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EVOLUTION OF SV-WAVE WITH GAUSSIAN PROFILE

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The evolution of a vertically polarized plane transverse wave (SV-wave) propagating in a nonlinear elastic medium is analyzed. The deformation process is described by the Murnaghan model, where the cubic nonlinearity is taken into account. Two approximate approaches are used and the solutions of the corresponding nonlinear wave equations are found for the first two approximations. Numerical examples are shown and commented

Keywords: SV-wave, cubic nonlinear elasticity, initial Gaussian profile, wave evolution

Introduction. The subject of this study is a vertically polarized plane transverse wave (SV-wave) propagating in a nonlinear elastic material whose deformation is described by the Murnaghan model [1–4]. The motion of this wave is set by a nonlinear wave equation that takes into account quadratic and cubic nonlinearities [8, 10, 11]:

$$\rho u_{3,tt} - \mu u_{3,11} = N_2 (u_{3,11} u_{1,1} + u_{1,11} u_{3,1}) + N_4 u_{3,11} (u_{3,1})^2 + N_5 u_{3,11} (u_{1,1})^2 + N_6 u_{3,11} (u_{2,1})^2,$$

$$N_2 = \lambda + 2\mu + (1/2)A + B, \quad N_4 = \lambda + 2\mu + (1/2)(5A + 14B + 4C),$$

$$N_5 = (3/2)(\lambda + 2\mu + A + 2B), \quad N_6 = 3A + 10B + 4C. \tag{1}$$

Equation (1) has been studied in detail for harmonic wave. The cases of quadratic (Eq. (2)) and cubic (Eq. (3)) nonlinearities have been considered:

$$\rho u_{3,tt} - \mu u_{3,11} = N_2 (u_{3,11} u_{1,1} + u_{1,11} u_{3,1}), \tag{2}$$

$$\rho u_{3,tt} - \mu u_{3,11} = N_4 u_{3,11} (u_{3,1})^2 + N_5 u_{3,11} (u_{1,1})^2 + N_6 u_{3,11} (u_{2,1})^2. \tag{3}$$

If only an SV-wave has been excited in the material (i. e., P-wave and SH-wave are not excited), then Eqs. (2) and (3) are become simpler:

$$\rho u_{3,tt} - \mu u_{3,11} = 0, \tag{4}$$

$$\rho u_{3,tt} - \mu u_{3,11} = N_4 u_{3,11} (u_{3,1})^2. \tag{5}$$

Finally, Eq. (4) becomes linear, while Eq. (5) is still nonlinear. This is usually given as a fact that, in the context of quadratic nonlinearity, the SV-wave does not generate itself (is not self-excited) and the wave profile evolution cannot be described, while the description of the self-excitation of the wave and the wave profile evolution is possible in the context of cubic nonlinearity.

The cases of non-harmonic profiles are considered only for P-waves [2–5, 9–12].

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2. SV-wave with initial Gaussian profile. Let only an SV-wave be excited. Then, its motion can be described by cubic nonlinear wave equation (5). Let us preset the initial profile in the form of Gaussian function:

$$u_3(x_1, 0) = u_3^o e^{-(x_1^2/2)} \tag{6}$$

(function $e^{-(x^2/2)}$ is also the Chebyshev–Hermite function of zero degree $\psi_0(x)$).

Let us suppose that the initial non-periodic profile (6) forms a solitary (non-periodic) wave in the form of

$$u_3(x_1, t) = u_3^o e^{-(\sigma^2/2)}, \tag{7}$$

where the phase variable of the wave is denoted by $\sigma = \sigma^o(x_1 - c_3 t)$, while $c_3 = \sqrt{\mu/\rho}$ is the constant phase velocity of the SV-wave. The parameter σ in the solitary wave corresponds to the wave number k in harmonic wave: a change in the wave number results in a change of the wavelength, while a change in parameter σ^o results in a change of the trough of the solitary wave.

Obviously, wave (7) is a representation of a D'Alembert simple wave [1, 2, 4–6, 9–12] and satisfies linear Eq. (4), which is also a linear part of Eq. (5).

Then, let us consider two approximate methods of nonlinear wave equation. *Method 1* is a classic (in linear acoustics) approximate method of nonlinear solution to wave Eq. (1) [1, 2, 9]. *Method 2* is based on an approximate decomposition of the variable wave velocity [1, 2, 12].

Method 1. Let us consider linear solution (4) as the first approximation of nonlinear solution (5)

$$u_3^{(1)}(x_1, t) = u_3^o e^{-(\sigma^2/2)}. \tag{8}$$

Let us define the second approximation $u_3^{(2)}(x, t)$ as the solution of inhomogeneous equation

$$\rho u_{3,tt}^{(2)} - \mu u_{3,11}^{(2)} = N_4\, u_{3,11}^{(1)} (u_{3,1}^{(1)})^2 \rightarrow \tag{9}$$

$$\rho u_{3,tt}^{(2)} - \mu u_{3,11}^{(2)} = N_4 (-u_3^o (1-\sigma^2) e^{-(\sigma^2/2)})(-u_3^o \sigma e^{-(\sigma^2/2)})^2 \rightarrow$$

$$u_{3,tt}^{(2)} - c_3^2 u_{3,11}^{(2)} = -(N_4/\rho)(u_3^o)^3 \sigma^2 (1-\sigma^2) e^{-3(\sigma^2/2)}. \tag{10}$$

Since the third harmonic $e^{-3(\sigma^2/2)}$ is the solution of Eq. (4), then, the solution of Eq. (10) is of resonant type:

$$u_3(x_1, t) = u_3^{(1)}(x_1, t) + u_3^{(2)}(x_1, t) = u_3^o e^{-(\sigma^2/2)} + x_1 (N_4/\rho)(u_3^o)^3 \frac{\sigma^2 (1-\sigma^2)}{3(1+\sigma-3\sigma^2)} e^{-3(\sigma^2/2)},$$

$$u_3(x_1, t) = u_3^o e^{-(\sigma^o)^2 (x_1 - c_3 t)^2/2} + x_1 \alpha_3 (c_3)^2 (u_3^o)^3 \frac{(\sigma^o)^2 (x_1 - c_3 t)^2 [1-(\sigma^o)^2 (x_1 - c_3 t)^2]}{3(1+\sigma^o (x_1 - c_3 t) - 3(\sigma^o)^2 (x_1 - c_3 t)^2)} e^{-3(\sigma^o)^2 (x_1 - c_3 t)^2/2}. \tag{11}$$

Comments 1 on Solution (11). This solution strongly depends on the phase σ: it changes in different ways at different profile points. When $\sigma = 0$ (bell top), there are no changes, i.e., the crest amplitude of the profile remains unchanged. There are no changes at the point $\sigma = 1$, where the nonlinear growth changes the sign from positive to negative. This can be commented as follows: the central part of the profile ($\sigma \in [-1; 1]$) enlarges ("get bigger"), while the tail part of profile narrows ("gets thinner"). Herewith, the tail "is getting thinner" in different ways on the left and right.

Comment 2 on Solution (11). The solution consists of two parts: one part corresponds to the classical single Gaussian wave (conventionally, the first harmonic wave) with constant parameters, while the other part conditionally corresponds to the third harmonic wave with variable amplitude. This amplitude deserves special attention: it is linearly dependent on the wave propagation distance and material properties. This dependence is typical of the approach adopted and has been earlier observed in other types of waves. It is the dependence that controls the profile evolution, when the wave moves.

Method 2. Let us describe Eq. (5) as

$$u_{3,tt} - u_{3,11}\{c_3^2[1+(N_4/\mu)(u_{3,1})^2] \equiv v_3^2\} = 0. \tag{12}$$

Subject to more general, than (6), initial condition $u_3(x,0) = F_3(x_1)$, the solution of Eq. (10) can be found in the form of a D'Alembert wave

$$u_3(x,t) = F_3(x_1 - v_3 t) \tag{13}$$

with unknown rate $v_3 = c_3\sqrt{1+\alpha_3(u_{3,1})^2}$, $\alpha_3 = (N_4/\mu)$. Let us take the assumption

$$|\alpha_3(u_{3,1})^2| < 1, \tag{14}$$

which allows describing approximate solution (13) as follows:

$$u_3(x_1,t) \cong F_3[x_1 - c_3 t - (1/2)\alpha_3 c_3 (u_{3,1})^2 t]. \tag{15}$$

The accuracy of approximation (15) depends on the precision of condition (14) including the assumptions for two parameters: parameter $\alpha_3 = (\lambda + 2\mu)/\mu + (5A + 14B + 4C)/2\mu$ and displacement gradient square $(u_{3,1})^2$.

Let us introduce an additional parameter $\delta = -(1/2)\alpha_3 c_3 (u_{3,1})^2 t$ and assume solution (15) in the form of the Taylor's theorem

$$u_3(x_1,t) \approx F_3(\sigma+\delta) \approx F_3(\sigma) + F_3'(\sigma)\delta + (1/2)F_3''(\sigma)\delta^2 + L,$$

$$\sigma = \sigma^o(x_1 - c_3 t). \tag{16}$$

Let us reduce the analysis to the first two members of decomposition (16), assuming $|\delta| < 1$. Since the smallness $|\alpha_3(u_{1,1})^2|$ is already assumed in (14), then, it is really the condition for the $c_3 t$ wave travel distance. Thus, we have

$$u_3(x_1,t) \approx F_3(\sigma) + F_3'(\sigma)\Big[\delta = -(1/2)\alpha_3 c_3 (u_{1,1})^2 t\Big] = F_3(\sigma) - (1/2)\alpha_3 c_3 t[F_3'(\sigma)]^3. \tag{17}$$

The description of solution (17) is of common nature and, for different selected profiles $F_3(x_1)$ it will describe the nonlinear wave effect, consisting in the inception of the third harmonic or similar new components and, finally, in the wave profile evolution.

Now, let us assume that we have a wave with Gaussian profile: the wave is set by Eq. (7). Then, Eq. (17) acquires more concrete form

$$u_3(x_1,t) \approx u_3^o e^{-(\sigma^o)^2(x_1-c_3t)^2/2} + (1/2)t\alpha_3 c_3 (\sigma^o)^3 (x_1 - c_3 t)^3 (u_3^o)^3 e^{-3(\sigma^o)^2(x_1-c_3t)^2/2}. \tag{18}$$

Comment on Solution (18). This solution depends on the phase σ significantly: it changes in different points in different ways; however, the nonlinear growth is always antisymmetric. When $\sigma = 0$ (bell top), there are no changes, i.e., the crest amplitude of the profile remains unchanged. However, at other profile points (symmetrical to the bell top), the profile changes asymmetrically: the right part of the profile enlarges ("gets bigger"), while the left part of profile narrows ("gets thinner").

The comparison of Eqs. (11) and (18) shows that they describe the cubic nonlinearity and, consequently, the wave evolution in different ways; moreover, they are obtained under different assumptions.

Based on Eqs. (11) and (18), we plotted 2D graphs with "displacement u_3–wave travel distance x_1" coordinates corresponding to the following parameter values: aluminum, $L = 30$, $\sigma^o = 60$, $u_3^o = 1\cdot 10^{-3}$ — (for (11), Figs. 1–3) and $L = 30$, $\sigma^o = 400$, $u_3^o = 1\cdot 10^{-3}$ — (for (18), Figs. 4–6).

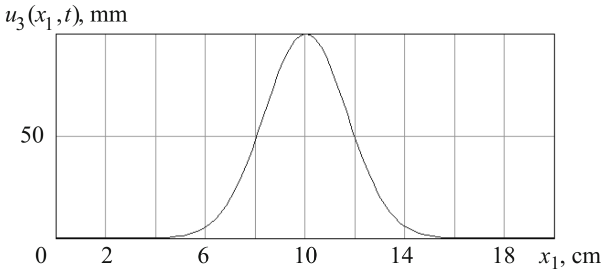

Fig. 1

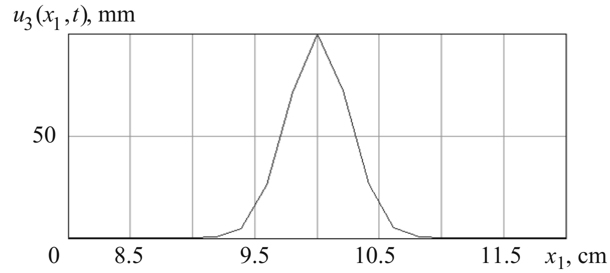

Fig. 4

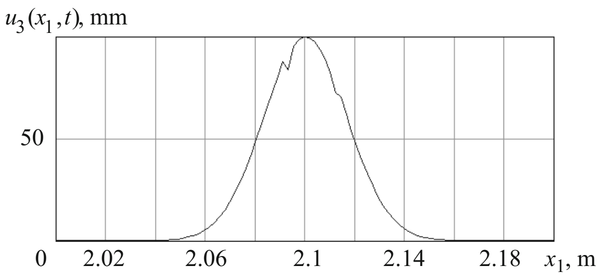

Fig. 2

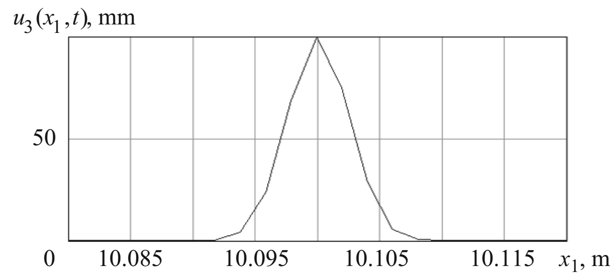

Fig. 5

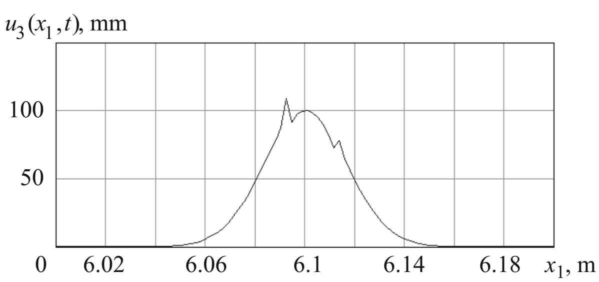

Fig. 3

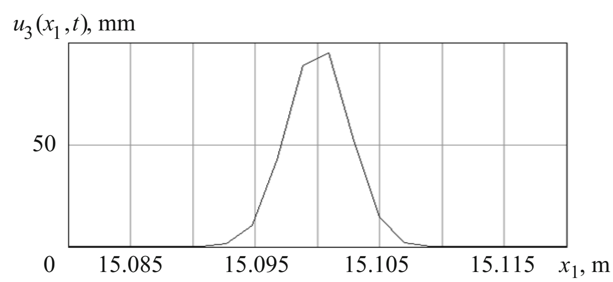

Fig. 6

Figures 1–3 show that the initial stage of the wave profile evolution is characterized by asymmetrical change in the symmetrical profile. Herewith, the crest amplitude on the trailing edge increases. Single hump tends to transform into three (the central hump remains and new humps appear on the trailing and leading edges. The trough of the curve remains virtually unchanged.

Figures 4–6 show that the initial stage of wave profile evolution is described slower and the profile change occurs slower. All changes observed in Figs. 1–3 are also seen in Figs. 4–6 (trough stability, formation of three humps, and asymmetry in trailing and leading edges), however, for longer distances. The difference can be explained by a short-time examination of evolution.

Conclusions. It has been established that a change in the wave trough significantly affects the evolution of the wave profile. It has been found out a difference in the evolution of the central and tail parts of the profile.

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