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NONSTATIONARY VIBRATIONS OF TRANSVERSELY REINFORCED ELLIPTIC CYLINDRICAL SHELLS ON AN ELASTIC FOUNDATION

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The forced vibrations of transversely reinforced elliptic cylindrical shells on an elastic foundation under nonstationary loads are studied using the Timoshenko-type theory of shells and rods. A numerical algorithm for solving problems of this class is developed. A numerical example for the case of distributed impulsive loading is given

Keywords: transversely reinforced cylindrical shell, elastic foundation, Timoshenko-type theory of shells and rods, nonstationary process, numerical method

Introduction. The dynamic behavior of a reinforced shell on an elastic foundation can be studied by solving two problems: (a) influence of the elastic foundation on plates and shells without reinforcement [4, 10–12, 15, 16, 19] and (b) influence of reinforcement on the inhomogeneous structure [3, 8, 9, 14, 16]. The influence of both elastic medium and reinforcement on the stress–strain state of inhomogeneous structures is addressed in [1, 3, 4, 13, 15, 17, 18].

The dynamic behavior of discretely reinforced cylindrical and spherical shells as a special case of reinforced shells of elliptic cross-section on an elastic foundation, was studied in [3].

Here we will solve the problem of the forced vibrations of a discretely reinforced elliptic cylindrical shell on a Pasternak two-parameter foundation under a distributed impulsive load. The dynamic behavior of the reinforced inhomogeneous shell will be analyzed using a geometrically linear theory of shells and rods and the Timoshenko hypotheses. The problem posed will be solved with the finite-difference method [3, 6]. Numerical results will be obtained depending on the geometrical and mechanical parameters of the structure and the elastic foundation.

1. Problem Statement. Basic Equations. Consider a reinforced elliptic cylindrical shell under a distributed internal load $P_3(s_1, s_2, t)$, where s_1, s_2, and t are the space and time coordinates. The coefficients of the first quadratic form and the curvatures of the coordinate surface of the shell are given by

$$A_1 = 1, \quad k_1 = 0,$$

$$A_2 = (a^2 \cos^2 \alpha_2 + b^2 \sin^2 \alpha_2)^{1/2},$$

$$k_2 = ab(a^2 \cos^2 \alpha_2 + b^2 \sin^2 \alpha_2)^{-3/2}, \tag{1.1}$$

where a and b are the semiaxes of the elliptical cross-section of the cylindrical shell.

To derive the vibration equations for a reinforced cylindrical shell on an elastic foundation, we will use the Hamilton–Ostrogradsky variational principle [1, 3, 5]. Performing standard transformations over the variational functional, we obtain two groups of equations:

(i) the equations of vibration of a smooth elliptic cylindrical shell

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$$\frac{\partial T_{11}}{\partial s_1}+\frac{\partial S}{\partial s_2}=\rho h\frac{\partial^2 u_1}{\partial t^2},\qquad \frac{\partial S}{\partial s_1}+\frac{\partial T_{22}}{\partial s_2}+k_2T_{23}=\rho h\frac{\partial^2 u_2}{\partial t^2},$$

$$C_2\left(\frac{\partial^2 u_3}{\partial s_1}+\frac{\partial^2 u_3}{\partial s_2}\right)+\frac{\partial T_{13}}{\partial s_1}+\frac{\partial T_{23}}{\partial s_2}-k_2T_{22}-C_1u_3+P_3(s_1,s_2,t)=\rho h\frac{\partial^2 u_3}{\partial t^2},$$

$$\frac{\partial M_{11}}{\partial s_1}+\frac{\partial H}{\partial s_2}-T_{13}=\rho\frac{h^3}{12}\frac{\partial^2 \varphi_1}{\partial t^2},\qquad \frac{\partial H}{\partial s_1}+\frac{\partial M_{11}}{\partial s_2}-T_{23}=\rho\frac{h^3}{12}\frac{\partial^2 \varphi_2}{\partial t^2},\tag{1.2}$$

(ii) the equations of vibration of the *j*th transverse rib aligned with the s_2-axis

$$\frac{\partial T_{21j}}{\partial s_2}+[T_{11}]_j=\rho_jF_j\left(\frac{\partial^2 u_1}{\partial t^2}\pm h_{cj}\frac{\partial^2 \varphi_1}{\partial t^2}\right),$$

$$\frac{\partial T_{22j}}{\partial s_2}+k_{2j}T_{23j}+[S]_j=\rho_jF_j\left(\frac{\partial^2 u_2}{\partial t^2}\pm h_{cj}\frac{\partial^2 \varphi_2}{\partial t^2}\right),$$

$$\frac{\partial T_{23j}}{\partial s_2}+k_2T_{22}+[T_{13}]_j=\rho_jF_j\frac{\partial^2 u_3}{\partial t^2},$$

$$\frac{\partial M_{21j}}{\partial s_2}\pm h_{cj}\frac{\partial T_{21j}}{\partial s_2}+[M_{11}]=\rho_jF_j\left[\pm h_{cj}\frac{\partial^2 u_1}{\partial t^2}+\left(h_{cj}^2+\frac{I_{torj}}{F_j}\right)\frac{\partial^2 \varphi_1}{\partial t^2}\right],$$

$$\frac{\partial M_{22j}}{\partial s_2}\pm h_{cj}\frac{\partial T_{22j}}{\partial s_2}-T_{23j}+[H]=\rho_jF_j\left[\pm h_{cj}\frac{\partial^2 u_1}{\partial t^2}+\left(h_{cj}^2+\frac{I_{2j}}{F_j}\right)\frac{\partial^2 \varphi_2}{\partial t^2}\right],\tag{1.3}$$

where $u_1, u_2, u_3, \varphi_1, \varphi_2$ are the components of the generalized displacement vector of the mid-surface; ρ and ρ_j are the densities of the shell and the *j*th rib, respectively; *h* is the thickness of the shell; $h_{cj}=0.5(h+h_j)$; h_j is the cross-sectional height of the *j*th rib; $[f]_i=f^+-f^-$, where $f^\pm$ are the values of the functions on the right and on the left of the *j*th discontinuity line (projection of the center of gravity of the *j*th rib onto the mid-surface of the shell); C_1 is the Winkler coefficient of subgrade reaction; C_2 is the Pasternak coefficient of subgrade reaction.

The forces/moments in the vibration equations (1.2) for the shell are related to the strains by

$$T_{11}=B_{11}(\varepsilon_{11}+\nu_2\varepsilon_{22}),\qquad T_{22}=B_{22}(\varepsilon_{22}+\nu_1\varepsilon_{11}),$$

$$M_{11}=D_{11}(\kappa_{11}+\nu_2\kappa_{22}),\qquad M_{22}=D_{22}(\kappa_{22}+\nu_1\kappa_{11}),\qquad H=D_{12}\kappa_{12},$$

$$\varepsilon_{11}=\frac{\partial u_1}{\partial s_1},\qquad \varepsilon_{22}=\frac{\partial u_2}{\partial s_2}+k_2u_3,$$

$$\varepsilon_{12}=\frac{\partial u_1}{\partial s_2}+\frac{\partial u_2}{\partial s_1},\qquad \varepsilon_{13}=\varphi_1+\frac{\partial u_3}{\partial s_1},\qquad \varepsilon_{23}=\varphi_2+\frac{\partial u_3}{\partial s_2}-k_2u_2,$$

$$\kappa_{11}=\frac{\partial \varphi_1}{\partial s_1},\qquad \kappa_{22}=\frac{\partial \varphi_2}{\partial s_2},\qquad \kappa_{12}=\frac{\partial \varphi_1}{\partial s_2}+\frac{\partial \varphi_2}{\partial s_1},\tag{1.4}$$

where

$$B_{11} = \frac{E_1 h}{1-\nu_1\nu_2}, \quad B_{22} = \frac{E_2 h}{1-\nu_1\nu_2}, \quad B_{12} G_{12} h, \quad B_{13} = G_{13} h, \quad B_{23} = G_{23} h,$$

$$D_{11} = \frac{E_1 h^3}{12(1-\nu_1\nu_2)}, \quad D_{22} = \frac{E_2 h^3}{12(1-\nu_1\nu_2)}, \quad D_{12} = G_{12}\frac{h^3}{12},$$

where $E_1, E_2, G_{12}, G_{13}, G_{23}, \nu_1, \nu_2$ are the mechanical parameters of the orthotropic material of the shell.

The forces/moments in the vibration equations (1.3) for the jth rib are related to the strains by

$$T_{22j} = E_j F_j \varepsilon_{22j}, \quad T_{12j} = G_j F_j \varepsilon_{12j}, \quad T_{13j} = G_j F_j \varepsilon_{13j},$$

$$M_{22j} = E_j I_{1j} \kappa_{22j}, \quad M_{12j} = G_j I_{\text{tor}j} \kappa_{12j},$$

$$\varepsilon_{22j} = \frac{\partial u_2}{\partial s_2} \pm h_{cj}\frac{\partial \varphi_2}{\partial s_2}, \quad \varepsilon_{12j} = \frac{\partial u_1}{\partial s_2} \pm h_{cj}\frac{\partial \varphi_1}{\partial s_2},$$

$$\varepsilon_{23j} = \varphi_2 + \frac{\partial u_3}{\partial s_2} - k_{2j} u_2, \quad \kappa_{22j} = \frac{\partial \varphi_2}{\partial s_2}, \quad \kappa_{12j} = \frac{\partial \varphi_1}{\partial s_2}, \tag{1.5}$$

where E_j and G_j are the mechanical parameters of the material of the jth rib; $F_j, I_{1j}, I_{\text{tor}j}$ are the geometrical parameters of the cross-section of the jth rib.

The vibration equations (1.2)–(1.5) are supplemented with appropriate boundary and initial conditions.

2. Numerical Problem-Solving Algorithm. The numerical algorithm for solving the initial–boundary-value problem (1.2)–(1.5) is based on the integro-interpolation method for the construction of difference schemes with respect to the space coordinates s_1 and s_2 and an explicit approximation with respect to the time coordinate t [3, 6].

Let us consider a domain $D = \{s_{10} \le s_1 \le s_{1N}, s_{20} \le s_2 \le s_{2N}\}$. Choosing a subdomain $D^1_{kl} \subset D$, $D^1_{kl} = \{s_{1k-1/2} \le s_1 \le s_{1k+1/2}, s_{2l-1/2} \le s_2 \le s_{2l+1/2}\}$ and integrating the vibration equations (1.2) over this subdomain, we obtain the following difference formulas for finding the solution at the $(n+1)$th time step:

$$\frac{T^n_{11k+1/2,l} - T^n_{11k-1/2,l}}{\Delta s_1} + \frac{S^n_{k,l+1/2} - S^n_{k,l-1/2}}{\Delta s_2} = \rho h (u^n_{1k,l})_{\bar{t}t},$$

$$\frac{S^n_{k,l+1/2} - S^n_{k,l-1/2}}{\Delta s_1} + \frac{T^n_{22k+1/2,l} - T^n_{22k-1/2,l}}{\Delta s_2} + \frac{k_{2k,l}}{2}(T^n_{23k,l-1/2} + T^n_{23k,l+1/2}) = \rho h (u^n_{2k,l})_{\bar{t}t},$$

$$\frac{T^n_{13k+1/2,l} - T^n_{13k-1/2,l}}{\Delta s_1} + \frac{T^n_{22k,l+1/2} - T^n_{22k,l-1/2}}{\Delta s_2} + \frac{k_{2k,l}}{2}(T^n_{22k,l-1/2} + T^n_{22k,l+1/2}) + P^n_{3k,l} = \rho h (u^n_{3k,l})_{\bar{t}t},$$

$$\frac{M^n_{11k+1/2,l} - M^n_{11k-1/2,l}}{\Delta s_1} + \frac{H^n_{k,l+1/2} - H^n_{k,l-1/2}}{\Delta s_2} - \frac{1}{2}(T^n_{13k+1/2,l} + T^n_{13k-1/2,l}) = \rho\frac{h^3}{12}(\varphi^n_{1k,l})_{\bar{t}t},$$

$$\frac{H^n_{k+1/2,l} - H^n_{k-1/2,l}}{\Delta s_1} + \frac{M^n_{22k,l+1/2} - M^n_{22k,l-1/2}}{\Delta s_2} - \frac{1}{2}(T^n_{23k,l+1/2} + T^n_{23k,l-1/2}) = \rho\frac{h^3}{12}(\varphi^n_{2k,l})_{\bar{t}t}. \tag{2.1}$$

Thus, in the difference equations, the generalized displacements $u_1, u_2, u_3, \varphi_1, \varphi_2$ are referred to the integer nodes of the difference mesh, while the forces/moments (and strains) to half-integer nodes $(k \pm 1/2, l)$, $(k, l \pm 1/2)$. To derive the generalized difference equations for the forces/moments, Eqs. (1.4) are integrated over the following domains:

$$D^2_{kl} = \{s_{1k-1} \le s_1 \le s_{1k}, s_{2l-1/2} \le s_2 \le s_{2l+1/2}\},$$

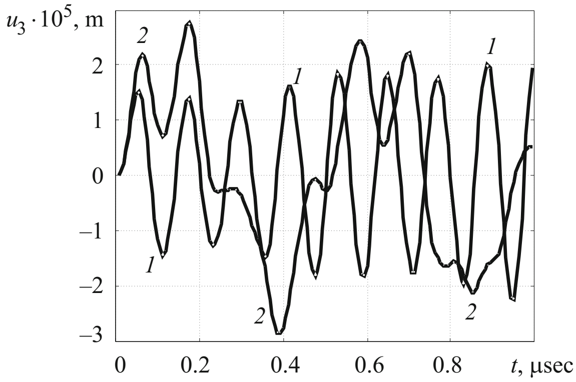

Fig. 1

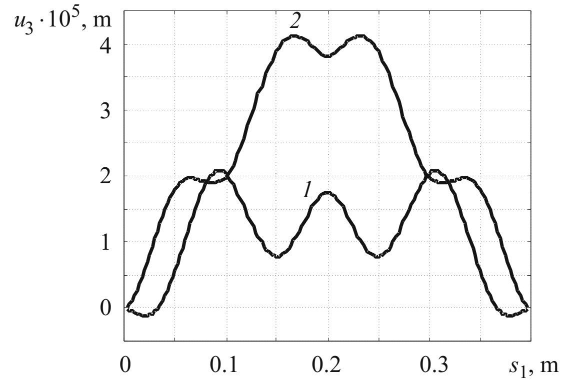

Fig. 2

$$D_{kl}^2 = \{s_{1k} \le s_1 \le s_{1k+1}, s_{2l-1/2} \le s_2 \le s_{2l+1/2}\}$$

etc. In (2.1), difference derivatives are denoted as in [3]. The difference formulas for the vibration equations (1.3), (1.5) for the *j*th rib are derived in a similar way [3, 6].

3. Numerical Results. Let us consider, as a numerical example, the dynamic behavior of a transversely reinforced elliptic cylindrical shell under a distributed internal impulsive load. All sides of the shell are clamped.

The distributed impulsive load $P_3(s_1, s_2, t)$ is defined by

$$P_3(s_1, s_2, t) = A \cdot \sin\frac{\pi t}{T}[\eta(t) - \eta(t - T)],$$

where A is the amplitude of the load; T is the duration of the load. In the example: $A = 10^6$ Pa and $T = 50$ μsec.

The geometrical and mechanical parameters of the shell: $E_1 = E_2 = 7 \cdot 10^{10}$ Pa, $\nu_1 = \nu_2 = 0.3$, $h = 10^{-2}$ m, $L = 0.4$ m. The aspect ratio of the elliptic cross-section: (i) $a = b = 0.1$; (ii) $a = 1.2b$. The domain $D = \left\{0 \le s_1 \le L,\ 0 \le s_2 \le A_2 \frac{\pi}{8}\right\}$ and the time interval $0 \le t \le 80T$ were used in the numerical analysis. Transverse ribs are located in the sections $s_{1j} = 0.25jL$, $j = \overline{1,3}$. The Winkler coefficients: $C_1^1 = 10^9$ N/m^3, $C_1^2 = 2 \cdot 10^9$ N/m^3, $C_1^3 = 3 \cdot 10^9$ N/m^3.

Figures 1 and 2 show numerical values of u_3 for C_1^1. Curves *1* correspond to $a/b = 1$, while curves *2* to $a/b = 1.2$. Figure 1 shows the variation in u_3 with time t at the point $s_1 = 0.375L$, $s_2 = 0$ (the middle between the first and second ribs in the section $s_2 = 0$).

It can be seen that the ellipticity of the cross-section of the shell is the reason why its amplitude and frequency are so much different from those of the shell of circular-section.

Figure 2 shows the variation in u_3 with the coordinate s_1 in the section $s_2 = 0$ at $t = 3.5T$ (the instant the u_3 reaches its maximum when $a/b = 1$).

The considerable difference between the values of u_3 in the two cases is analyzed in Fig. 1. The calculated results for C_1^2 and C_1^3 are similar qualitatively and quantitatively.

Conclusions. The forced vibrations of discretely reinforced elliptic cylindrical shells on an elastic foundation under a distributed impulsive load have been studied using the Timoshenko-type theory of shells and rods.

The problem posed has been solved with the finite-difference method.

Numerical results have been presented. The influence of the elastic foundation on the stress–strain state of the reinforced structure has been analyzed.

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