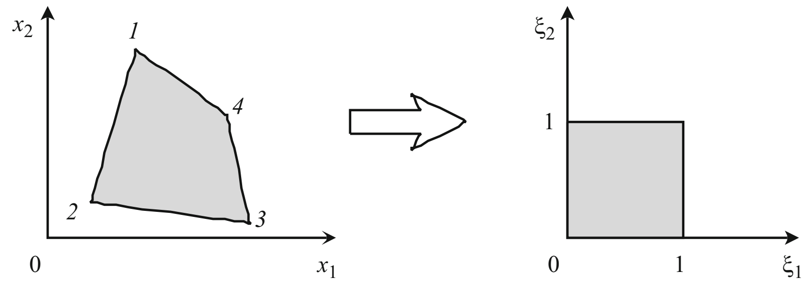

Fig. 1

For the moments and shearing forces, elasticity relations are valid. They, as applied to an orthotropic plate whose axes are aligned with the coordinate axes, take the form

$$M_1 = D_{11}\kappa_1 + D_{12}\kappa_2, \quad M_2 = D_{22}\kappa_2 + D_{12}\kappa_1, \quad M_{12} = 2D_{66}\kappa_{12},$$

$$Q_1 = K_1\gamma_1, \quad Q_2 = K_2\gamma_2,$$

where κ_1, κ_2, and κ_{12} are the strains of the midsurface bending and twisting, which with

$$\kappa_1 = \psi_{1,1}, \quad \kappa_2 = \psi_{2,2}, \quad 2\kappa_{12} = \psi_{1,2} + \psi_{2,1},$$

$$\gamma_1 = \psi_1 - \theta_1, \quad \gamma_2 = \psi_2 - \theta_2, \quad -\theta_1 = w_{,1}, \quad -\theta_2 = w_{,2}$$

can be expressed in terms of angles of rotation of the surface element ψ_1, ψ_2, angles of rotation of the normal θ_1, θ_2 disregarding shear, and angles of rotation γ_1, γ_2 of the normal caused by shear. The stiffness coefficients K_i and D_{ij} are determined by

$$D_{11} = \frac{E_1 h^3}{12(1-\nu_1\nu_2)}, \quad D_{12} = \nu_2 D_{11}, \quad D_{22} = \frac{E_2 h^3}{12(1-\nu_1\nu_2)},$$

$$D_{66} = \frac{G_{12} h^3}{12}, \quad K_1 = \frac{5}{6} h G_{13}, \quad K_2 = \frac{5}{6} h G_{23},$$

where E_i, G_{ij}, ν_i are the elastic and shear moduli and Poisson's ratios, respectively.

Considering the elasticity relations and strains expressed in terms of the angles ψ_1, ψ_2 and plate deflection w, the initial equilibrium equations become

$$K_1\psi_{1,1} + K_1 w_{,11} + K_2\psi_{2,2} + K_2 w_{,22} = -q,$$

$$D_{11}\psi_{1,11} + D_{12}\psi_{2,12} + D_{66}\psi_{1,22} + D_{66}\psi_{2,12} - K_1\psi_1 - K_1 w_{,1} = 0,$$

$$D_{22}\psi_{2,22} + D_{12}\psi_{1,12} + D_{66}\psi_{2,11} + D_{66}\psi_{1,12} - K_2\psi_2 - K_2 w_{,2} = 0. \tag{1.1}$$

The following boundary conditions at the edge $x_1 =$ const are $w = 0$, $\psi_1 = 0$, $\psi_2 = 0$ if clamped and $w = 0$, $\psi_{1,1} = 0$, $\psi_2 = 0$ if hinged. At the edge $x_2 =$ const, the boundary conditions are similar.

The governing system of equations (1.1) for the deflection w and angles of rotation ψ_1, ψ_2 together with other boundary conditions at the sides $x_i =$ const forms a two-dimensional boundary-value problem in a refined statement.

2. Basic Ideas of the Approach. Let us consider a domain in coordinates x_1, x_2, bounded by the sides of a convex quadrangle and map it onto a normalized domain $[0 \le \xi_1 \le 1]$, $]0 \le \xi_2 \le 1]$ in a new coordinate system $\xi_1\xi_2$ (Fig. 1). Such mapping is possible if

$$\bar{x} = T \cdot \bar{\varepsilon}, \tag{2.1}$$

where $\bar{x}$ is a vector with components $\{x_1, x_2\}$, $\bar{\varepsilon}$ is a vector with components $\{1, \xi_1, \xi_2, \xi_1\xi_2\}$. Note that the components t_{ij} of the matrix T depend on the geometry of the quadrangular plate. If the quadrangle apexes are at the points (x_{11}, x_{21}), (x_{12}, x_{22}), (x_{13}, x_{23}), (x_{14}, x_{24}), the components of the matrix T become:

$$t_{11} = x_{12}, \quad t_{12} = x_{13} - x_{12}, \quad t_{13} = x_{11} - x_{12}, \quad t_{14} = x_{14} - x_{13} + x_{12} - x_{11},$$

$$t_{21} = x_{22}, \quad t_{22} = x_{23} - x_{22}, \quad t_{23} = x_{21} - x_{22}, \quad t_{24} = x_{24} - x_{23} + x_{22} - x_{21}.$$

Considering relation (2.1) that describes the geometry of the quadrangle, we will represent the governing system of equations (1.1) in new coordinates. To this end, we will introduce a vector $\bar{f}$ with 18 components $\{\psi_1, \psi_{1,1}, \psi_{1,2}, \psi_{1,11}, \psi_{1,22}, \psi_{1,12}, \psi_2, \ldots, w_{,12}\}$ and corresponding 3×18 matrix of coefficients S. Then, Eqs. (1.1) become

$$S \cdot \bar{f} = \bar{q}, \tag{2.2}$$

where $\bar{q} = \{-q, 0, 0\}$. For the nonzero components of the matrix S, we have

$$s_{12} = K_1, \quad s_{19} = K_2, \quad s_{1.16} = K_1, \quad s_{1.17} = K_2,$$

$$s_{21} = -K_1, \quad s_{24} = D_{11}, \quad s_{25} = D_{66}, \quad s_{2.12} = D_{12} + D_{66}, \quad s_{2.14} = -K_1,$$

$$s_{36} = D_{12} + D_{66}, \quad s_{37} = -K_2, \quad s_{3.10} = D_{66}, \quad s_{3.11} = D_{22}, \quad s_{3.15} = -K_2.$$

To determine the elements of the matrix $\widetilde{S}$, which is similar to S and consists of the coefficients of Eqs. (1.1) in the new coordinate system, it is necessary, with transformation (2.1), to derive expressions for all the components of the vector $\bar{f}$. We will derive the relations for partial derivatives, using as an example, the deflection function $w(x_1, x_2)$.

The first derivatives can be obtained from the system of the equations composed of the known expressions for a partial derivative of a complex function (hereafter the derivatives with respect to ξ_i are denoted by indices after semicolon):

$$w_{;1} = w_{,1}x_{1;1} + w_{,2}x_{2;1}, \quad w_{;2} = w_{,1}x_{1;2} + w_{,2}x_{2;2}.$$

Its solution is

$$w_{,1} = Aw_{;1} + Bw_{;2}, \quad w_{,2} = Cw_{;1} + Dw_{;2}, \tag{2.3}$$

where A, B, C, D (expressions for ξ_1, ξ_2) are defined by

$$A = x_{2;2} / (x_{1;1}x_{2;2} - x_{1;2}x_{2;1}), \quad B = -x_{2;1} / (x_{1;1}x_{2;2} - x_{1;2}x_{2;1}),$$

$$C = -x_{1;2} / (x_{1;1}x_{2;2} - x_{1;2}x_{2;1}), \quad D = x_{1;1} / (x_{1;1}x_{2;2} - x_{1;2}x_{2;1}),$$

or in an explicit form:

$$A = (t_{24}\xi_1 + t_{23}) / \chi, \quad B = -(t_{24}\xi_2 + t_{22}) / \chi,$$

$$C = -(t_{14}\xi_1 + t_{13}) / \chi, \quad D = (t_{14}\xi_2 + t_{12}) / \chi, \tag{2.4}$$

where $\chi = (t_{12}t_{24} - t_{22}t_{14})\xi_1 + (t_{14}t_{23} - t_{24}t_{13})\xi_2 + (t_{12}t_{23} - t_{22}t_{13})$.

The second partial derivatives can be determined using the first derivatives (2.3) and replacing the function w in the right-hand side of (2.3) by either $w_{,1}$ or $w_{,2}$:

$$w_{,11} = (AA_{;1} + BA_{;2})w_{;1} + (AB_{;1} + BB_{;2})w_{;2} + A^2 w_{;11} + B^2 w_{;22} + 2ABw_{;12},$$

$$w_{,22} = (CC_{;1} + DC_{;2})w_{;1} + (CD_{;1} + DD_{;2})w_{;2} + C^2 w_{;11} + D^2 w_{;22} + 2CDw_{;12},$$

$$w_{,12} = (AC_{;1} + BC_{;2})w_{;1} + (AD_{;1} + BD_{;2})w_{;2} + ACw_{;11} + BDw_{;22} + (AD + BC)w_{;12}. \tag{2.5}$$

Consider a vector $\overline{m}$ with components $\{w_{,1}, w_{,2}, w_{,11}, w_{,22}, w_{,12}\}$, which are derivatives of the function $w(x_1, x_2)$ in the initial coordinate system, a corresponding vector $\overline{m}^*$ with components $\{w_{;1}, w_{;2}, w_{;11}, w_{;22}, w_{;12}\}$ in new coordinates, and a transformation matrix L such that

$$\overline{m} = L \cdot \overline{m}^* . \tag{2.6}$$

Deriving from (2.3) and (2.5) relations between the components of $\overline{m}$ and $\overline{m}^*$, we determine the nonzero elements of the transformation matrix L:

$$l_{11} = A, \quad l_{12} = B, \quad l_{21} = C, \quad l_{22} = D,$$

$$l_{31} = AA_{;1} + BA_{;2}, \quad l_{32} = AB_{;1} + BB_{;2}, \quad l_{33} = A^2, \quad l_{34} = B^2, \quad l_{35} = 2AB,$$

$$l_{41} = CC_{;1} + DC_{;2}, \quad l_{42} = CD_{;1} + DD_{;2}, \quad l_{43} = C^2, \quad l_{44} = D^2, \quad l_{45} = 2CD,$$

$$l_{51} = AC_{;1} + BC_{;2}, \quad l_{52} = AD_{;1} + BD_{;2}, \quad l_{51} = AC, \quad l_{54} = BD, \quad l_{55} = AD + BC.$$

The expressions for A, B, C, and D are given in (2.4). Their derivatives $A_{;1}$, $A_{;2}$, …, $D_{;2}$ take the following form:

$$A_{;1} = (t_{24}\chi - (t_{24}\xi_1 + t_{23})(t_{12}t_{24} - t_{22}t_{14}))/\chi^2, \quad A_{;2} = -(t_{24}\xi_1 + t_{23})(t_{14}t_{23} - t_{24}t_{13})/\chi^2,$$

$$B_{;1} = (t_{24}\xi_2 + t_{22})(t_{12}t_{24} - t_{22}t_{14})/\chi^2, \quad B_{;2} = -(t_{24}\chi - (t_{24}\xi_2 + t_{22})(t_{14}t_{23} - t_{24}t_{13}))/\chi^2,$$

$$C_{;1} = -(t_{14}\chi - (t_{14}\xi_1 + t_{13})(t_{12}t_{24} - t_{22}t_{14}))/\chi^2, \quad C_{;2} = (t_{14}\xi_1 + t_{13})(t_{14}t_{23} - t_{24}t_{13})/\chi^2,$$

$$D_{;1} = -(t_{14}\xi_2 + t_{12})(t_{12}t_{24} - t_{22}t_{14})/\chi^2, \quad D_{;2} = (t_{14}\chi - (t_{14}\xi_2 + t_{12})(t_{14}t_{23} - t_{24}t_{13}))/\chi^2.$$

By analogy with the vector $\bar{f}$ for the initial coordinate system, we introduce a vector $\bar{f}^*$ with 18 components $\{\psi_1, \psi_{1;1}, \psi_{1;2}, \psi_{1;11}, \psi_{1;22}, \psi_{1;12}, \psi_2, \ldots, w_{;12}\}$ and transformation matrix P such that

$$P \cdot \bar{f}^* = \bar{f}. \tag{2.7}$$

The components of the vectors $\bar{f}$ and $\bar{f}^*$ include derivatives of three functions, relation (2.6) applying to each of them. Let us introduce the following notion: O is a 5×5 zero matrix; o_c is a zero column vector of five components; o_r is a zero row vector of five components. Then the matrix P becomes:

$$P = \begin{pmatrix} 1 & o_r & 0 & o_r & 0 & o_r \\ o_c & L & o_c & O & o_c & O \\ 0 & o_r & 1 & o_r & 0 & o_r \\ o_c & O & o_c & L & o_c & O \\ 0 & o_r & 0 & o_r & 1 & o_r \\ o_c & O & o_c & O & o_c & L \end{pmatrix}.$$

Considering (2.7), we write Eqs. (2.2) in the new coordinate system as

$$S \cdot (P \cdot \bar{f}^*) = \bar{q}^*$$

or

$$\widetilde{S} \cdot \bar{f}^* = \bar{q}^*, \tag{2.8}$$

where $\bar{f}^*$ and $\bar{q}^*$ are analogs of the vectors $\bar{f}$ and $\bar{q}$ in the new coordinate system while the nonzero components of the matrix $\tilde{S} = S \cdot P$ are

$$\tilde{s}_{12} = s_{12} l_{11}, \quad \tilde{s}_{13} = s_{12} l_{12}, \quad \tilde{s}_{18} = s_{19} l_{21}, \quad \tilde{s}_{19} = s_{19} l_{22}, \quad \tilde{s}_{1.14} = s_{1.16} l_{31} + s_{1.17} l_{41},$$

$$\tilde{s}_{1.15} = s_{1.16} l_{32} + s_{1.17} l_{42}, \quad \tilde{s}_{1.16} = s_{1.16} l_{33} + s_{1.17} l_{43}, \quad \tilde{s}_{1.17} = s_{1.16} l_{34} + s_{1.17} l_{44},$$

$$\tilde{s}_{1.18} = s_{1.16} l_{35} + s_{1.17} l_{45}, \quad \tilde{s}_{21} = s_{21}, \quad \tilde{s}_{22} = s_{24} l_{31} + s_{25} l_{41}, \quad \tilde{s}_{23} = s_{24} l_{32} + s_{25} l_{42},$$

$$\tilde{s}_{24} = s_{24} l_{33} + s_{25} l_{43}, \quad \tilde{s}_{25} = s_{24} l_{34} + s_{25} l_{44}, \quad \tilde{s}_{26} = s_{24} l_{35} + s_{25} l_{45}, \quad \tilde{s}_{28} = s_{2.12} l_{51},$$

$$\tilde{s}_{29} = s_{2.12} l_{52}, \quad \tilde{s}_{2.10} = s_{2.12} l_{53}, \quad \tilde{s}_{2.11} = s_{2.12} l_{54}, \quad \tilde{s}_{2.12} = s_{2.12} l_{55}, \quad \tilde{s}_{2.14} = s_{2.14} l_{11},$$

$$\tilde{s}_{2.15} = s_{2.14} l_{12}, \quad \tilde{s}_{32} = s_{36} l_{51}, \quad \tilde{s}_{33} = s_{36} l_{52}, \quad \tilde{s}_{34} = s_{36} l_{53}, \quad \tilde{s}_{35} = s_{36} l_{54}, \quad \tilde{s}_{36} = s_{36} l_{55},$$

$$\tilde{s}_{37} = s_{37}, \quad \tilde{s}_{38} = s_{3.10} l_{31} + s_{3.11} l_{41}, \quad \tilde{s}_{39} = s_{3.10} l_{32} + s_{3.11} l_{42}, \quad \tilde{s}_{3.10} = s_{3.10} l_{33} + s_{3.11} l_{43},$$

$$\tilde{s}_{3.11} = s_{3.10} l_{34} + s_{3.11} l_{44}, \quad \tilde{s}_{3.12} = s_{3.10} l_{35} + s_{3.11} l_{45}, \quad \tilde{s}_{3.14} = s_{3.15} l_{21}, \quad \tilde{s}_{3.15} = s_{3.15} l_{22}.$$

Equations (2.8) represent the governing system of equations (1.1) in the coordinate system $\xi_1 \xi_2$ and describe geometry of the quadrangular plate. Since the initial domain in the form of an arbitrary quadrangle in the new coordinates transforms into a square, the boundary-value problem can be solved using the discrete-othogonalization and spline-collocation methods.

It should be noted that in solving the problem under the boundary conditions that include derivatives of the deflection function w and angles ψ_1, ψ_2 (hinged or free edge), it is necessary to take into account the changes caused by passing, in accordance with (2.6), to the new coordinate system. Particularly, if the sides $\xi_1 = \text{const}$ are hinged, the above boundary conditions (see Sec. 1) take the form $w = 0$, $A\psi_{1;1} + B\psi_{1;2} = 0$, $\psi_2 = 0$ with similar corrections for ψ_2 at the sides $\xi_2 = \text{const}$. It is assumed that expressions for A and B are similar to those in (2.4).

3. Problem-Solving Methods. To validate the approach proposed, we will use the well-known discrete-othogonalization and spline-collocation methods [1, 2, 4]. Since the governing system of equations was derived within the framework of the refined theory of plates and the equations include the partial derivatives of the searched functions up to the second order, the spline-approximation will be carried out using cubic B-splines. Then, we will search the solution for the deflection function $w(x_1, x_2)$, for example, in the form

$$w(x_1, x_2) = \sum_{i=0}^{N} w_i(x_1)\varphi_i(x_2), \tag{3.1}$$

where w_i are the unknown functions, φ_i are linear combinations of cubic B-splines. The approximating functions φ_i are calculated using the boundary conditions at the edges as follows [2]:

$$\varphi_0(x_2) = \alpha_{11} B_3^{-1} + \alpha_{12} B_3^0, \quad \varphi_1(x_2) = B_3^{-1} + \alpha_{21} B_3^0 + \alpha_{22} B_3^1,$$

$$\varphi_i(x_2) = B_3^i, \quad i = 2, \ldots, N-2, \quad \varphi_{N-1}(x_2) = B_3^{N+1} + \beta_{21} B_3^N + \beta_{22} B_3^{N-1},$$

$$\varphi_N(x_2) = \beta_{11} B_3^{N+1} + \beta_{12} B_3^N.$$

The spline-functions B_3^i are constructed on a uniform mesh of nodes Δ with spacing $x_2^{i+1} - x_2^i$; the coefficients α_{ij} and β_{ij} with notation

$$A_\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \quad A_\beta = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix},$$

are equal to:

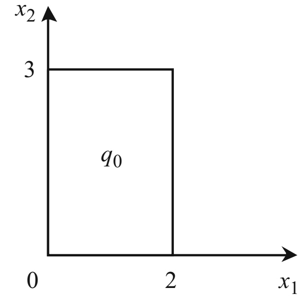

Fig. 2

$$A_\alpha = A_\beta = \begin{bmatrix} -4 & 1 \\ -1/2 & 1 \end{bmatrix} \qquad \text{if the edges } x_2 = \text{const are clamped and}$$

$$A_\alpha = A_\beta = \begin{bmatrix} 0 & 1 \\ -1/2 & 1 \end{bmatrix} \qquad \text{if the edges are hinged.}$$

In order to increase the accuracy of the approximation, we choose collocation points within the intervals between mesh nodes x_2^{2i}, x_2^{2i+1} as $\varepsilon_{2i} = x_2^{2i} + t_1(x_2^{2i+1} - x_2^{2i})$, $\varepsilon_{2i+1} = x_2^{2i} + t_2(x_2^{2i+1} - x_2^{2i})$, where t_1 and t_2 are the roots of the second-order Legendre polynomial on the segment [0; 1], $i = 0, 1, \ldots, n$.

With (3.1), the initial two-dimensional boundary-value problem reduces to a one-dimensional one for the system of ordinary high-order differential equations.

4. Calculated Results. To test the approach proposed, we will analyze the SSS of a rectangular plate using the numerical method of discrete orthogonalization for 500 points of integration. To decrease the dimension of the initial two-dimensional boundary-value problem, we will employ the spline-approximaion method with 20 pairs of collocation points.

Consider a rectangular isotropic plate (Fig. 2) with sizes $a = 2$, $b = 3$, $h = 0.1$ for two variants of fixation of edges: clamped and hinged ones. The surface load $q = q_0$ is constant and uniformly distributed. This makes it possible to compare the solutions of the boundary-value problem obtained in the coordinates x_1, x_2 for system (2.2) and in the coordinates ξ_1, ξ_2 for system (2.8). In the latter case, the problem was solved using the proposed transformation to the normalized domain $[0 \le \xi_1 \le 1]$, $[0 \le \xi_2 \le 1]$ with the plate geometry (Fig. 2) being shown in the coefficients of the governing system of equations.

The physical parameters of the plate, mesh dimensions, and the number of integration points are the same for both variants. Since the goal of the comparison is to validate the numerical results, the absolute values of the parameters are of secondary importance.

As a result, the values of the deflection w in the form $\hat{w} = wE / q_0$ in the plate center that have been get for the scheme with direct description of the domain $[0 \le x_1 \le 2]$, $[0 \le x_2 \le 3]$ being studied have coincided up to the 11th sign with those obtained by the scheme proposed above with the transformation to $[0 \le \xi_1 \le 1]$, $[0 \le \xi_2 \le 1]$. The discrepancy in the last digits may be due to the computer roundoff error.

To evaluate the quality of describing complex-shaped objects, we will solve numerically some problems for a number of plates in the form of a parallelogram and trapezium with clamped edges. The plates are acted upon by a uniformly distributed load $q = q_0$. The shape of some of them and form of the surface of the bending function $\hat{w}$ are shown in Figs. 3–6 (plate thickness $h = 0.1$, Poisson's ratio is 0.3). The coordinates of the apexes of the plate-forming quadrangles are summarized in Table 1. In calculating, we have used 30 pairs of collocation points for splines and 1500 integration points for the discrete -orthogonalization method.

The results obtained are compared with data presented in [6], where the above plates were considered within the framework of the classical Kirchhoff–Love theory. In problem solving, the discrete-orthogonalization method with parameters (the number of collocation points for splines and the number of integration points) different from those in the present work were used. To outline the complex shape, the authors have employed, depending on the plate shape, different transformations such as transition to three-angular or oblique coordinates. The results obtained with the finite-element method (FEM) are presented in the same place.

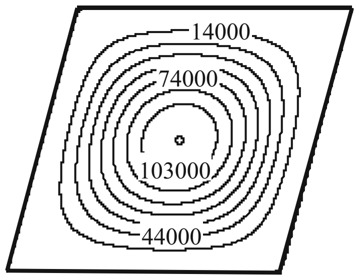

Fig. 3

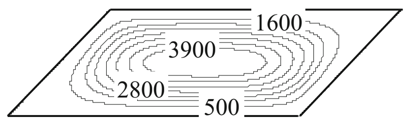

Fig. 4

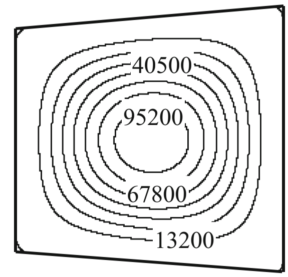

Fig. 5

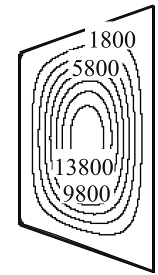

Fig. 6

TABLE 1

Object	x_{11}	x_{21}	x_{12}	x_{22}	x_{13}	x_{23}	x_{14}	x_{24}
1	2.59	9.66	0	0	10.00	0	12.59	9.66
2	7.07	7.07	0	0	10.00	0	17.07	7.07
3	1.29	4.83	0	0	10.00	0	11.29	4.83
4	3.54	3.54	0	0	10.00	0	13.54	3.54
5	47.15	4.13	47.15	–4.13	57.15	–5.00	57.15	5.00
6	8.66	2.32	8.66	–2.32	18.66	–5.00	18.66	5.00
7	52.15	4.56	52.15	–4.56	57.15	–5.00	57.15	5.00
8	8.74	3.18	8.74	–3.18	13.74	–5.00	13.74	5.00

The values of the maximum deflection $\hat{w}$ obtained with the above scheme using the transformation (transl. 1×1), data from [6] for the discrete-orthogonalization (d.ort.), and finite-element (FEM) methods as well as the associated values of the relative distinction, i.e., divergence of the results, are collected in Table 2. The data calculated are in a good agreement: the relative difference δ in the values of the deflection $\hat{w}$ for the most objects being studied is within the limits of several percents.

The data in Figs. 3–6 indicate that the surfaces of the deflection function $w(x_1, x_2)$ have the similar shape, which varies in accordance with the geometrical parameters of the plates. The values of the maximum deflection (Table 2) decrease with the object area subject to the load q.

Since the results obtained with different methods are similar, we can conclude that the approach proposed for describing the geometry of quadrangular complex-shaped plates can be employed for solving static problems with the discrete-orthogonalization and spline- collocation methods. In spite of some distinctions in the theory applied and in the methods and calculation parameters, the above scheme makes it possible to obtain results being in agreement with those of other authors [6].

As advantages of the approach proposed, we can note a larger class of potentially solvable problems that have not been considered earlier due to the complexity in the description of the domain being studied. The discrete-orthogonalization and spline-collocation methods make it possible to solve the problems for plates made of orthotropic materials including those with variable thickness and acted upon by differently distributed loads.

Conclusions. The present paper proposes an approach that widens the capabilities for analyzing the SSS of quadrangular complex-shaped plates with the discrete-orthogonalization and spline-collocation methods. The calculation scheme based on the transformation of the coordinates was tested by comparing numerical results with those obtained by direct

TABLE 2

No.	$\hat{w}$, trans. 1×1	$\hat{w}$, d. opt.	$\hat{w}$, FEM	δ, d. opt.	δ, FEM
1	118160	122400	123100	3.46%	4.01%
2	40205	41060	41460	2.08%	3.03%
3	15077	15150	15180	0.48%	0.68%
4	4483	4510	4505	0.60%	0.49%
5	108889	112400	113000	3.12%	3.64%
6	59309	60960	60760	2.71%	2.39%
7	17128	17030	17120	0.58%	0.05%
8	15628	15630	15650	0.01%	0.14%

solving of the problem in the case of a rectangular domain. The results obtained with the approach developed are in a good agreement with literature data.

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