DOI 10.1007/s10778-016-0752-8
International Applied Mechanics, Vol. 52, No. 3, May, 2016

STABILITY AND POST-BUCKLING BEHAVIOR OF ORTHOTROPIC CYLINDRICAL SHELLS WITH LOCAL DEFLECTIONS

N. P. Semenyuk and N. B. Zhukova

A procedure for analytical solution of the problem of the stability and post-buckling behavior of orthotropic cylindrical shells under external pressure or axial compression with allowance for transverse shears is developed. The shells are geometrically imperfect due to the presence of a local deflection. The problem is solved by analyzing the interaction of the modes that represent the critical loads of the perfect shell and using the Byskov–Hutchinson method. Equilibrium curves for both shells are plotted using the method of continuous loading

Keywords: buckling, post-buckling, imperfection, laminated shell, mode interaction, Cauchy problem, external pressure, axial compression

Introduction. A method of stability analysis of reinforced imperfect cylindrical shells that takes into account the interaction of local and overall buckling modes is outlined in [9, 10]. It differs from that proposed earlier in [13] by being used irrespective of whether the eigenvalues of the homogeneous problem solved to determine the critical loads of perfect shells are equal, close, or different. This is why this method can also be used to solve stability problems for thin-walled structures with imperfections [1, 6, 16].

In what follows, we will outline an approach, based on the Byskov–Hutchinson method, to solving the problem of the nonlinear deformation of orthotropic cylindrical shells with a deflection bounded by segments of the coordinate lines. The local deflection is described as a product of two trigonometric functions that are equal to zero on its boundary. Such products are represented by double Fourier series. Each term in them is an eigenfunction of the homogeneous stability problem. Formally, a Fourier series is an expansion of initial imperfections into a series of the eigenvectors of the nonperturbed problem (a procedure of using the coefficients of these series is outlined below). It is important to determine the number of interacting modes sufficient to obtain results with satisfactory accuracy.

The influence of multimodal imperfections not related to local deflections on the critical loads and post-buckling behavior of composite shells was discussed in [6, 16]. The results presented below supplement and develop those reported in the works cited above.

The stability of isotropic shells with local deflections bounded by the coordinate axes was analyzed analytically and experimentally in [4, 7, 8, 11, 12, 14].

1. Problem Formulation. Governing Equations. Consider a laminated cylindrical shell of radius R, thickness t, and length L acted upon by forces proportional to some parameter λ.

Let us derive the required equations using the Timoshenko–Mindlin shell theory [2]. The virtual-work principle is expressed as

$$\int_0^L \int_0^{2\pi R} \Big[T_{11}\delta\varepsilon_1 + T_{22}\delta\varepsilon_2 + T_{12}^*\delta\omega_1 + T_{21}^*\delta\omega_2 + T_{13}^*\delta\theta_1 + T_{23}\delta\theta_2 + T_{13}\delta\theta + T_{23}^*\delta\psi + M_{11}\delta k_1$$

S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterova St., Kyiv, Ukraine 03057, e-mail: compos@inmech.kiev.ua. Translated from Prikladnaya Mekhanika, Vol. 52, No. 3, pp. 92–104, May–June, 2016. Original article submitted July 13, 2015.

$$+M_{22}\delta k_2 + M_{12}\delta t_1 + H(\delta t_1 + \delta t_2)\big]dxdy - \delta A = 0, \tag{1}$$

where x and y are the longitudinal and circumferential coordinates of the reference cylindrical surface; T_{ij} and M_{ij} are the forces and moments statically equivalent to the acting stresses. The projections of the forces onto the undeformed midsurface axes are given by

$$T_{12}^* = S + T_{11}\omega_1, \quad T_{21}^* = S + \frac{1}{R}H + T_{22}\omega_2,$$

$$T_{13}^* = T_{13} + T_{11}(\theta_1 + \bar{\theta}_1) + S(\theta_2 + \bar{\theta}_2), \quad T_{23}^* = T_{23} + S(\theta_1 + \bar{\theta}_1) + T_{22}(\theta_2 + \bar{\theta}_2). \tag{2}$$

Let us introduce parameters needed to express the nonlinear strains in terms of the displacements:

$$\varepsilon_1 = \frac{\partial u}{\partial x}, \quad \omega_1 = \frac{\partial v}{\partial x}, \quad \theta_1 = \frac{\partial w}{\partial x}, \quad \varepsilon_2 = \frac{\partial v}{\partial y} - \frac{w}{R}, \quad \omega_2 = \frac{\partial u}{\partial y}, \quad \theta_2 = \frac{\partial w}{\partial y} + \frac{v}{R},$$

$$k_1 = \frac{\partial \theta}{\partial x}, \quad k_2 = \frac{\partial \psi}{\partial y}, \quad t_1 = \frac{\partial \psi}{\partial x}, \quad t_2 = \frac{\partial \theta}{\partial y}. \tag{3}$$

Substituting them into the expressions for strains, we get

$$\varepsilon_{11} = \varepsilon_1 + \frac{1}{2}(\omega_1^2 + \theta_1^2 + 2\theta_1\bar{\theta}_1), \quad \varepsilon_{22} = \varepsilon_2 + \frac{1}{2}(\omega_2^2 + \theta_2^2 + 2\theta_2\bar{\theta}_2),$$

$$\varepsilon_{12} = \omega_1 + \omega_2 + \theta_1\theta_2 + \theta_1\bar{\theta}_2 + \bar{\theta}_1\theta_2, \quad k_{11} = k_1, \quad k_{22} = k_2 + \frac{\varepsilon_2}{R},$$

$$k_{12} = t_1 + t_2 + \frac{\omega_2}{R}, \quad \varepsilon_{13} = \theta + \theta_1, \quad \varepsilon_{23} = \psi + \theta_2. \tag{4}$$

The functions with overbar denote the angles of rotation of the surface caused by the initial geometrical imperfections such as deflections $\bar{w}$.

The variation of the work of external forces δA depends on how they are distributed over the faces or lateral surface. If the shell is subject to an axial compressive force T_{11}^0, then

$$\delta A = -\int_0^{2\pi R} T_{11}^0 \delta u\Big|_0^L dy. \tag{5}$$

If the shell is subject to uniform external pressure of intensity q, then

$$\delta A = \int_0^L \int_0^{2\pi R} q\big[-\theta_1\delta u - \theta_2\delta v + (1 + \varepsilon_1 + \varepsilon_2)\delta w\big]dxdy. \tag{6}$$

Assume that the shell consists of N plies of a fibrous composite symmetric about the midsurface. Then the constitutive equations are

$$T_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22}, \quad T_{22} = C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22}, \quad S = C_{66}\varepsilon_{12}, \quad T_{13} = C_{55}\varepsilon_{13},$$

$$T_{23} = C_{44}\varepsilon_{23}, \quad M_{11} = D_{11}k_{11} + D_{12}k_{22}, \quad M_{22} = D_{12}k_{11} + D_{22}k_{22}, \quad H = D_{66}k_{12}, \tag{7}$$

where C_{ij} and D_{ij} are stiffnesses,

$$C_{kl} = \sum_{i=1}^N C_{kl}^i, \quad D_{kl} = \sum_{i=1}^N (D_{kl}^i + z_i^2 C_{kl}^i), \tag{8}$$

where z_i is the coordinate of the midsurface of the ith ply.

Equations (1)–(8) make it possible to solve different nonlinear problems of the deformation of shells with initial geometrical imperfections. The asymptotic method proposed in [9, 10] can be applied to these equations. Assume that the pre-buckling stress–strain state of the shell is linear and the displacement field at $\lambda = 1$ is characterized by a vector U_0. Linearizing Eqs. (1)–(8) about the bifurcation load, we obtain a system of equations to determine the critical values of the load parameter λ_i and the modes U_i corresponding to the ith buckling mode (here U_i is the vector with components (u, v, w, θ, ψ)).

The linearized variational equation (1) for perfect cylindrical shells is

$$\int_0^L \int_0^{2\pi R} \left\{ T_{11}^i \delta\varepsilon_1 + S^i \delta\omega_1 + T_{22}^i \delta\varepsilon_2 + \left(S^i + \frac{2}{R} H^i \right) \delta\omega_2 + T_{13}^i \left(\delta\theta_1 + \delta\theta \right) \right.$$

$$+ H^i \delta k_{12} + T_{23}^i (\delta\theta_2 + \delta\psi) + M_{11}^i \delta k_{11} + M_{22}^i \delta k_{22} + \lambda_i \Big[T_{11}^0 \left(\omega_1^i \delta\omega_1 + \theta_1^i \delta\theta_1 \right)$$

$$\left. + T_2^0 (\omega_2^i \delta\omega_2 + \theta_2^i \delta\theta_2) + S^0 (\theta_1^i \delta\theta_2 + \theta_2^i \delta\theta_1) \Big] \right\} dxdy = 0. \tag{9}$$

The equilibrium equations about the critical point at the beginning of the post-buckling region can be written for displacements by substituting (3) and (4) into (7) and the result into these equations [2, 17]. We obtain five equations for five unknown functions $u_1 = u, u_2 = v, u_3 = w, u_4 = \theta, u_5 = \psi$. Their concise form is

$$\sum_{j=1}^{5} \Delta_{ij} u_j - \lambda (m_i) \begin{pmatrix} T_{11}^0 \\ 2S^0 \\ T_{22}^0 \end{pmatrix} = 0 \quad (i = 1, \ldots, 5), \tag{10}$$

where m_i are row vectors whose components are derivatives of displacements:

$$m_1 = \left(\frac{\partial^2 u}{\partial x^2}, \ \frac{\partial^2 w}{\partial x \partial y}, \ \frac{\partial^2 u}{\partial y^2} - \frac{\partial w}{\partial x} \right), \qquad m_2 = \left(\frac{\partial^2 v}{\partial x^2}, \ \frac{\partial^2 v}{\partial x \partial y} - \frac{1}{R}\frac{\partial w}{\partial x}, \ \frac{\partial^2 v}{\partial y^2} - \frac{2}{R}\frac{\partial w}{\partial y} - \frac{v}{R^2} \right),$$

$$m_3 = \left(\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y} + \frac{1}{R}\frac{\partial v}{\partial x}, \ \frac{\partial^2 w}{\partial y^2} + \frac{2}{R}\frac{\partial v}{\partial y} - \frac{w}{R^2} \right), \qquad m_4 = (0,\ 0,\ 0), \qquad m_5 = (0,\ 0,\ 0), \tag{11}$$

$T_{11}^0, 2S^0, T_{22}^0$ are the subcritical compressive and torsional forces proportional to the coefficient λ; $\Delta_{ij} u_j$ are differential operators of the displacements u, v, w, θ, ψ.

The amplitudes of the buckling modes U_i are denoted by ξ_i. They remain undetermined in solving the homogeneous problem (9)–(10) and can only be determined by solving the original nonlinear problem (1)–(8). To this end, the displacement vector is asymptotically expanded:

$$U = \lambda U_0 + \xi_i U_i + \xi_i \xi_j U_{ij}. \tag{12}$$

The vector of initial geometrical imperfections is represented as the sum $\overline{U} = \overline{\xi}_i U_i$, where $\overline{\xi}_i$ is the amplitude of an imperfection in the form of the ith mode. Hereafter, the summation is over repeated indices.

The vectors U_{ij} are orthogonal to the buckling modes:

$$\int_0^L \int_0^{2\pi R} {}_i \Big[T_{11}^0 \left(\omega_1^i \omega_1^{kl} + \theta_1^i \theta_1^{kl} \right) + T_2^0 (\omega_2^i \omega_2^{kl} + \theta_2^i \theta_2^{kl}) + S^0 (\theta_1^i \theta_2^{kl} + \theta_2^i \theta_1^{kl}) \Big] \Big\} dxdy = 0. \tag{13}$$

According to the Byskov–Hutchinson method in the Timoshenko–Mindlin shell theory, the variational equation for second-order variables is

$$\int_0^L \int_0^{2\pi R} \Big\{ T_{11}^{ij}\delta\varepsilon_1 + S^{ij}\delta\omega_1 + T_{22}^{ij}\delta\varepsilon_2 + \left(S^{ij} + \frac{2}{R}H^{ij} \right)\delta\omega_2 + T_{13}^{ij}(\delta\theta_1 + \delta\theta)$$

$$+T_{23}^{ij}(\delta\theta_2 + \delta\psi) + M_{11}^{ij}\delta k_{11} + M_{22}^{ij}\delta k_{22} + +H^{ij}\delta k_{12} + \lambda_i \Big[T_{11}^0 (\omega_1^{ij}\delta\omega_1$$

$$+ \theta_1^{ij}\delta\theta_1) + T_2^0(\omega_2^{ij}\delta\omega_2 + \theta_2^{ij}\delta\theta_2) + S^0(\theta_1^{ij}\delta\theta_2 + \theta_2^{ij}\delta\theta_1) \Big] \Big\} dxdy$$

$$= -\frac{1}{2}\int_o^L \int_0^{2\pi R} \Big[(T_{11}^i \omega_1^{ji} + T_{11}^j \omega_1^i)\delta\omega_1 + (T_{22}^i \omega_2^j + T_{22}^j \omega_2^i)\delta\omega_2$$

$$+ (T_{11}^i\theta_1^j + S^i\theta_2^j + T_{11}^j\theta_1^i + S^j\theta_2^i)\delta\theta_1 + (T_{22}^i\theta_2^j + S^i\theta_2^j + T_{22}^j\theta_{21}^i + S^j\theta_2^i)\delta\theta_2 \Big] dxdy. \quad (14)$$

To derive the system of equations convenient for obtaining the analytic solution, we will nondimensionalize all variables, unknown functions, and stiffnesses [2], assuming that $x = R\alpha_1$, $y = R\alpha_2$,

$$(\tilde{u}, \tilde{v}) = \frac{R}{t^2}(u, v), \quad \tilde{w} = \frac{1}{t}w, \quad (\tilde{\theta}, \tilde{\psi}) = \frac{R}{t}(\theta, \psi), \quad (\tilde{\varepsilon}_{11}, \tilde{\varepsilon}_{22}, \tilde{\varepsilon}_{12}) = \frac{R^2}{t^2}(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}),$$

$$(\tilde{\varepsilon}_{13}, \tilde{\varepsilon}_{23}) = \frac{R}{t}(\varepsilon_{13}, \varepsilon_{23}), \quad (\tilde{k}_{11}, \tilde{k}_{22}, \tilde{k}_{12}) = \frac{R^2}{t}(k_{11}, k_{22}, k_{12}),$$

$$t_{ij} = \frac{T_{ij}R^2}{C_0 t^2}, \quad m_{ij} = \frac{M_{ij}R^2}{C_0 t^2}, \quad m_q = q\frac{T_{ij}R^3}{C_0 t^2}, \quad h = \frac{t}{R}.$$

We will not use dimensional functions below; therefore, new notation will not be used for dimensionless functions.

Let us represent expression (14) in the form

$$\int_0^{L/R} \int_0^{2\pi} (P_1\delta u + P_2\delta v + P_3\delta w + P_4\delta\theta + P_5\delta\psi)\, d\alpha_1 d\alpha_2 = 0, \quad (15)$$

where P_i are differential operators,

$$P_1 = t_{11}^{ij}\frac{\partial}{\partial\alpha_1} + s^{ij}\frac{\partial}{\partial\alpha_2} + \frac{1}{2}\left(t_{22}^i \frac{\partial u_j}{\partial\alpha_2} + t_{22}^j \frac{\partial u_i}{\partial\alpha_2} \right)\frac{\partial}{\partial\alpha_2},$$

$$P_2 = t_{22}^{ij}\frac{\partial}{\partial\alpha_2} + s^{ij}\frac{\partial}{\partial\alpha_1} + t_{23}^{ij} + hm_{12}^{ij}\frac{\partial}{\partial\alpha_1} + hm_{22}^{ij}\frac{\partial}{\partial\alpha_2} - \lambda_c h^2 \frac{\partial v_{ji}}{\partial\alpha_1}\frac{\partial}{\partial\alpha_1} + \frac{1}{2}\left(t_{11}^i h^2 \frac{\partial v_j}{\partial\alpha_1} + t_{11}^j h^2 \frac{\partial v_i}{\partial\alpha_1} \right)$$

$$+\frac{1}{2}\left[t_{22}^i h\left(\frac{\partial w_j}{\partial\alpha_2} + hv_j \right) + t_{22}^j h\left(\frac{\partial w_i}{\partial\alpha_2} + hv_i \right) \right] + \frac{1}{2}\left(s^i h\frac{\partial w_j}{\partial\alpha_1} + s^j h\frac{\partial w_i}{\partial\alpha_1} \right) + m_q h\left(\frac{\partial v_{i_j}}{\partial\alpha_2} - \frac{w_{i_j}}{h} \right),$$

$$P_3 = \frac{1}{h}t_{13}^{ij}\frac{\partial}{\partial\alpha_1} + \frac{1}{h}t_{23}^{ij}\frac{\partial}{\partial\alpha_2} - \frac{1}{h}(t_{22}^{ij} + hm_{22}^{ij}) - \lambda_c \frac{\partial w}{\partial\alpha_1}\frac{\partial}{\partial\alpha_1} + \frac{1}{2}\left(t_{11}^i \frac{\partial w_j}{\partial\alpha_1} + t_{11}^j \frac{\partial w_i}{\partial\alpha_1} \right)\frac{\partial}{\partial\alpha_1}$$

$$+\frac{1}{2}\left[t_{22}^i\left(\frac{\partial w_j}{\partial\alpha_2} + hv_j \right) + t_{22}^j\left(\frac{\partial w_i}{\partial\alpha_2} + hv_i \right) \right]\frac{\partial}{\partial\alpha_2} + \frac{1}{2}\left\{ s^i\left[\frac{\partial w_j}{\partial\alpha_1}\frac{\partial}{\partial\alpha_2} + \left(\frac{\partial w_j}{\partial\alpha_1} + hv_j \right)\frac{\partial}{\partial\alpha_1} \right] \right.$$

$$+s^i\left[\frac{\partial w_i}{\partial \alpha_1}\frac{\partial}{\partial \alpha_2}+\left(\frac{\partial w_i}{\partial \alpha_1}+hv_i\right)\frac{\partial}{\partial \alpha_1}\right]\Bigg\},$$

$$P_4=\frac{1}{h}t_{13}^{ij}+m_{11}^{ij}\frac{\partial}{\partial \alpha_1}+m_{12}^{ij}\frac{\partial}{\partial \alpha_2}, \qquad P_5=\frac{1}{h}t_{23}^{ij}+m_{22}^{ij}\frac{\partial}{\partial \alpha_{21}}+m_{12}^{ij}\frac{\partial}{\partial \alpha_1}. \tag{16}$$

In [9, 10], it is proposed to use the minimum of λ_j as λ in Eq. (16). Solving homogeneous (9)–(11) and inhomogeneous (14)–(16) boundary-value problems for the amplitudes ξ_i in Eq. (1) (assuming that $\delta U = U_1$), we arrive at the system of nonlinear algebraic equations

$$\xi_r\left(1-\frac{\lambda}{\lambda_r}\right)+\xi_i\xi_j a_{ijr}+\xi_i\xi_j\xi_k b_{ijkr}=\overline{\xi}_r\frac{\lambda}{\lambda_r} \qquad (r=1,\ldots,M), \tag{17}$$

$$\left[a_{ijr}=-\frac{A_{ijr}}{2D},\ b_{ijr}=-\frac{B_{ijkr}}{D}\right], \tag{18}$$

$$A_{ijr}=\int_0^{2\pi}\int_0^{L/R}\Big[2t_{11,i}(h^2\omega_{1,j}\omega_{1,r}+\theta_{1,j}\theta_{1,r})+2t_{22,i}\left(h^2\omega_{2,j}\omega_{2,r}+\theta_{2,j}\theta_{2,r}\right)$$

$$+2s_i(\theta_{1,j}\theta_{2,r}+\theta_{2,j}\theta_{1,r})+t_{11,r}(\omega_{1,i}\omega_{1,j}+\theta_{1,i}\theta_{1,j})$$

$$+t_{22,r}(\omega_{2,i}\omega_{2,j}+\theta_{2,i}\theta_{2,j})+s_r(\theta_{1,i}\theta_{2,j}+\theta_{2,i}\theta_{1,j})\Big]d\alpha_1 d\alpha_2,$$

$$B_{ijkr}=\int_0^{2\pi}\int_0^{L/R}\Bigg\{\frac{1}{2}\Big[2t_{11,i}(\omega_{1,j}\omega_{1,kr}+\theta_{1,j}\theta_{1,kr})+2t_{22,i}(\omega_{2,j}\omega_{2,kr}+\theta_{2,j}\theta_{2,kr})+2s_i\left(\theta_{1,j}\theta_{2,kr}+\theta_{2,j}\theta_{1,kr}\right)$$

$$+t_{11,i}(h^2\omega_{1,jk}\omega_{1,r}+\theta_{1,jk}\theta_{1,r})+t_{22,i}(h^2\omega_{2,jk}\omega_{2,r}+\theta_{2,jk}\theta_{2,r})+s_i(\theta_{1,jk}\theta_{2,r}+\theta_{2,jk}\theta_{1,r})$$

$$+t_{11,r}(h^2\omega_{1,i}\omega_{1,jk}+\theta_{1,i}\theta_{1,jk})+t_{22,r}(h^2\omega_{2,i}\omega_{2,jk}+\theta_{2,i}\theta_{2,jk})+s_r(\theta_{1,i}\theta_{2,r}+\theta_{1,ij}\theta_{,,k})$$

$$+t_{11,ii}(h^2\omega_{1,k}\omega_{1,r}+\theta_{1,k}\theta_{1,r})+t_{22,ii}(h^2\omega_{2,k}\omega_{2,r}+\theta_{2,k}\theta_{2,r})+s_{ii}(\theta_{2,k}\theta_{1,r}+\theta_{1,k}\theta_{2,r})$$

$$+t_{11,ri}(h^2\omega_{1,j}\omega_{1,k}+\theta_{1,j}\theta_{1,k})+t_{22,r}(h^2\omega_{2,j}\omega_{2,k}+\theta_{2,j}\theta_{2,k})+s_{ri}(\theta_{2,i}\theta_{1,k}+\theta_{2,k}\theta_{1,i})\Big]\Bigg\}d\alpha_1 d\alpha_2,$$

$$D=\int_0^{2\pi}\int_0^{L/R}\Bigg(t_{11,r}\varepsilon_{1,r}+t_{12,r}\omega_{1,r}+t_{22,r}\varepsilon_{2,r}+t_{21,r}\omega_{2,r}+\frac{1}{h}t_{13,r}\varepsilon_{13,r}$$

$$+\frac{1}{h}t_{23,r}\varepsilon_{23,r}+m_{11,r}k_{11,r}+m_{22,r}k_{22,r}+m_{12}k_{12,r}\Bigg)d\alpha_1 d\alpha_2.$$

Equations (17) can be used to study the nonlinear pre-buckling deformation of imperfect structures, to calculate the critical (ultimate) loads, and to analyze the post-buckling behavior of structures with local imperfections in the form of deflections.

Moreover, system (17) can also be used when modes coincide, are similar, or differ considerably.

This feature of the Byskov–Hutchinson method will be used below to develop a procedure for analyzing the stability and post-buckling behavior of laminated composite cylindrical shells with multimodal imperfections, such as those described by Fourier series.

2. Solution for Cylindrical Shells. The local deflection of the cylindrical surface is described by a function of two variables x and y that coincide within some local domain with the global coordinates $0 \le x \le L, -\pi R \le y \le \pi R$, where L is the length of the cylinder, $2\pi R$ is the length of its circumference. The local domain has the boundary $L_1 - l_1 \le x \le L_1 + l_1$, $-y_1 \le y \le y_1$. The initial deflection within the domain is described by

$$\overline{w} = \xi \sin \frac{\pi(x - L_1)}{l_1} \cos \frac{\pi y}{2 y_1}. \tag{19}$$

Since the amplitude of each function in (19) is equal to unity, the amplitude of the initial deflection $\overline{w}$ is equal to ξ. The solution of the perturbed problem can be expanded into Fourier series because trigonometric functions are eigenfunctions for the homogeneous problem. Then

$$\sin \frac{\pi(x - L_1)}{l_1} = \sum_m B_m \sin \frac{m\pi x}{L_1} \tag{20}$$

$$\left[B_m = \frac{2}{L} \left(\alpha \cos \frac{\pi L_1}{l_1} - \beta \sin \frac{\pi L_1}{l_1} \right) \right],$$

$$\left[\alpha = \frac{1}{2} \left\{ \frac{1}{\gamma_1} \left[\sin \gamma_1 (L_1 + l_1) - \sin \gamma_1 L_1 \right] - \frac{1}{\gamma_2} \left[\sin \gamma_2 \left(L_1 + l_1 \right) - \sin \gamma_2 L_1 \right] \right\}, \right.$$

$$\beta = \frac{1}{2} \left\{ \frac{1}{\gamma_1} \left[\cos \gamma_1 (L_1 + l_1) - \cos \gamma_1 L_1 \right] - \frac{1}{\gamma_2} \left[\cos \gamma_2 (L_1 + l_1) - \cos \gamma_2 L_1 \right] \right\},$$

$$\left. \gamma_1 = \frac{\pi}{l_1} - \frac{m\pi}{L}, \quad \gamma_2 = \frac{\pi}{l_1} + \frac{m\pi}{L} \right].$$

Since $y = R\alpha_2$, $y_1 = R\alpha'_2$, we have

$$\cos \frac{\pi y}{2 y_1} = \cos \frac{\pi \alpha_2}{2\alpha'_2} = A_0 + \sum_n A_n \cos n\varphi \quad \left[A_0 = \frac{2\alpha'_2}{\pi^2}, \quad A_n = \cos n\alpha'_2 / \alpha_2 \left[\left(\frac{\pi}{2\alpha'_2} \right)^2 - n^2 \right] \right].$$

Using these expansions, we represent the initial deflection as a double Fourier series:

$$\overline{w} = \xi \sum_{m=1,} \sum_{n=0,1,} B_m A_n \sin \frac{m\pi x}{L} \cos n\alpha_2. \tag{21}$$

The coefficients of the two-dimensional matrix $\xi B_m A_n$ are used to form a one-dimensional vector with components $\overline{\xi}_i$ so that $\overline{\xi}_1 > \overline{\xi}_2 > \overline{\xi}_3 > \ldots > \overline{\xi}_m$.

Below (see the examples) we will show that the wave numbers m and n of the buckling modes of the perfect shells are consistent with those of the trigonometric functions of Fourier series.

Let us use the above procedure to analyze laminated cylindrical shells under external pressure or axial compression.

In solving the homogeneous problem (10), we expand the deflections into trigonometric series that term-wise satisfy the hinged boundary conditions at the ends:

$$u_i = A^i_{m,n} \cos l_m \alpha_1 \cos n_i \alpha_2, \quad v_i = B^i_{m,n} \sin l_m \alpha_1 \sin n_i \alpha_2, \quad w_i = C^i_{m,n} \sin l_m \alpha_1 \cos n_i \alpha_2,$$

$$\theta_i = D^i_{m,n} \cos l_m \alpha_{12} \cos n_i \alpha_2, \quad \psi_i = E^i_{m,n} \sin l_m \alpha_1 \sin n_i \alpha_2 \tag{22}$$

$$\left(l_m = m\pi R / L, \quad \alpha_1 = x / R, \quad \alpha_2 = y / R \right).$$

Substituting (22) into (10), we get a system of homogeneous algebraic equations. Enumerating the wave numbers m and n, we identify the spectrum of eigenvalues λ_i and associated eigenvectors normalized so that $C^i_{m,n}i = 1$.

The solution of the system derived from the variational equation (15) is represented as follows considering the form of their right-hand sides:

$$u_{ij} = \sum_k \left[A^{ij}_{k,1} \cos(n_i - n_j)\alpha_2 + A^{ij}_{k,2} \cos(n_i + n_j)\alpha_2 \right] \cos l_k \alpha_1,$$

$$v_{ij} = \sum_k \left[B^{ij}_{k,1} \sin(n_i - n_j)\alpha_2 + B^{ij}_{k,2} \sin(n_i + n_j)\alpha_2 \right] \sin l_k \alpha_1,$$

$$w_{ij} = \sum_k \left[C^{ij}_{k,1} \cos(n_i - n_j)\alpha_2 + C^{ij}_{k,2} \cos(n_i + n_j)\alpha_2 \right] \sin l_k \alpha_1,$$

$$\theta_{ij} = \sum_k \left[D^{ij}_{k,1} \cos(n_i - n_j)\alpha_2 + D^{ij}_{k,2} \cos(n_i + n_j)\alpha_2 \right] \cos l_k \alpha_1,$$

$$\psi_{ij} = \sum_k \left[E^{ij}_{k,1} \sin(n_i - n_j)\alpha_2 + E^{ij}_{k,2} \sin(n_i + n_j)\alpha_2 \right] \sin l_k \alpha_1. \tag{23}$$

To solve system (17) for the initial value of λ, we will employ the Newton–Kantorovich method and the method of incremental loading. The solution obtained at the ith step of loading is used as the initial value of the load at the $(i+1)$th step. To find the solution at the points where the Jacobian of system (17) is equal to zero, the method of continuous loading is used [4, 5, 15]. A $(M+1)$-vector $\overline{X}$ with components $(\xi_1, \ldots, \xi_M, \lambda)^T$ is introduced. Then system (17) can be represented in compact form:

$$F_r(\overline{X}) = 0 \quad (r = 1, \ldots, M). \tag{24}$$

Differentiating (17) with respect to the parameter s, we arrive at a system of M linear homogeneous equations for $M+1$ unknowns:

$$\sum_{j=1}^{M+1} F_{r,j} \frac{d\xi_j}{ds} = 0 \quad (r = 1, \ldots, M), \tag{25}$$

where $\overline{J} = [F_{r,j}] = [\partial F_r / \partial \xi_i]$ is the Jacobian matrix of system (24). The rank $[\overline{J}] = M$ at regular and limit points.

The solution of system (25) can be represented in the form of a Cauchy problem:

$$\frac{d\overline{X}}{ds} = \mathrm{ort}(J, Q) \tag{26}$$

with initial condition $\overline{X}(s_0) = \overline{X}_0$.

The operation $\mathrm{ort}(J, Q)$ proposed in [5] orthogonalizes the row vectors of the matrix $\overline{J}$ and defines the ort that completes the basis of dimension $(M+1)$. The solution obtained with the Newton–Kantorovich method for $\lambda << \lambda_s$ is used as an initial value $\overline{X}(s_0)$. This procedure corresponds to the method of continuous loading. It proved to be very effective in solving many nonlinear problems [4, 5, 11, 12].

3. Analysis of the Numerical Results. We will use the procedure outlined above to analyze some features of the nonlinear deformation of cylindrical shells made of composites with local geometrical imperfections. Let the shell consist of 10 elementary glassfiber-reinforced plastic plies.

In the examples below, we will use the following common layup sequence: 0°/45°/90°/135°/180°/–180°/–135°/–90°/–45°/0°. The mechanical characteristics of the plies are: $E_1 = 0.43415 \cdot 10^7$ MPa, $E_2 = 0.11338 \cdot 10^6$ MPa, $G_{12} = 0.52888 \cdot 10^5$ MPa, $G_{13} = 0.52888 \cdot 10^5$ MPa, $G_{23} = 0.42830 \cdot 10^5$ MPa, $\nu_1 = 0.28266$.

The first example is a shell under uniform external pressure, while the second example is a shell subject to axial forces uniformly distributed over its ends.

TABLE 1

λ_r	$\bar{\xi}_i$	m	n	λ_i	m_1	n_1
0.12438	0.012772	1	6	0.12438	1	6
0.14021	0.015001	1	5	0.12759	1	7
0.21296	0.016014	1	4	0.13821	1	8
0.50015	0.018384	1	3	0.14021	1	5
1.6988	0.010947	4	6	0.15191	1	9
1.9322	0.019309	1	2	0.16680	1	10
2.3741	0.012857	4	5	0.18193	1	11
3.6327	0.014497	4	4	0.19681	1	12
4.0304	0.012550	6	6	0.21113	1	13
5.69132	0.014740	6	5	0.21296	1	4
6.3609	0.015756	4	3	0.22473	1	14
8.7451	0.016620	6	4	0.23752	1	15
11.987	0.019625	1	1	0.24949	1	16
14.062	0.016549	4	4	0.26062	1	17

The geometrical parameters of the shells are: $L / R = 1$, $t / R = 0.05$, $L_1 / L = 0.36$, $l_1 / L = 0.1$, $t = 0.005$. Such deflections result in an asymmetric (about the midlength) distribution of strains and stresses for $L_1 / L = 0.36$ and in a symmetric distribution for $L_1 / L = 0.5$. It will be shown below that this fact affects the stability of imperfect shells.

If the limits of variation in the wave numbers m and n in (21) and (22) are equal, then each eigenvalue $\lambda_{m,n}$ corresponds to the parameters $\bar{\xi}_{m,n}$. However, it makes no sense to use system (26) for the whole set of λ_{mn} and ξ_{mn} because of the very wide difference in their values. A possible way to do computations is to form two one-dimensional arrays λ_r and $\bar{\xi}_i$. Table 1 shows parts of these arrays for the shell under external pressure. One of the columns contains the eigenvalues λ_i in ascending order. The difference between λ_r and λ_i is due to the difference of the natural modes.

In Table 1: λ_r are the eigenvalues calculated in two stages. At the first stage, we use the coefficients of the matrix ξ_{mn} to form three one-dimensional arrays, one containing the coefficients of the matrix arranged in descending order, the second and third arrays containing the wave numbers m and n, respectively. At the second stage, we form an array of eigenvalues λ_{mn} with m and n taken from the second and third columns, the coefficients of this array being arranged in ascending order in the first column ($\bar{\xi}_i$ are the coefficients of the Fourier series for imperfections; λ_i are the eigenvalues for the perfect shell; m and n are the wave numbers of the functions in the Fourier series; m_1 and n_1 are the wave numbers of the eigenfunctions of the original homogeneous problem).

The accuracy of the results obtained in solving the system of equations (26) depends on the maximum value of the index r, which also indicates the number of coefficients in the columns of Table 1. The equilibrium curves in Figs. 1*a* and 1*b* represent symmetric shells under external pressure or axial compression, respectively. In these figures, the abscissa axis indicates the deflection/thickness ratio for $\varphi = 0$ and $\xi = 0.5$ in the case of external pressure and the shortening/thickness ratio in the case of

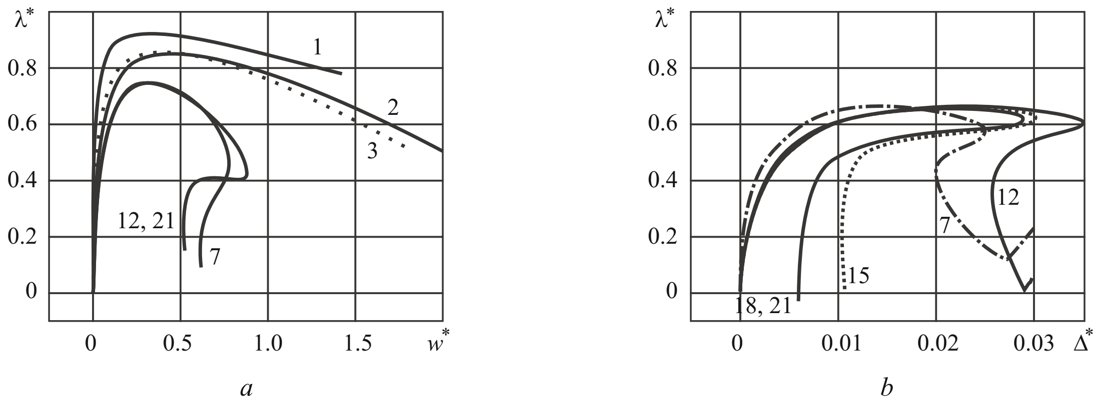


Fig. 1

TABLE 2

λ_r	$\bar{\xi}_i$	m	n	λ_i	m_1	n_1
0.30869	0.016914	1	4	0.30869	1	4
0.32936	0.015001	1	5	0.32156	2	4
0.35100	0.016820	4	1	0.32257	2	5
0.35205	0.016549	4	2	0.32936	1	5
0.35451	0.015756	4	3	0.33779	3	3
0.35925	0.014497	4	4	0.33844	3	4
0.35714	0.012857	4	5	0.34094	3	2
0.37897	0.010947	4	6	0.34097	2	3

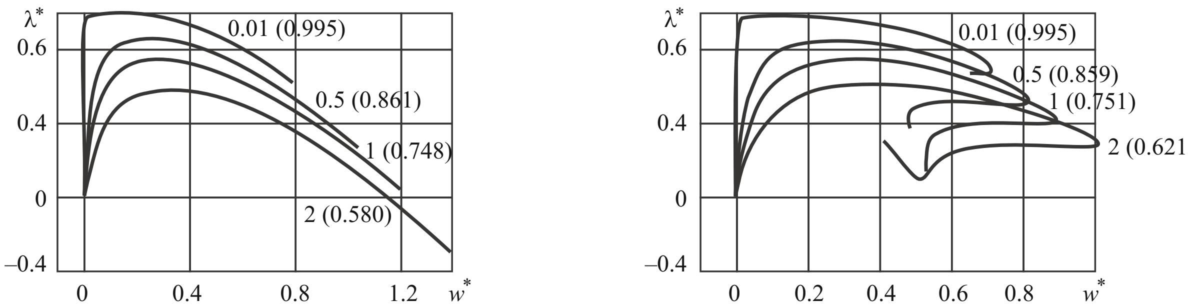
axial compression, while the ordinate axis indicates $\lambda^* = \lambda_r / \lambda_c$, where λ_c is the minimum of λ_i, in both cases. The numbers near the curves are the maximum values of the index *r*, which determines the number of equations in (26).

These results were obtained with 21 equations kept in system (26). The equilibrium curves in Figs. 2*a* and 2*b* represent nonsymmetric and symmetric shells under external pressure, respectively (the numbers near the curves are the values of ξ). A number in the second column multiplied by this parameter is the amplitude of an imperfection ξ with the corresponding wave numbers. The relative critical loads are indicated in brackets near the above-mentioned numbers.

The eigenvalues and associated modes for axial compression are summarized in Table 2 (the notation is the same as in Table 1).

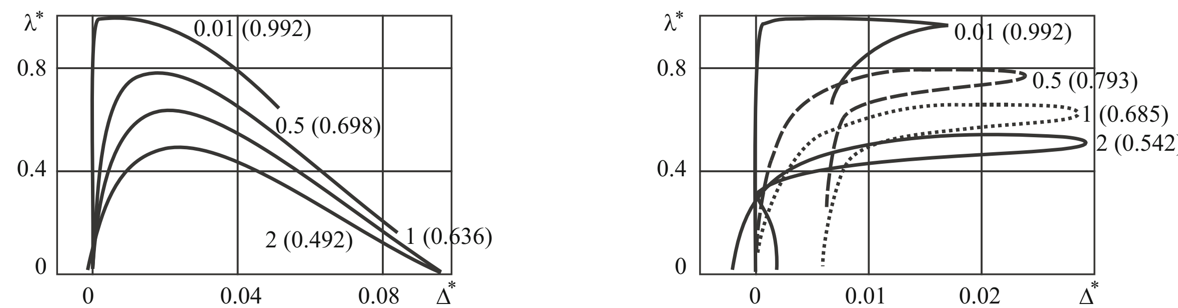
The equilibrium curves for the shell under axial compression are shown in Figs. 3*a* and 3*b*.

Comparing Figs. 2*a*, *b* and Figs. 3*a*, *b* reveals that the shells of medium dimensions under either external pressure or axial compression are sensitive to the local initial deflection, the symmetric shells being more sensitive than the asymmetric shells. However, the critical loads for external pressure and axial compression change differently. If the amplitude of the initial deflection is equal to the thickness of the shell, then the critical load is 0.748 of the critical load of the perfect shell under external pressure and 0.636 under axial compression. The pressure reaches the critical level at substantially larger deflections in the former case than the shortening does in the latter case. The post-buckling behavior of the nonsymmetric shell illustrated by Fig. 2*a* is characterized by a smooth increase in the deflection amplitude and a decrease in the pressure. The equilibrium curves in



a *b*

Fig. 2



a *b*

Fig. 3

Figs. 2*b* and 3*b* are different, their form depending on ξ. The shortening-vs.-load curve for the symmetric shells under axial compression smoothly turns to the unstable branch and does not turn back, unlike the curves for thin isotropic shells [8]. These features are due to the imperfections, the material properties of the shell, and the limited capabilities of the asymptotic method.

Conclusions. It has been established that the Byskov–Hutchinson method can be used to study the nonlinear deformation (including determination of the critical loads) of orthotropic cylindrical shells with local initial deflections with allowance for transverse shear strains. Such imperfections are described by double Fourier series. It has been shown that a limited number of terms should be kept in these series, according to the procedure ensuring the validity of the results obtained.

The stability and post-buckling behavior of a medium-size fiberglass-reinforced plastic shell subject to external pressure or axial compression have been analyzed. It has been revealed that not only the presence of local deflections, but also their position affect the critical loads and, especially, the shape of the post-buckling equilibrium curves. For example, if the deflection amplitude is of the order of the shell thickness, then the critical load is 0.6–0.8 of that for the perfect shell.

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