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BIFURCATION PROCESSES IN A PHYSICAL MODEL

N. V. Nikitina[1] and V. N. Sidorets[2]

The bifurcations in a three-dimensional system that models an electric circuit with an arc are analyzed. A qualitative analysis is made of the limit cycles, strange attractor, and the fixed point approached by the representative point

Keywords: bifurcation, limit cycle, chaos

1. Introduction. Problem Statement. The birth of attractors in three-dimensional systems was discussed in [2, 5–12]. The presence of an attractor in a dynamic three-dimensional system suggests the existence of motions with complex trajectories. The coexistence of homoclinic and periodic trajectories was analyzed in [4]. A nonperiodic trajectory of a dynamic system is called homoclinic if the α-limit and ω-limit sets of the trajectory coincide and are a saddle cycle [4]. Three-dimensional attractors play an important role in bifurcation theory.

The present study is related to the monograph [3] where an electric circuit with an arc was modeled mathematically and the equations of motion were analyzed qualitatively. Our goal here is to develop a method for the qualitative analysis of bifurcation processes and to apply it to the study of an electric circuit with an arc.

Consider the system

$$\frac{dx}{dt} = F(x), \quad x(t) \in R^m, \tag{1}$$

where $F(x)$ is a smooth function, $m = 3$. Introduce a small deviation δx_i $(i = 1, 2, \ldots, m)$ in the neighborhood of the partial solutions $\bar{x}_i(t)$ of Eqs. (1) $\delta x_i = x_i(t) - \bar{x}_i(t)$ $(i = 1, 2, \ldots, m)$. Let δx_i be new coordinates. The linear system corresponding to system (1) in the coordinates δx_i

$$d\delta x / dt = A(\bar{x})\,\delta x, \quad \delta x \in R^m, \tag{2}$$

where $A(\bar{x}) = \partial F / \partial x|_{x=\bar{x}}$, is called a system of variational equations [11]. By analyzing the roots of the characteristic equation of the matrix $A(\bar{x})$, we can gain an insight into the mechanism of formation of an attractor. It is also possible to explain the occurrence of a multiple period in a regular attractor and the transformation of a regular attractor into a strange one in terms of a bifurcation process.

We make the following assumptions on system (1).

Assumption 1. System (1) has two singular points. The saddle-focus has characteristic exponents $\operatorname{Re}\lambda_1 > 0$, $\operatorname{Re}\lambda_2 > 0$, $\lambda_3 < 0$ and saddle value $\sigma = \operatorname{Re}\lambda_1 + \operatorname{Re}\lambda_2 + \lambda_3 < 0$. The saddle-node has characteristic exponents $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0$ and $\lambda_1 + \lambda_2 + \lambda_3 < 0$.

Assumption 2. On one coordinate plane and planes parallel to it, system (1) has a circular trajectory around a singular point, representing damped oscillations. On the other two planes and planes parallel to them, the trajectory does not go to ∞.

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[1]S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterova St., Kyiv, Ukraine 03057, e-mail: center@inmech.kiev.ua. [2]E. O. Paton Institute of Electric Welding, National Academy of Sciences of Ukraine, 11 Bozhenko St., Kyiv, Ukraine 03680, e-mail: sidvn@ua.fm. Translated from Prikladnaya Mekhanika, Vol. 52, No. 3, pp. 135–144, May–June, 2016. Original article submitted December 30, 2014.

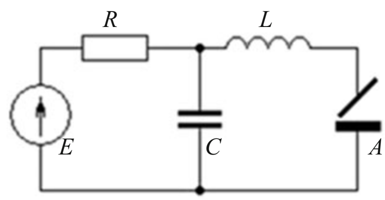


Fig. 1

Statement. Let the differential system (1) satisfy Assumptions 1 and 2. Then an attractor exists in the neighborhood of the singular point (saddle-focus).

Proof. According to Assumption 1, the circular trajectory of system (1) around the singular point, which is a saddle-focus, is unstable. According to Assumption 2, the existence of a damped circular trajectory on one of the coordinate planes is indicative of the dissipative nature of the motion. Assumption 2 suggests that the trajectory does not go to infinity, but rather remains in some neighborhood of the saddle-focus. The effect of the saddle-node on the trajectory of system (1) can be seen as the asymmetry of its projections onto the coordinate planes. Thus, in system (1), there is an attracting trajectory that does not go to ∞ and can form an attractor in the neighborhood of the saddle-focus.

2. Equations of an RLC-circuit with an Arc. We modeled an electric circuit with an arc to generalize several models [3]. The circuit consists of a constant-voltage source E, a resistor R, a reactor L, a capacitor C, and an arc A (Fig. 1). An RLC-arc is described by the following dimensionless differential equations:

$$\begin{cases} dx/dt = \dfrac{1}{L}(y - xz^{(n-1)/2}), \\ dy/dt = \dfrac{1}{RC}(1 + R - y - Rx), \\ dz/dt = x^2 - z, \end{cases} \tag{3}$$

where $x = i/I_0$, $y = u/U_0$, $z = i_\theta^2/I_0^2$, $t = \tau/\theta$ are independent variables and time (n is the exponent in the power fit of the static arc current-voltage (i–u) characteristic), u and i are the capacitor voltage and the reactor current; i_θ is the arc current; τ is time; U_0 and I_0 are constants interpreted as the coordinates of a point on the static arc characteristic; θ is the time constant of the arc; R, L, and C are the resistance, inductance, and capacitance of the circuit. The singular points of system (3) are determined from the following equation [3]:

$$1 + R - Rx = x^n. \tag{4}$$

Consider two real positive solutions of Eq. (4). They correspond to the singular point S with coordinates (1, 1, 1) and the singular point N with coordinates (x_N, x_N^n, x_N^2), where x_N is determined by a nonunit root of Eq. (4).

In deriving the variational equations, we expend the expression $(\bar{z} + \delta z)^{(n-1)/2}$ in the first equation of system (3) into a power series:

$$(\bar{z} + \delta z)^{(n-1)/2} = \bar{z}^{(n-1)/2}\left(1 + \frac{\delta z}{\bar{z}}\right)^{(n-1)/2} = \bar{z}^{(n-1)/2}\left(1 + \frac{(n-1)}{2}\frac{\delta z}{\bar{z}} + \ldots\right) = \bar{z}^{(n-1)/2} + \frac{(n-1)}{2}\bar{z}^{(n-3)/2}\delta z + \ldots .$$

The variational equations (2) of system (3) become

$$\begin{cases} \delta\dot{x} = \dfrac{1}{L}\left(-\bar{z}^{(n-1)/2}\delta x + \delta y - \dfrac{n-1}{2}\bar{x}\bar{z}^{(n-3)/2}\delta z\right), \\ \delta\dot{y} = \dfrac{1}{RC}(-\delta y - R\delta x), \\ \delta\dot{z} = 2\bar{x}\delta x - \delta z. \end{cases} \tag{5}$$

System (5) has the following characteristic equation:

$$CLR\lambda^3 + (CLR + CR\bar{z}^{(n-1)/2} + L)\lambda^2 + (\bar{z}^{(n-1)/2} + CR(\bar{z}^{(n-1)/2} - \bar{x}^2\bar{z}^{(n-3)/2})$$

$$+ CRn\bar{x}^2\bar{z}^{(n-3)/2} + L + R)\lambda + \bar{z}^{(n-1)/2} + (n-1)\bar{x}^2\bar{z}^{(n-3)/2} + R = 0. \quad (6)$$

Equation (6) can be used to find the characteristic numbers of any point of the three-dimensional space of system (3). Substituting the coordinates of the singular points $S(1, 1, 1)$ and $N(x_N, x_N^n, x_N^2)$ of system (3) into Eq. (6), we obtain the characteristic equations of the singular points:

$$CLR\lambda^3 + (CLR + CR + L)\lambda^2 + (L + R + 1 + CRn)\lambda + R + n = 0, \quad (7)$$

$$CLR\lambda^3 + (CLR + CRx_N^{n-1} + L)\lambda^2 + (L + R + x_N^{n-1} + CRnx_N^{n-1})\lambda + R + nx_N^{n-1} = 0.$$

Equations (7) coincide with Eqs. (3.25) and (3.16) in [3].

3. Limit Cycle, Multiple-Period Cycle, and Strange Attractor. Let us choose parameter values of system (3) from the numerical analysis of Hopf bifurcation in [3, p. 76]:

$$(C, L, R, n) = (2.25,\ 1,\ 15,\ -0.4). \quad (8)$$

System (3) has two singular points: saddle-focus S and saddle-node N (according to Assumption 1). Let us analyze Assumption 2. Consider a coordinate plane xy on which the equations of motion for system (3) have the form

$$\begin{cases} dx/dt = -\dfrac{1}{L}y, \\ dy/dt = \dfrac{1}{RC}(1 + R - y - Rx). \end{cases}$$

This system describes a dissipative oscillator:

$$C\ddot{y} + \dot{y}/R + y/L = 0. \quad (9)$$

The equations of motion on the plane xz can be represented as

$$\begin{cases} dx/dt = -\dfrac{1}{L}xz^{(n-1)/2}, \\ dz/dt = x^2 - z. \end{cases} \quad (10)$$

The characteristic exponents of a singular point on the plane xz of system (10) are $\lambda_1 = 0$, $\lambda_2 = -1$. The variational equations and the characteristic equation of system (10) are

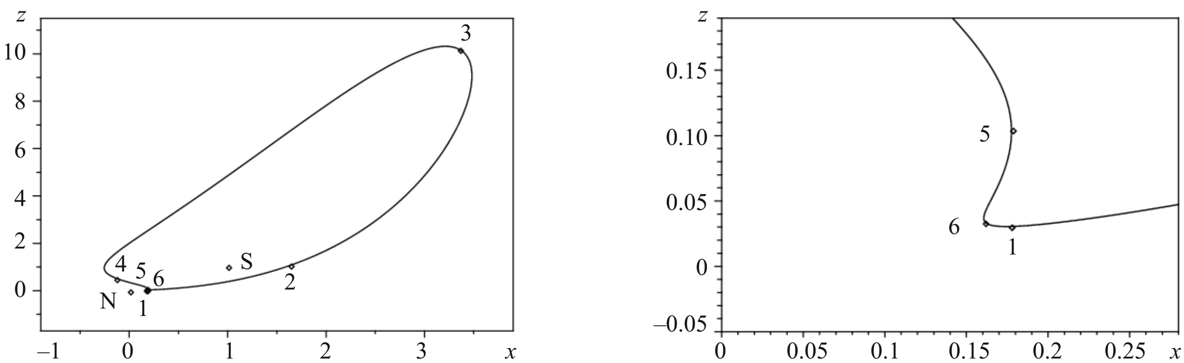
$$\begin{cases} \delta\dot{x} = \dfrac{1}{L}\left(-\bar{z}^{(n-1)/2}\delta x - \dfrac{n-1}{2}\bar{x}\bar{z}^{(n-3)/2}\delta z\right), \\ \delta\dot{z} = 2\bar{x}\delta x - \delta z, \end{cases}$$

$$\bar{\lambda}^2 + \bar{\lambda}\left(1 + \frac{\bar{z}^{(n-1)/2}}{L}\right) + \frac{\bar{z}^{(n-1)/2}}{L} + \frac{(n-1)\bar{x}^2\bar{z}^{(n-3)/2}}{L} = 0.$$

The characteristic equation has the roots

$$\bar{\lambda}_{1,2} = -\frac{1}{2}\left(1 + \frac{\bar{z}^{(n-1)/2}}{L}\right) \pm \sqrt{\frac{1}{4}\left(1 + \frac{\bar{z}^{(n-1)/2}}{L}\right)^2 - \frac{\bar{z}^{(n-1)/2}}{L} - \frac{(n-1)\bar{x}^2\bar{z}^{(n-)3/2}}{L}}. \quad (11)$$

The inequality $\bar{\lambda}_1 + \bar{\lambda}_2 < 0$ holds on the boundary of the domain in (11). The equations of motion on the plane yz are uncoupled:



a *b*

Fig. 2

TABLE 1

No.	Name and CE	Bifurcation	CE
1	saddle-focus ($\mathrm{Re}\,\lambda_{1,2} > 0, \lambda_3 < 0$)	birth of saddle-node	$\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$
2	saddle-node ($\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$)	birth of saddle-focus	$\mathrm{Re}\,\lambda_{1,2} > 0, \lambda_3 < 0$
3	node-focus ($\mathrm{Re}\,\lambda_{1,2} > 0, \lambda_3 < 0$)	birth of node-focus	$\mathrm{Re}\,\lambda_{1,2} < 0, \lambda_3 < 0$
4	node-focus ($\mathrm{Re}\,\lambda_{1,2} < 0, \lambda_3 < 0$)	birth of node	$\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$
5	node ($\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$)	birth of node-focus	$\mathrm{Re}\,\lambda_{1,2} < 0, \lambda_3 < 0$
6	node-focus ($\mathrm{Re}\,\lambda_{1,2} < 0, \lambda_3 < 0$)	birth of saddle-focus	$\mathrm{Re}\,\lambda_{1,2} > 0, \lambda_3 < 0$

$$dy/dt = \frac{(1+R-y)}{RC}, \qquad dz/dt = -z.$$

Assumption 1, Assumption 2, and the Statement on the existence of an attractor hold for system (3).

The attractor can be regular or strange, or can be an attracting fixed point in a three-dimensional space to which the trajectory of system (3) tends. For symmetric dynamic systems, it is possible to prove the theorem of the existence of a regular attractor (a symmetric limit cycle) [2]. Closed trajectories also exist in the neighborhood of the space of parameters responsible for the symmetry. Here no symmetric attractor forms because the singular point N breaks symmetry. According to [3], there is a regular attractor (parameter values (8)) in system (3). Figure 2*a*, *b* shows an attractor (three-dimensional limit cycle) projected onto the coordinate plane xz. In what follows, we will analyze the bifurcations described by Eq. (6) and numerical solution $(\bar{x}, \bar{z})$.

Table 1 shows the bifurcation pattern of the limit cycle with parameter values (8). The first column numbers (No.) the singular points consistently with the notation on the projection onto xz (Fig. 2*a*). The second column names the points and gives their characteristic exponents (CEs). The third column names the bifurcation of the birth of a new point; the CEs of this point are given in the fourth column. For example, the saddle-focus ($\mathrm{Re}\,\lambda_{1,2} > 0$, $\lambda_3 < 0$) transforms into the saddle-node ($\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$) at point 1. The bifurcation at point 1 is the birth of a saddle-node from a saddle-focus. The saddle value < 0 at all points of the closed trajectory. Bifurcation points 1, 5, 6 are close to each other and visually coincide in Fig. 2*a* (see Fig. 2*b*). The qualitative changes of the trajectory near points 1, 4, 5, 6 are caused by the singular point N. The bifurcations of points 1, 4, 5, 6 take longer time than it takes the representative point to traverse sections 2–3 and 3–4. On section 4–6, the representative point moves much more slowly. This makes the whole motion non-uniform.

In [2], it was pointed out that a multiple period results from the nonuniform motion of the representative point. For example, the representative point of a Chua system visits the domain where its velocity drops to nearly zero. As the parameter C

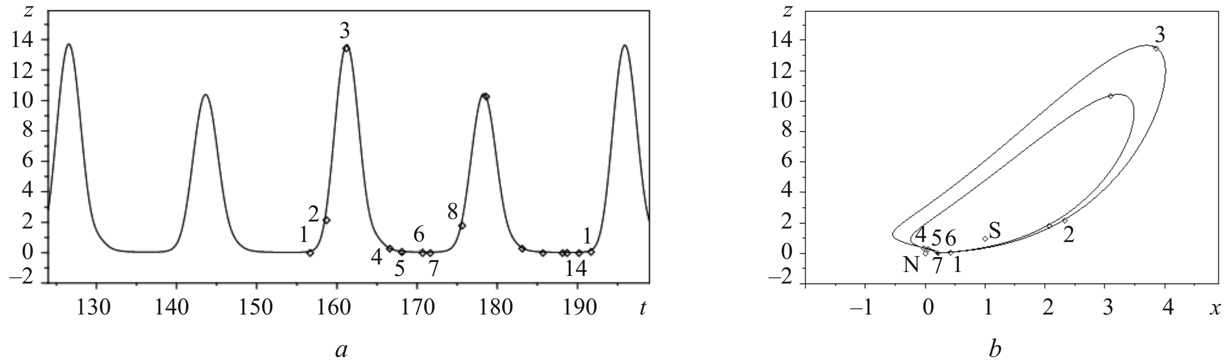


Fig. 3

TABLE 2

No.	Name and CE	Bifurcation	CE
1	saddle-node ($\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0$)	birth of saddle-node	$\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$
2	saddle-node ($\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$)	birth of saddle-focus	$\operatorname{Re}\lambda_{1,2} > 0, \lambda_3 < 0$
3	saddle-focus ($\operatorname{Re}\lambda_{1,2} > 0, \lambda_3 < 0$)	birth of node-focus	$\operatorname{Re}\lambda_{1,2} < 0, \lambda_3 < 0$
4	node-focus ($\operatorname{Re}\lambda_{1,2} < 0, \lambda_3 < 0$)	birth of node	$\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$
5	node ($\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$)	birth of node-focus	$\operatorname{Re}\lambda_{1,2} < 0, \lambda_3 < 0$
6	node-focus ($\operatorname{Re}\lambda_{1,2} < 0, \lambda_3 < 0$)	birth of saddle-focus	$\operatorname{Re}\lambda_{1,2} > 0, \lambda_3 < 0$
7	saddle-focus ($\operatorname{Re}\lambda_{1,2} > 0, \lambda_3 < 0$)	birth of saddle-node	$\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$
8	saddle-node ($\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$)	birth of saddle-focus	$\operatorname{Re}\lambda_{1,2} > 0, \lambda_3 < 0$

of the electric circuit with an arc increases, the absolute saddle value of the point S decreases. The domain in which the trajectory exists, forming an attractor, can be expected to grow.

Figure 3*a* shows a double-period attractor with $(C, L, R, n) =$ (2.4, 1, 15, –0.4). The points at which the solution bifurcates are indicated. The saddle value < 0 at all points of the closed trajectory. The results are presented in Table 2. The motion is even more nonuniform here. This nonuniformity is observed in the neighborhood of points 4–7 and is manifested as period doubling. The period doubling in system (3) can be detailed.

In Fig. 3*a*, the period of the cycle can be measured by bifurcation points. We begin with point 1 on the left in Fig. 3*a*. If the trajectory closed in one cycle, then closure point 1 would appear on the right between points 7 and 8. However, there is no bifurcation point 1 between points 7 and 8. This means that the trajectory does not close after one turn. The trajectory is unstable during the first turn. After another turn of the trajectory, point 1 appears after point 14. The trajectory becomes closed.

Increasing the parameter C, we obtain a cascade of multiple-period limit cycles.

Consider a limit cycle (Fig. 4*a*, *b*) with period tripling $(C, L, R, n) =$ (2.8, 1, 15, –0.4). It takes a long time for the motion to become periodic. On the section $m - n$ (Fig. 4), all points are nodes with characteristic exponents $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$.

Figure 4*b* shows by heavy lines two sections corresponding to nodes ($\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$) and saddle-nodes. The node-foci and saddle-foci are shown by thin lines. Here $\sigma = \lambda_1 + \lambda_2 + \lambda_3 < 0$ for all points of the attractor.

The formation of regular attractors is complete at some parameter value at which the trajectory is not closed and forms a strange attractor (Fig. 5*a*, *b*, *c*) $(C, L, R, n) =$ (2.81, 1, 15, –0.4). The process remains periodic. The orbital instability is due to the bifurcation pattern nonrepeated during a period. In Fig. 5*b*, the nodes ($\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$) and saddle-nodes are shown by

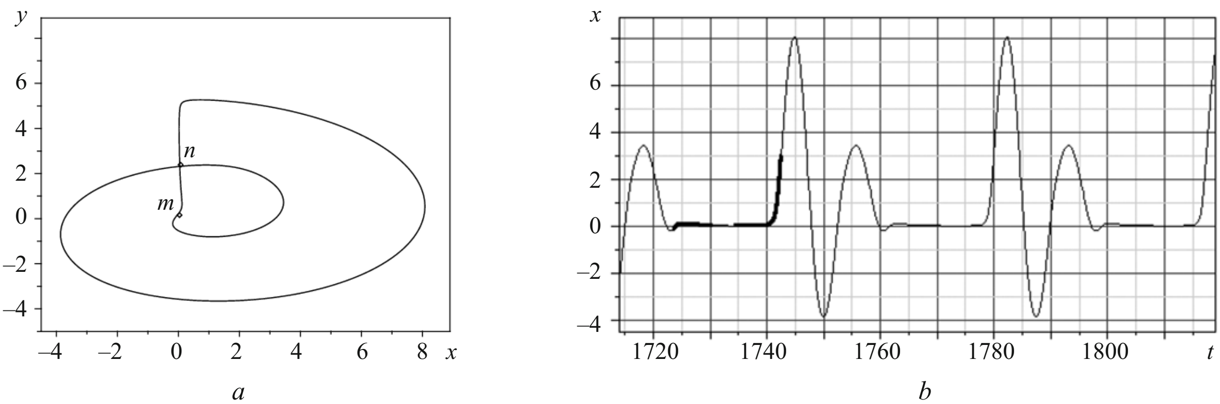


Fig. 4

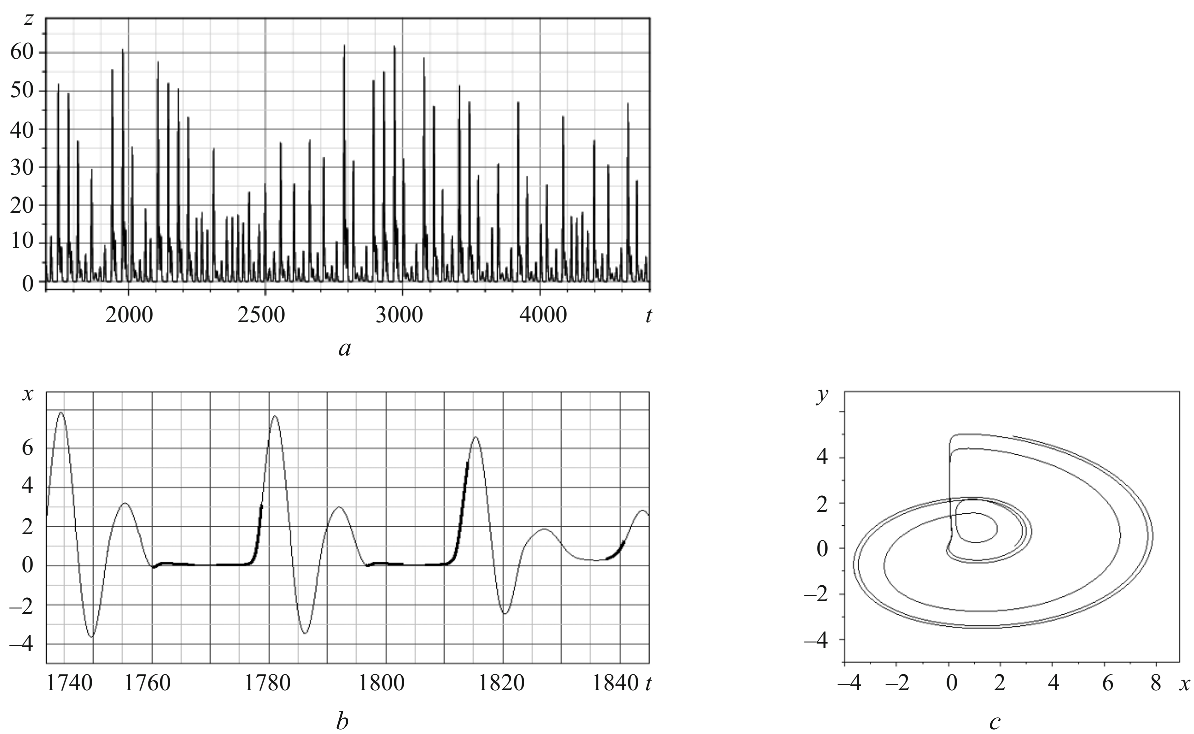


Fig. 5

heavy lines. The node-foci and saddle-foci are shown by thin lines. The thin lines represent the node-foci and saddle-foci. The trajectory is orbitally unstable during each turn; however, $\sigma = \lambda_1 + \lambda_2 + \lambda_3 < 0$ for all points of the trajectory in Fig. 5*b*, *c*.

4. Bifurcation Giving Birth to an Attracting Fixed Point. This state of the dynamic system (3) corresponds to the extinction of the arc. Let us clarify the physics behind this phenomenon. Note that the limit cycle (Fig. 2*a*, *b*) and the multiple-period cycles (Figs. 3*a*, *b* and Fig. 4*a*, *b*) have sections of nodes on the stable trajectory, where $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$. These are points 4–5, according to Tables 1 and 2, and the interval $m-n$ (Fig. 4). Denote $|\sigma| = |\lambda_1 + \lambda_2 + \lambda_3|$. In the cases listed above, $|\sigma| > 1$ is greater, but not much, than unity. In this case, the trajectory misses the sections of nodes.

Let us introduce parameters $(C, L, R, n) = (2.814,\ 1,\ 15,\ -0.4)$ and plot the phase portrait for these parameter values (Fig. 6). On the interval $m-n$, $|\sigma| >> 1$ and increases to a high order exceeding the existing one by three orders of magnitude and more. It is this fact that is interpreted as the existence of an attracting fixed point.

Conclusions. Three types of attractors have been considered: a limit cycle, a strange attractor, and a fixed point approached by the representative point. The characteristic equation (6) of the variational system has been used to analyze the bifurcations of points of the trajectory and the physics behind the occurrence of the multiple-period limit cycle and the strange attractor. One singular point is a saddle-focus forming a circular trajectory. The other singular point of system (3) is a

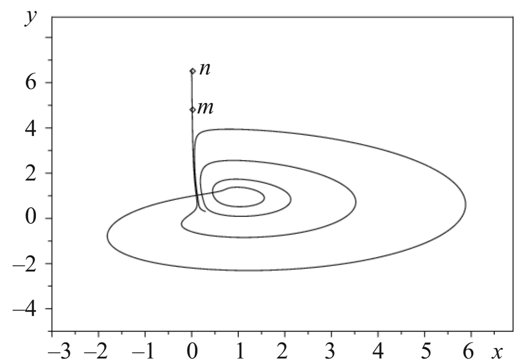


Fig. 6

saddle-node. This point influences the configuration of the trajectory, but a circular trajectory does not occur around this point. A strange attractor is born during loss of stability of the orbit of the system in the neighborhood of the singular point S and is due to the nonrepeating bifurcation pattern.

The Lyapunov characteristic exponents of system (1) have been calculated using the technical approach of identifying an attractor which makes sense in system (1) for $m > 3$. The numerical algorithms date back to the Italian computational school of the 1970s (see [1] for the results). The analysis of bifurcations and the qualitative analysis answer the following questions: How does a regular attractor occur? and How does qualitative changes of bifurcations affect the orbital stability and the occurrence of a strange attractor? Of practical interest is to identify and qualitatively analyze the conditions for the extinction of the arc.

An alternative is the Lorentz system in which a strange attractor occurs when the representative point jumps from the attraction domain of one singular point to the attraction domain of the other point. The Lorentz system has two equivalent singular points (saddle-foci) around which orbits may occur with loss of stability.

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