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NUMERICAL ANALYSIS OF THE EVOLUTION OF PLANE LONGITUDINAL NONLINEAR ELASTIC WAVES WITH DIFFERENT INITIAL PROFILES

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Nonlinear plane longitudinal elastic waves with different profiles are studied using the Murnaghan model. The novelty is that the waves are analyzed using the same approximate method and the solutions of the nonlinear wave equations are similar in form. The distortion of the initial wave profile described by cosinusoidal, Gaussian, and Whittaker functions is described theoretically and numerically. About 80 variants of initial parameters are studied numerically: three analytical representations of the initial profile, three materials (aluminum, copper, steel), three wave lengths, three initial maximum amplitudes. For each variant, four (cosine) and five (Gauss, Whittaker) two-dimensional graphs of wave shape versus traveled distance are plotted to demonstrate the distortion of the wave profile

Keywords: nonlinear plane longitudinal wave, approximate method, wave profiles, cosine, Gauss function, Whittaker function, initial wave profile, distortion

1. Introduction. Analytic Formulas for Numerical Analysis of the Evolution of a Plane Wave with Initial Profile Described by a Smooth Function.

Let the nonlinearity of the material in which a wave propagates be described by the Murnaghan potential [4, 15, 16]:

$$W(\varepsilon_{ik}) = (1/2)\lambda(\varepsilon_{mm})^2 + \mu(\varepsilon_{ik})^2 + (1/3)A\varepsilon_{ik}\varepsilon_{im}\varepsilon_{km} + B(\varepsilon_{ik})^2\varepsilon_{mm} + (1/3)C(\varepsilon_{mm})^3, \tag{1}$$

where ε_{ik} are the components of the nonlinear Cauchy–Green strain tensor

$$\varepsilon_{nm} = (1/2)(u_{n,m} + u_{m,n} + u^{k,n}u_{k,m}), \tag{2}$$

u_k are the components of the displacement vector; λ, μ, A, B, C are the elastic constants in the Murnaghan model.

Let us choose a model that keeps the second and third powers in the expression of the Murnaghan potential in terms of the displacement gradients:

$$W = (1/2)\lambda(u_{m,m})^2 + (1/4)\mu(u_{i,k} + u_{k,i})^2 + \left(\mu + (1/4)A\right)u_{i,k}u_{m,i}u_{m,k}$$

$$+(1/2)(\lambda + B)u_{m,m}(u_{i,k})^2 + (1/12)Au_{i,k}u_{k,m}u_{m,i} + (1/2)Bu_{i,k}u_{k,i}u_{m,m} + (1/3)C(u_{m,m})^3. \tag{3}$$

Expression (3) can be used to derive various nonlinear wave equations for analyzing harmonic (periodic) or solitary (acyclic) waves. Let us consider three plane longitudinal waves with different profiles: (i) profile described by a cosine function (harmonic wave), (ii) profile described by a Gauss function (solitary wave), (iii) profile described by a Whittaker function (solitary wave).

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Harmonic nonlinear elastic waves have been much studied [3, 7, 10, 15, 16, 18]. Solitary waves in materials have been actively studied in recent years [4, 5, 10, 11, 14]. A nonlinear elastic wave with cosine profile was studied in different ways by different scientists [3, 4, 12, 14]. Here we will analyze this wave using the approximate approach to compare to the curves of evolution of the initial wave profile plotted earlier and to solitary waves of complex profile. A wave with Gaussian profile (bell-shaped solitary wave) was earlier studied numerically on an IBM286 computer with very limited graphics capabilities [3]; therefore, we can only use the results of this study to compare the general trends in wave profile evolution. A wave with profile described by the Whittaker functions was studied using the mixture model [4]. Because of the significant difference of the models, the results of this study can only be used to compare the general trends in wave profile evolution.

Without loss of generality, we will analyze a plane wave in the case where the displacements $u_k = u_k(x_1, t)$ depend only on one space coordinate of time (displacements along the Ox_1-axis of the Cartesian coordinate system $Ox_1x_2x_3$). Then potential (3) becomes simpler:

$$W = (1/2)\left[(\lambda + 2\mu)(u_{1,1})^2 + \mu\left[(u_{2,1})^2 + (u_{3,1})^2\right]\right]$$

$$+\left[\mu + (1/2)\lambda + (1/3)A + B + (1/3)C\right](u_{1,1})^3 + (1/2)(\lambda + B)u_{1,1}\left[(u_{2,1})^2 + (u_{3,1})^2\right]. \tag{4}$$

From (4), we obtain the simplest nonlinear wave equations—quadratic nonlinear wave equations for three elastic polarized (longitudinal, transverse horizontal, transverse vertical) plane waves. The equation for a longitudinally polarized wave has the form

$$\rho u_{1,tt} - (\lambda + 2\mu)u_{1,11} = N_1 u_{1,11}u_{1,1} + N_2(u_{2,11}u_{2,1} + u_{3,11}u_{3,1}), \tag{5}$$

$$N_1 = \left[3(\lambda + 2\mu) + 2(A + 3B + C)\right], \quad N_2 = \lambda + 2\mu + (1/2)A + B. \tag{6}$$

We will restrict the analysis to the so-called first standard problem where a longitudinal wave is only generated [2] and the main nonlinear phenomenon is the self-generation of a wave. Then the nonlinear equation (5) becomes

$$\rho u_{1,tt} - (\lambda + 2\mu)u_{1,11} = N_1 u_{1,11}u_{1,1} \to u_{1,tt} - (c_L)^2 u_{1,11} = (N_1/\rho)u_{1,11}u_{1,1}, \tag{7}$$

where the velocity of a linear longitudinal plane wave is denoted by $c_L = \sqrt{(\lambda + 2\mu)/\rho}$.

2. Approximate Approach to the Analysis of the Evolution of the Initial Wave Profile Using the Nonlinear Equation (7).

Let us represent Eq. (7) in the form

$$u_{1,tt} - \{(c_L)^2 + (N_1/\rho)u_{1,1}\}u_{1,11} = 0 \to u_{1,tt} - \{1 + \alpha u_{1,1}\}(c_L)^2 u_{1,11} = 0, \tag{8}$$

$$\alpha = \left[N_1/(\lambda + 2\mu)\right]. \tag{9}$$

Let us consider a D'Alembert wave with initial profile described by a sufficiently smooth function $u(x_1, t = 0) = F(x_1)$:

$$u(x_1, t) = F(x_1 - vt), \tag{10}$$

where the velocity of the wave is defined by

$$v = c_L\sqrt{1 + \alpha u_{1,1}}. \tag{11}$$

We will consider the following three variants of the function $F(x_1)$:

(i) $F(x_1) = \cos k_L x_1$ or $F(x_1) = e^{-ik_L x_1}$ (harmonic wave),

(ii) $F(x_1) = e^{-((ax_1)^2/2)}$ (solitary wave),

(iii) $F(x_1) = W_{5/4,1/4}(ax_1)$ (solitary wave), where k_L determines the length of the harmonic wave and a determines the "length" of the solitary wave.

Assume that

$$|\alpha u_{1,1}| << 1, \tag{12}$$

which allows expansion of the root in (11) into a series:

$$\sqrt{1+\alpha u_{1,1}} = (1+\alpha u_{1,1})^{1/2} = 1+(1/2)\alpha u_{1,1} - (1/8)(\alpha u_{1,1})^2 + \ldots$$

and an approximate representation of solution (12):

$$u(x_1, t) \cong F\left[x_1 - c_L t - (1/2)\alpha u_{1,1} t\right]. \tag{13}$$

The accuracy of (13) depends on how accurately condition (12) is satisfied, which includes constraints for two parameters: $\alpha = 3+2(A+3B+C)/(\lambda+2\mu)$ and $u_{1,1}$. Let us restrict the analysis to the class of structural materials to which the Murnaghan model can safely be applied and that are soft-nonlinear (i.e., $A+3B+C$ is always negative). For example, the constant α is approximately equal to –9, –4, and –8 for aluminum, copper, and steel, respectively. For many metallic materials, $2 < \alpha < 19$ [1]. Hence, the displacement gradient $u_{1,1}$ can only be considered small. Then it is sufficient to make the assumption, popular in the theory of elasticity, that the strains are small:

$$u_{1,1} << 1 \tag{14}$$

(small displacement gradient). This is consistent with the well-known fact that the Murnaghan potential describes nonlinear deformation assuming small strains. Next it is necessary to define an acceptable error of formula (11). Let it be 0.1%. Based on $\sqrt{1+0.094} = 1.00459\ldots$, $\sqrt{1-0.094} = 0.9518\ldots$, $\sqrt{1\pm 0.094} \approx 1 \pm (1/2)\times 0.094 = 1.0047,\ 0.953$, an accuracy of 0.1% is reached when the product $|\alpha u_{1,1}| \le 0.094$ is maximum. Hence, at an accuracy of 0.1%, constraints (12) and (14) can be corrected: $|\alpha u_{1,1}| \le 0.001$, $|u_{1,1}| \le 0.0005 = 5\cdot 10^{-4}$.

When the profile of a wave is known, constraint (14) can be assigned a geometrical meaning: $u_{1,1}$ can be considered as a tangent to the wave profile and its smallness corresponds to the smallness of the tangent of the angle between the tangent and the abscissa axis. The tangent of an angle is equal to the angle when $0 < \gamma < 5° \equiv 0.0873$.

Then constraint (14) means that the wave profile is such that the wavelength exceeds the maximum amplitude by one to two orders of magnitude. This constraint should be tested in each specific case.

Denoting the phase of a wave with constant phase velocity by $\sigma = a(x_1 - c_L t)$ and the additional small parameter by $\delta = -(1/2)\alpha a c_L u_{1,1} t$, we expand solution (13) into a Taylor series:

$$u(x_1, t) \approx F(\sigma+\delta) \approx F(\sigma) + F'(\sigma)\delta + (1/2)F''(\sigma)\delta^2 + \ldots\ . \tag{15}$$

Let us keep the first two terms in (15), assuming that δ is small, i.e., $|\delta| = |-(1/2)\alpha a c_L u_{1,1} t| < 1$ (since the smallness of $|\alpha u_{1,1}|$ was already assumed in (12), this inequality actually restricts $a c_L t$, i.e., the class of structural materials: the typical size a (length) of the wave must be much less than the distance (from 1 to 10 m) traveled by the wave). Since the equality below follows from (15):

$$u_{1,1}(x_1, t) \approx F'_\sigma(\sigma+\delta)\cdot\sigma'_{x_1} = F'_\sigma(\sigma)a + F''_{\sigma\sigma}(\sigma)\left(-(1/2)\alpha a^2 c_L u_{1,11} t\right) \approx F'_\sigma(\sigma),$$

solution (15) can be represented as

$$u(x_1, t) \approx F(\sigma) + F'(\sigma)a\left[\delta = -(1/2)\alpha a c_L u_{1,1} t\right] = F(\sigma) - (1/2)\alpha a^2 c_L t[F'(\sigma)]^2. \tag{16}$$

The approximate solution (16) is general and describes the same nonlinear wave effect irrespective of the functions chosen: occurrence of the second harmonic or the like and increase in the amplitude of the second harmonic with time.

Formula (16) allows describing the evolution of a solitary wave of any profile described by a function with finite weight whose derivative can be expressed analytically.

3. Parameters of the Material and Wave in Numerical Simulation. We choose three structural materials with the following parameters in the Murnaghan model (SI system) [18]:

Aluminum: $\rho = 2.7\cdot10^3$, $\lambda = 5.2\cdot10^{10}$, $\mu = 2.7\cdot10^{10}$, $A = -0.65\cdot10^{10}$, $B = -2.05\cdot10^{11}$, $C = -3.7\cdot10^{11}$, $c_L = 6.27\cdot10^3$, $\alpha = -16.81$;
Copper: $\rho = 8.93\cdot10^3$, $\lambda = 10.7\cdot10^{10}$, $\mu = 4.8\cdot10^{10}$, $A = -2.8\cdot10^{11}$, $B = -1.72\cdot10^{11}$, $C = -2.4\cdot10^{11}$, $c_L = 4.77\cdot10^3$, $\alpha = -7.207$;
Steel: $\rho = 7.8\cdot10^3$, $\lambda = 9.4\cdot10^{10}$, $\mu = 7.9\cdot10^{10}$, $A = -3.25\cdot10^{11}$, $B = -3.1\cdot10^{11}$, $C = -8{,}0\times10^{11}$, $c_L = 5.68\cdot10^3$, $\alpha = -13.31$.

The parameters of the harmonic wave are the following: initial frequency ω determined from the given velocity $c_L = (\omega / k_L)$ and wavelength $L = (2\pi / k_L)$ determined from the given wave number $k_L = (\omega / c_L)$ (for each material);

$\omega = 1.6\cdot10^5$, $L = 0.246$ Aluminum; $L = 0.187$ Copper; $L = 0.233$ Steel;
$\omega = 3.2\cdot10^5$, $L = 0.246$ Aluminum; $L = 0.187$ Copper; $L = 0.233$ Steel;
$\omega = 4.8\cdot10^5$, $L = 0.246$ Aluminum; $L = 0.187$ Copper; $L = 0.233$ Steel.

For solitary waves described by Gauss and Whittaker functions (which are of finite weight), we assume that their length L is the interval (distance) beyond which the area under the wave profile is negligibly small.

Then, according to the 3σ-rule, the length of the Gaussian (bell-shaped) wave $e^{-(x^2/2\sigma^2)} = e^{-[(x/\sigma)^2/2]}$ is equal to 6σ. Therefore, the parameter a in $F(x_1) = e^{-((ax_1)^2/2)}$ is the wave length defined by the formula $\sigma = (1/a)$. For all the three materials, the initial wave length is the same identical, and three variants are considered: $L = \{0.09, 0.15, 0.20\}$ ($a = \{167, 40, 25.9\}$).

The Whittaker wave $W_{5/4,1/4}(x)$ has length $L = 20$. Therefore, the parameter a in $F(x_1) = W_{5/4,1/4}(ax_1)$ is the wave length defined by the formula $\sigma = (1/a)$. For all the three materials, the initial wave length is the same identical $L = 0.20$ ($a = 50$), and three variants are considered: $L = \{0.143, 0.20, 0.25\}$ ($a = \{70, 50, 40\}$).

Three initial amplitudes were selected for each material and for each initial wave length: $a^o = 5.0\cdot10^{-5}$, $7.5\cdot10^{-5}$, $1.0\cdot10^{-4}$ (for harmonic and bell-shaped profiles) and $a^o = 1.0\cdot10^{-4}$, $6.0\cdot10^{-3}$, $1.5\cdot10^{-3}$ (for Whittaker profile).

Thus, 81 variants of waves (three materials, three types of profile, three wavelengths, three initial amplitudes) were analyzed and about 400 two-dimensional evolution curves were plotted.

4. Numerical Analysis of the Harmonic Wave. The initial wave profile is described by $F(x_1) = e^{-ik_L x_1}$, and formula (16) takes the form

$$u_1(x_1, t) = a^o e^{-ik_L(x_1 - c_L t)} - (1/2)c_L t\alpha(k_L)^2 (a^o)^2 e^{-2ik_L(x_1 - c_L t)}. \tag{17}$$

The approximate solution (17) is identical to the solutions of the nonlinear wave equation (9) (up to a constant factor, which has no qualitative effect on the evolution of the wave) obtained by the method of successive approximations [3, 4, 16]:

$$u_1(x_1, t) = u_{1o}\cos(k_L x_1 - \omega t) + (1/8\rho)x_1\alpha(k_L)^2 (u_{1o})^2 \cos 2(k_L x_1 - \omega t). \tag{18}$$

Formula (17) was used to plot two-dimensional curves of displacement u_1 versus traveled distance x_1. There were 27 sets of plots (three materials, three wavelengths, three maximum initial amplitudes). Each set includes four graphs of a wave profile for different distances: from the initial position of the wave to the position at a distance of approximately 20 wavelengths where the effect of nonlinearity and the distortion of the wave profile are significant. The fifth graph shows the shape on only the second term in (18).

Figure 1 shows the curves of one set for the following parameters: aluminum, $L = 0.246$, $k_L = 1.2\cdot10^{-2}$, $a_o = 5.0\cdot10^{-5}$. It can be seen that the initial wave profile evolves asymmetrically: the positive portion of the profile as if keeps the shape of the first harmonic, whereas the negative portion tends to transform into the profile of the second harmonic.

The curves in Fig. 1 corresponding to formula (17) and the curves in [3, 4, 16] corresponding to formula (18) are in qualitative agreement. Hence, formula (17) is applicable to a harmonic wave. This allows us to assume that the formulas for the other profiles derived from the general formula (16) are applicable as well.

5. Numerical Analysis of the Gaussian Wave. The initial wave profile is described by $F(x_1) = e^{-((ax_1)^2/2)}$, and formula (16) takes the form

$$u_1(x_1, t) = A^o e^{-[a^2(x_1 - c_L t)^2/2]} - (1/2)t\alpha c_L a^2 (x_1 - c_L t)^2 (A^o)^2 e^{-a^2(x_1 - c_L t)^2}. \tag{19}$$

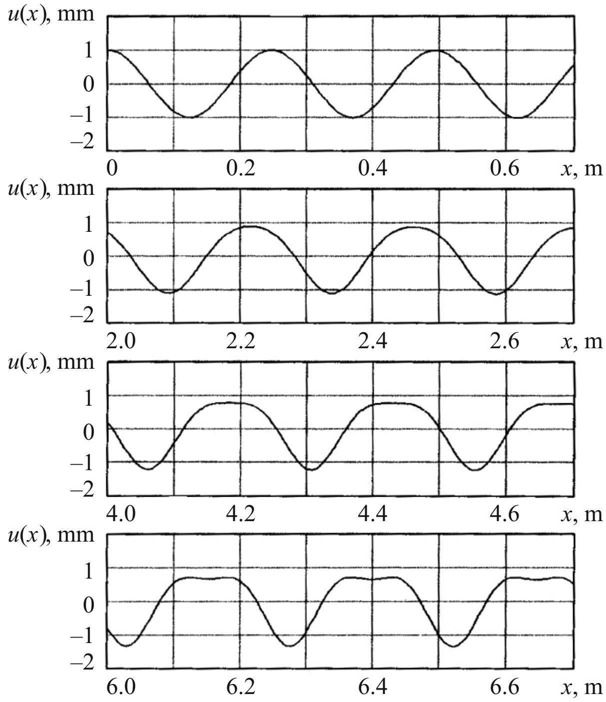

Fig. 1

The concepts of first and second harmonics are inapplicable to profile (19), and the functions $e^{-[a^2(x_1-c_L t)^2/2]}$ and $e^{-a^2(x_1-c_L t)^2}$ can be considered the first and second harmonics very approximately. However, the approximate solution (19) is very similar to solution (19). The obvious difference between solutions (17) and (19) is in that the nonlinear term does not explicitly depend on the wave phase $\sigma = k_L x_1 - \omega t$ for wave (17), whereas the squared wave phase $\sigma = a(x_1 - c_L t)$ appears explicitly in the expression for the amplitude of wave (19).

Formula (19) was used to plot two-dimensional curves of displacement u_1 versus traveled distance x_1. There were 27 sets of plots (three materials, three wavelengths, three maximum initial amplitudes). Each set includes five graphs of a wave profile for different distances: four curves of evolution from the initial position of the wave to the position at a distance of approximately 80 wavelengths where the effect of nonlinearity and the distortion of the wave profile are significant, and one curve for the second (nonlinear) component showing the effect of nonlinearity.

Figure 2 shows the curves of one set for the following parameters: aluminum, $L = 0.15$, $a = 40$, $a_o = 5.0 \cdot 10^{-3}$.

It can be seen that the initial wave profile, which is symmetric, evolves symmetrically: the maximum amplitude slowly increases and one hump tends to transform into two humps (which is demonstrated by the fifth curve in Fig. 2) and the middle portion of the initial profile widens, i.e., the profile as if bloats, but its length remains constant.

Thus, by allowing for the nonlinearity in analyzing the propagation of a Gaussian solitary wave, we can describe the distortion of its profile.

6. Numerical Analysis of the Whittaker Wave. Let the initial wave profile be described by $F(x_1) = W_{5/4,3/4}(ax_1)$. Then formula (16) takes the form

$$u_1(x_1,t) = a^o W_{5/4,3/4}(ax_1) - (1/8)t\alpha(k_L)^2(a^o)^2\left(W'_{5/4,3/4}(x_1)\right)^2.$$

Since the function $W_{\kappa,\mu}(z)$ is differentiated as follows [8, 13]:

$$\frac{d}{dz}W_{\lambda,\mu}(z) = \left(\frac{\lambda}{z} - \frac{1}{2}\right)W_{\lambda,\mu}(z) - \frac{1}{z}\left[\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right]W_{\lambda-1,\mu}(z)$$

and the first derivative of $W_{5/4,3/4}(x_1)$ is

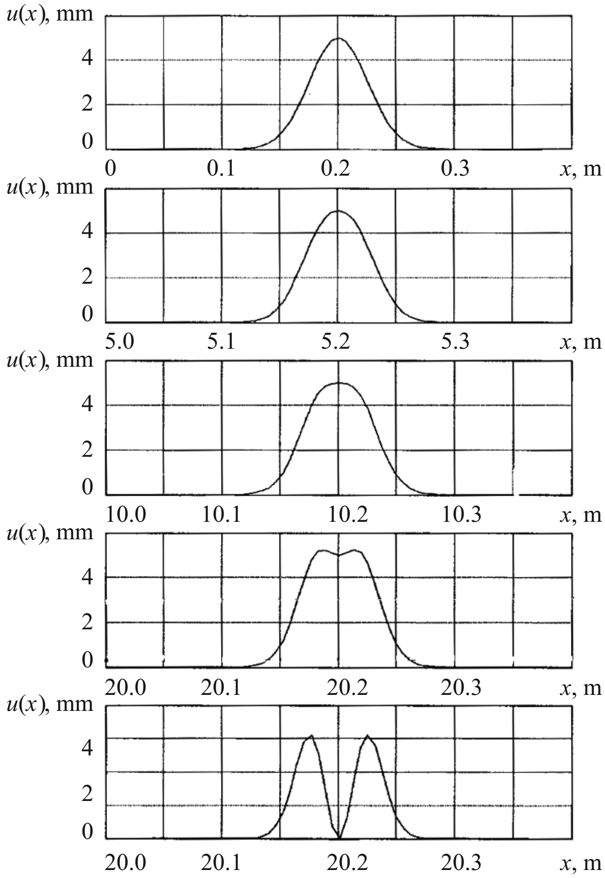

Fig. 2

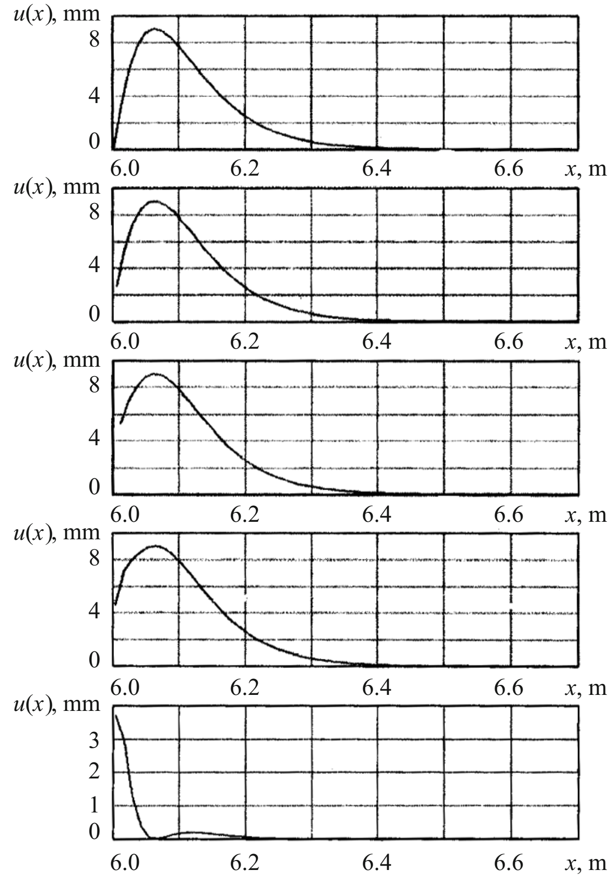

Fig. 3

$$(W_{5/4,3/4}(\sigma))' = \left(\frac{5}{4\sigma} - \frac{1}{2}\right) W_{5/4,3/4}(\sigma),$$

solution (17) takes the form

$$u_1(x_1, t) = a^o W_{5/4,3/4}\big(a(x_1 - c_L t)\big) - (1/2) t\alpha c_L (a)^2 (a^o)^2 \big(W'_{5/4,3/4}(x_1)\big)^2, \tag{20}$$

$$u_1(x_1, t) = a^o W_{5/4,3/4}\big(a(x_1 - c_L t)\big) - (1/2) t\alpha c_L (a)^2 (a^o)^2 \left(\left(\frac{5}{4a(x_1 - c_L t)} - \frac{1}{2}\right) W_{5/4,3/4}\big(a(x_1 - c_L t)\big)\right)^2.$$

Solution (20) describes the change in the solitary wave profile (due to the time dependence of the nonlinear component) and the spreading of the wave profile (due to the presence of the nonlinear component).

Formula (20) was used to plot two-dimensional curves of wave shape u_1 versus traveled distance x_1. There were 27 sets of plots (three materials, three wavelengths, three maximum initial amplitudes). Each set includes five graphs of a wave profile for different distances: four curves of evolution from the initial position of the wave to the position at a distance of approximately 40 wavelengths where the effect of nonlinearity and the distortion of the wave profile are significant, and one curve for the second (nonlinear) component showing the effect of nonlinearity.

Figure 3 shows the curves of one set for the following parameters: aluminum, $a = 1.2 \cdot 10^{-2}$, $a_o = 1.0 \cdot 10^{-5}$. It can be seen that the asymmetric wave profile evolves in three different ways: the maximum amplitude slowly increases, the left portion of the profile becomes shallower, and the right portion becomes shallower much faster. The wave length corresponds to the length of the nonlinear term (which is demonstrated by the fifth curve in Fig. 3), the middle portion of the wave profile as if bloats, but its length remains constant. Thus, by allowing for the nonlinearity in analyzing the propagation of a Whittaker solitary wave, we can describe the distortion of its profile.

Conclusions. Harmonic, symmetric solitary, and asymmetric solitary nonlinear elastic longitudinal plane waves $u(x_1, t)$ have been analyzed numerically. The wave profiles are described by trigonometric function $\cos x_1$ (e^{ix_1}), Gaussian function $e^{-x_1^2/2}$, and Whittaker function $W_{5/4,3/4}(x_1)$, respectively. What all the three wave profiles have in common is that they distort during propagation due to the nonlinear interaction of the wave with itself. However, these wave profiles distort differently. The harmonic wave does not originally change its length and tends to form two humps instead of one, which can result in transformation of the first harmonic into the second harmonic and decrease in the wavelength to half the initial value. The bell-shaped solitary wave remains symmetric. Like the harmonic wave, this wave does not originally change its length and tends to form two humps instead of one, which can lead to formation of a two-hump profile (the fifth curve in Fig. 2) with the same length as that of the single-hump profile. The middle portion of the bell spreads, the profile as if bloats, but its length remains constant. The asymmetric Whittaker wave profile evolves in three different ways: the maximum amplitude slowly increases, the left portion of the profile becomes shallower, and the right portion becomes shallower much faster. The wavelength does not change at the initial stage (the fifth curve in Fig. 3); the middle portion of the profile spreads, the profile itself as if bloats, but its length remains constant.

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