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DEFORMATION AND SHORT-TERM DAMAGE OF A PHYSICALLY NONLINEAR UNIDIRECTIONAL FIBROUS COMPOSITE

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A structural theory of the coupled processes of deformation and short-term damage of unidirectional fibrous composites with physically nonlinear components whose stress–strain curves have a descending section is developed. The damage process is modeled as the collapse of dispersed microvolumes followed by the formation of quasispherical micropores. A strain-based failure criterion formulated for the second invariant of the deviatoric macrostrain tensor is used as a condition for the short-term damage of a microvolume of the material. The effect of the fiber volume fraction on the deformation and damage of a unidirectional fibrous composite is analyzed

Keywords: unidirectional fibrous composite, short-term damage, physical nonlinearity, stochastic structure, effective characteristics, damage (porosity) balance equation

Introduction. Fracture mechanics uses not only the theory of cracks [1, 6, 10, 13], but also the theory of accumulation of damages [2–5, 7–9, 11, 12, 14], which, coalescing, give rise to main cracks. There are models based on ideas about the structure of material [3, 7, 9–12, 14] and models based on formal damage parameters and equations describing their evolution [2, 4, 5].

The structural models of the coupled processes of deformation and damage of homogeneous and composite materials [7, 9, 11, 12] are based on the idea of the stochastic micrononuniformity of the strength of a material because of which dispersed microdamages occur and accumulate under loading. Modeling microdamages by quasispherical pores allows the mechanics of stochastically inhomogeneous materials [8] to be used to describe the coupled processes of deformation and damage.

In [7, 11, 12], a microvolume is assumed to collapse in accordance with the Huber–Mises or Schleicher–Nadai failure criterion formulated for the corresponding microstresses. This makes it possible to study the dependence of damage (porosity) on the macrostresses and the nonlinearity of the macrostress–macrostrain relationship caused by damage for homogeneous and composite materials with given (linear and nonlinear) laws of deformation of the undamaged portion, provided that the microstress–microstrain relationship is unambiguous. It is clear that with a microstress-based failure criterion, the damage of a material can only be described on the ascending portion of the nonlinear stress–strain curve for a microvolume.

Here we will construct a structural model to describe the short-term damage of a unidirectional fibrous composite using a strain-based microfailure criterion for its components. It will allow us to describe the total service life of a composite, including the descending sections of the stress–strain curves of its components. We will use a strain-based failure criterion formulated for the second invariants of the deviatoric microstrain tensors as a condition for the short-term damage of microvolumes in the composite components. The ultimate strength is assumed to be a random function of coordinates whose one-point distribution is described by a power function on some interval or by the Weibull function.

The effective deformation properties and the stress–strain state of a material with random microdamages are determined from the stochastic equations of nonlinear elasticity of porous composites, given macrostrains for the porous components. The macrostrains for the porous components and the effective elastic characteristics of a fibrous composite are

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determined from the stochastic equations of nonlinear elasticity of a nonlinear elastic unidirectional fibrous composite, given its macrostrains. Considering the ergodicity of the random field of ultimate microstrength and the properties of distribution functions, we will derive the damage (porosity) balance equation to close the system of equations describing the coupled processes of deformation and damage over the entire range of microstrains in the undamaged portion of the composite components.

This theory will be used to solve the problems of the deformation and damage of a unidirectional composite. Its matrix is nonlinear elastic, has microdamages, is described by a stress–strain curve with a descending section, and is reinforced with linear elastic fibers without damages. We will analyze the effect of the volume fraction of fibers on the stress–strain curves of the composite and the variation of the elastic characteristics with the macrostrains.

1. Starting Equations. Consider a two-component composite with an isotropic matrix of volume fraction c_1 reinforced with unidirectional continuous isotropic fibers of volume fraction c_2. The components are physically nonlinear. Their elastic deformation is accompanied by the formation of damaged microvolumes because of the stochastic nonuniformity of microstrength. Destroyed microvolumes are modeled by quasispherical pores, their size and distances between them being negligible compared with the diameters of the fibers and the distances between them. The initial porosities and total porosities of the components are denoted by p_{10}, p_{20} and p_1, p_2, respectively, the bulk and shear moduli of the undamaged portions of the components by K_1, K_2 and μ_1, μ_2, respectively, and the effective bulk and shear moduli of the porous components by K_1^*, K_2^* and μ_1^*, μ_2^*, respectively.

Then the stress–strain state and effective characteristics of a unidirectional fibrous material with porous components can be determined by (i) finding the stresses and strains $\langle\sigma_{ij}^1\rangle$, $\langle\varepsilon_{ij}^1\rangle$, $\langle\sigma_{ij}^2\rangle$, $\langle\varepsilon_{ij}^2\rangle$ in the undamaged portions of the components and the effective characteristics K_1^*, μ_1^* and K_2^*, μ_2^* of the porous components, given the macrostrains $\langle\varepsilon_{ij}^{*1}\rangle$ $\langle\varepsilon_{ij}^{*2}\rangle$ and porosities p_1, p_2 of the porous components; and (ii) determining the stresses and strains $\langle\sigma_{ij}^{*1}\rangle$ $\langle\varepsilon_{ij}^{*1}\rangle$ $\langle\sigma_{ij}^{*2}\rangle$ $\langle\varepsilon_{ij}^{*2}\rangle$ of the porous components and the effective characteristics λ_{11}^*, λ_{12}^*, λ_{13}^*, λ_{33}^*, λ_{44}^* of the unidirectional fibrous composite, given its macrostrains $\langle\varepsilon_{ij}\rangle$.

Consider a two-component stochastic composite with perfectly bonded components, which can be regarded as microinhomogeneous physically nonlinear statistically homogeneous elastic material. The microstresses σ_{ij} and microstrains ε_{ij} at an arbitrary point are related by

$$\sigma_{ij} = \lambda_{ijmn}(\varepsilon_{\alpha\beta})\varepsilon_{mn}, \tag{1.1}$$

where λ_{ijmn} is the stiffness tensor deterministically depending on the strains $\varepsilon_{\alpha\beta}$ and being a statistically homogeneous random function of the coordinates x_r.

If a macrovolume is subject to macrohomogeneous deformation, the microstresses σ_{ij} and microstrains ε_{ij} are ergodic statistically homogeneous random functions of the coordinates. Their expectations $\langle\sigma_{ij}\rangle$ and $\langle\varepsilon_{ij}\rangle$ at an arbitrary point are equal to the macrostresses and macrostrains, respectively [8]. Using the equilibrium equations

$$\sigma_{ij,j} = 0, \tag{1.2}$$

the kinematic equations

$$\varepsilon_{ij} = u_{(i,j)} \equiv \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.3}$$

and formulas (1.1), we arrive at physically and statistically nonlinear equilibrium equations for the displacements u_i:

$$[\lambda_{ijmn}(\varepsilon_{\alpha\beta})u_{m,n}]_{,j} = 0. \tag{1.4}$$

Representing the random fields of stresses, strains, and displacements as sums of expectations and fluctuations

$$\sigma_{ij} = \langle\sigma_{ij}\rangle + \sigma_{ij}^0, \quad \varepsilon_{ij} = \langle\varepsilon_{ij}\rangle + \varepsilon_j^0, \quad u_i = \langle\varepsilon_{ij}\rangle x_j + u_i^0, \tag{1.5}$$

we reduce Eq. (1.4) to

$$\lambda^{c}_{ijmn} u^{0}_{m,nj} + \{[\lambda_{ijmn}(\varepsilon_{\alpha\beta}) - \lambda^{c}_{ijmn}]\varepsilon_{mn}\}_{,j} = 0, \tag{1.6}$$

where λ^{c}_{ijmn} is the stiffness tensor of some homogeneous comparison body. The boundary condition for the infinitely distant boundary S of the macrovolume V is as follows, according (1.5):

$$u^{0}_{i}\Big|_{S} = 0. \tag{1.7}$$

Using Green's function $G_{ij}(x^{(1)}_{r} - x^{(2)}_{r})$ satisfying the equation

$$\lambda^{c}_{ijmn} G_{mk,jn}(x^{(1)}_{r} - x^{(2)}_{r}) + \delta(x^{(1)}_{r} - x^{(2)}_{r})\delta_{ik} = 0, \tag{1.8}$$

we reduce the boundary-value problem (1.6), (1.7) to an integral equation for the strains:

$$\varepsilon^{(1)}_{ij} = \langle\varepsilon_{ij}\rangle + K_{ijpq}(x^{(1)}_{r} - x^{(2)}_{r})[\lambda^{(2)}_{pqmn}(\varepsilon^{(2)}_{\alpha\beta}) - \lambda^{c}_{pqmn}]\varepsilon^{(2)}_{mn}, \tag{1.9}$$

where K_{ijpq} is an integral operator defined by

$$K_{ijpq}(x^{(1)}_{r} - x^{(2)}_{r})\varphi^{(2)} = \int_{V^{(2)}} G_{(ip,j)q}(x^{(1)}_{r} - x^{(2)}_{r})(\varphi^{(2)} - \langle\varphi\rangle)dV^{(2)}, \tag{1.10}$$

where the superscript in parentheses denotes the corresponding point in space.

The nonlinear relations (1.1) for a point in the kth component have the form

$$\sigma^{k}_{ij} = \lambda^{k}_{ijmn}(\varepsilon^{k}_{\alpha\beta})\varepsilon^{k}_{mn}, \tag{1.11}$$

where the stresses and strains can be represent by sums of averages and fluctuations in the kth component,

$$\sigma^{k}_{ij} = \langle\sigma^{k}_{ij}\rangle + \sigma^{k0}_{ij}, \qquad \varepsilon^{k}_{ij} = \langle\varepsilon^{k}_{ij}\rangle + \varepsilon^{k0}_{ij}. \tag{1.12}$$

If the fluctuations σ^{k0}_{ij} and ε^{k0}_{ij} are neglected, relations (1.11) and (1.12) yield

$$\langle\sigma^{k}_{ij}\rangle = \lambda^{k}_{ijmn}(\langle\varepsilon^{k}_{\alpha\beta}\rangle)\langle\varepsilon^{k}_{mn}\rangle. \tag{1.13}$$

Averaging over the macrovolume, we derive an expression for the macrostresses of an N-component material:

$$\langle\sigma_{ij}\rangle = \sum_{k=1}^{N} c_{k}\lambda^{k}_{ijmn}(\langle\varepsilon^{k}_{\alpha\beta}\rangle)\langle\varepsilon^{k}_{mn}\rangle. \tag{1.14}$$

Let us average the integral equation (1.9) using conditional density $f(\varepsilon^{(1)}_{ij}, \varepsilon^{(2)}_{ij}, \lambda^{(2)}_{ijmn}|^{(1)}_{v})$ (distribution density of the strains at the points $x^{(1)}_{r}, x^{(2)}_{r}$ and the elastic moduli at the point $x^{(2)}_{r}$ provided that the point $x^{(1)}_{r}$ is in the vth component). Then, neglecting the fluctuations of strains within the component, we obtain a system of nonlinear algebraic equations for the average strains in the component:

$$\langle\varepsilon^{v}_{ij}\rangle = \langle\varepsilon_{ij}\rangle + \sum_{k=1}^{N} K^{vk}_{ijpq}[\lambda^{k}_{pqmn}(\langle\varepsilon^{k}_{\alpha\beta}\rangle) - \lambda^{c}_{pqmn}]\langle\varepsilon^{k}_{mn}\rangle \quad (v = 1, \ldots, N). \tag{1.15}$$

According to (1.10), the matrix operator K^{vk}_{ijpq} is defined by

$$K^{vk}_{ijpq} = K_{ijpq}(x^{(1)}_{r} - x^{(2)}_{r})p_{vk}(x^{(1)}_{r} - x^{(2)}_{r}) \quad (v, k = 1, \ldots, N), \tag{1.16}$$

where $p_{\nu k}(x_r^{(1)} - x_r^{(2)}) = f(\,^{(2)}_{k}|^{(1)}_{\nu})$ is the probability of transition from the point $x_r^{(1)}$ in the νth component to the point $x_r^{(2)}$ in the kth component,

$$c_k p_{\nu k}(x_r) = c_\nu p_{\nu k}(x_r), \quad p_{\nu k}(0) = \delta_{\nu k}, \quad p_{\nu k}(\infty) = c_k, \quad \sum_{k=1}^{N} p_{\nu k}(x_r) = 1. \tag{1.17}$$

Let us consider a two-component composite with an isotropic matrix and isotropic unidirectional quasispheroidal inclusions:

$$\lambda^k_{ijmn}(\langle\varepsilon^k_{\alpha\beta}\rangle) = \lambda_k(\langle\varepsilon^k_{\alpha\beta}\rangle)\delta_{ij}\delta_{mn} + 2\mu_k(\langle\varepsilon^k_{\alpha\beta}\rangle)I_{ijmn} \qquad (k = 1, 2), \tag{1.18}$$

$$\lambda^{c}_{ijmn} = \lambda_c\delta_{ij}\delta_{mn} + 2\mu_c I_{ijmn}, \qquad p_{\nu k} = c_k + (\delta_{\nu k} - c_k)\exp\left[-\sqrt{n_1^2(x_1^2 + x_2^2) + n_2^2 x_3^2}\,\right],$$

where λ_k, μ_k, λ_c, μ_c are the elastic moduli of the components and comparison body; $I_{ijmn} = (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})/2$ is a unit tensor; n_1 and n_2 are the reciprocal semiaxes of quasispheroidal inclusions. In this case, operator (1.16) is defined by

$$K^{\nu k}_{ijpq} = (\delta_{\nu k} - c_k)\Big\{a_1\delta_{ij}\delta_{pq} + a_2 I_{ijpq} + a_3\Big[\delta_{ij}\delta_{3p}\delta_{3q} + \delta_{i3}\delta_{j3}\Big(\delta_{pq} - 2\delta_{3p}\delta_{3q}\Big)\Big] + a_4\delta_{i3}\delta_{j3}\delta_{3p}\delta_{3q}$$

$$+ a_5(I_{i3pq}\delta_{j3} + I_{j3pq}\delta_{i3} - 2\delta_{i3}\delta_{j3}\delta_{3p}\delta_{3q})\Big\}, \tag{1.19}$$

$$a_1 = \frac{(\lambda_c + \mu_c)(1 - s_1 - s_2)}{8\mu_c(\lambda_c + 2\mu_c)}, \qquad a_2 = -\frac{(\lambda_c + 3\mu_c)(1 - s_1) + (\lambda_c + \mu_c)s_2}{4\mu_c(\lambda_c + 2\mu_c)},$$

$$a_3 = \frac{(\lambda_c + \mu_c)(s_1 + 5s_2 - 1)}{8\mu_c(\lambda_c + 2\mu_c)}, \qquad a_4 = \frac{\lambda_c + 5\mu_c - (\lambda_c + 13\mu_c)s_1 - 5(\lambda_c + \mu_c)s_2}{8\mu_c(\lambda_c + 2\mu_c)},$$

$$a_5 = \frac{\mu_c - (2\lambda_c + 5\mu_c)s_1 + 5(\lambda_c + \mu_c)s_2}{4\mu_c(\lambda_c + 2\mu_c)}, \qquad s_1 = \frac{1 - s}{1 - n^2}, \qquad s_2 = \frac{1 - (1 + 2n^2)s_1}{2(1 - n^2)},$$

$$n = \frac{n_1}{n_2}, \qquad s = \begin{cases} -\dfrac{n}{\sqrt{n^2 - 1}}\ln\Big(n - \sqrt{n^2 - 1}\Big), & n \ge 1, \\[2ex] \dfrac{n}{\sqrt{n^2 - 1}}\arcsin\sqrt{1 - n^2}, & n \le 1. \end{cases}$$

If $n = 0$, $n = \infty$, $n = 1$, we obtain the expressions of the operator for laminated, unidirectional fibrous, and particulate materials, respectively.

2. Short-Term Damage of a Unidirectional Fibrous Material. Let the bulk moduli of the fibers (K_1) and the matrix (K_2) be constant, while the shear moduli μ_1 and μ_2 be defined by the functions

$$\mu_i(J^i_\varepsilon) = \begin{cases} \mu_{i0}, & J^i_\varepsilon < \dfrac{k_i}{2\mu_{i0}}, \\[2ex] \mu'_i + \left(1 - \dfrac{\mu'_i}{\mu_{i0}}\right)\dfrac{k_i}{2J^i_\varepsilon}, & J^i_\varepsilon \ge \dfrac{k_i}{2\mu_{i0}}, \end{cases} \tag{2.1}$$

$$J^i_\varepsilon = (\langle\varepsilon^i_{pq}\rangle'\langle\varepsilon^i_{pq}\rangle')^{1/2} \qquad (i = 1, 2),$$

where $\langle\varepsilon^i_{pq}\rangle'$ is the deviatoric average-strain tensor in the undamaged portion of the ith component. According to (1.15)–(1.19) for $n = 1$, the strains $\langle\varepsilon^i_{pq}\rangle$ and the effective moduli of the porous fibers and matrix (K^*_ν, μ^*_ν) are defined by

$$\langle\varepsilon^{v}_{pq}\rangle=\left[\frac{\overline{K}_v}{\overline{K}_v+K_v p_v}V_{ijpq}+\frac{\overline{\mu}_v}{\overline{\mu}_v+\mu_v(J^v_\varepsilon)}D_{ijpq}\right]\langle\varepsilon^{*v}_{pq}\rangle,$$

$$K^*_v=\frac{4\mu_{v0}\xi_v(1-p_v)^2}{4+(3\xi_v-4)p_v},\quad \mu^*_v=\frac{\mu_{v0}\hat{\mu}_v(J^v_\varepsilon)(1-p_v)^2}{1+\left[\eta_v\hat{\mu}_v(J^v_\varepsilon)-1\right]p_v},\quad \mu^*_{v0}=\frac{\mu_{v0}(1-p_v)^2}{1+(\eta_v-1)p_v}, \tag{2.2}$$

$$\mu_v=\mu_{v0}\hat{\mu}_v(J^v_\varepsilon),\quad \overline{K}_v=\frac{4}{3}\mu_{v0}(1-p_v),\quad \overline{\mu}_v=\frac{1}{\eta_v}\mu_{v0}(1-p_v),\quad \xi_v=\frac{K_v}{\mu_{v0}},$$

$$\eta_v=\frac{6(K_v+2\mu_{v0})}{9K_v+8\mu_{v0}},\quad V_{ijpq}=\frac{1}{3}\delta_{ij}\delta_{pq},\quad D_{ijpq}=\frac{1}{2}\left(\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{jp}-\frac{2}{3}\delta_{ij}\delta_{pq}\right)\quad (v=1,2).$$

The average strains $\langle\varepsilon^{*v}_{ij}\rangle$ in the vth component are related to the macrostrains $\langle\varepsilon_{ij}\rangle$, according to (1.15)–(1.19) for $n=\infty$, by

$$\langle\varepsilon^{*v}_{11}\rangle=l^*_{v1}\langle\varepsilon_{11}\rangle+l^*_{v2}\langle\varepsilon_{22}\rangle+l^*_{v3}\langle\varepsilon_{33}\rangle,\quad \langle\varepsilon^{*v}_{12}\rangle=(l^*_{v1}-l^*_{v2})\langle\varepsilon_{12}\rangle,$$

$$\langle\varepsilon^{*v}_{22}\rangle=l^*_{v2}\langle\varepsilon_{11}\rangle+l^*_{v1}\langle\varepsilon_{22}\rangle+l^*_{v3}\langle\varepsilon_{33}\rangle,\quad \langle\varepsilon^{*v}_{13}\rangle=l^*_{v4}\langle\varepsilon_{13}\rangle,$$

$$\langle\varepsilon^{*v}_{33}\rangle=\langle\varepsilon_{33}\rangle,\quad \langle\varepsilon^{*v}_{23}\rangle=l^*_{v4}\langle\varepsilon_{23}\rangle, \tag{2.3}$$

where $l^*_{v1}, l^*_{v2}, l^*_{v3}, l^*_{v4}$ are defined by

$$l^*_{v1}=\frac{1}{2}\left[\frac{1}{k^*_v+\overline{k}^*}\left(\frac{c_1}{k^*_1+\overline{k}^*}+\frac{c_2}{k^*_2+\overline{k}^*}\right)^{-1}+\frac{1}{m^*_v+\overline{m}^*}\left(\frac{c_1}{m^*_1+\overline{m}^*}+\frac{c_2}{m^*_2+\overline{m}^*}\right)^{-1}\right],$$

$$l^*_{v2}=\frac{1}{2}\left[\frac{1}{k^*_v+\overline{k}^*}\left(\frac{c_1}{k^*_1+\overline{k}^*}+\frac{c_2}{k^*_2+\overline{k}^*}\right)^{-1}-\frac{1}{m^*_v+\overline{m}^*}\left(\frac{c_1}{m^*_1+\overline{m}^*}+\frac{c_2}{m^*_2+\overline{m}^*}\right)^{-1}\right],$$

$$l^*_{v3}=\frac{1}{2(k^*_v+\overline{k}^*)}\left[\left(\frac{c_1}{k^*_1+\overline{k}^*}+\frac{c_2}{k^*_2+\overline{k}^*}\right)^{-1}\left(\frac{c_1\lambda^*_1}{k^*_1+\overline{k}^*}+\frac{c_2\lambda^*_2}{k^*_2+\overline{k}^*}\right)-\lambda^*_v\right],$$

$$l^*_{v4}=\frac{1}{\mu^*_v+\overline{\mu}^*}\left(\frac{c_1}{\mu^*_1+\overline{\mu}^*}+\frac{c_2}{\mu^*_2+\overline{\mu}^*}\right)^{-1},\quad \lambda^*_v=K^*_v-\frac{2}{3}\mu^*_v,\quad k^*_v=K^*_v+\frac{1}{3}\mu^*_v,$$

$$m^*_v=\mu^*_v,\quad \overline{k}^*=m^*_c,\quad \overline{m}^*=\frac{k^*_c m^*_c}{k^*_c+2m^*_c},\quad m^*_c=\left(\frac{c_1}{\mu^*_{10}}+\frac{c_2}{\mu^*_{20}}\right)^{-1},$$

$$k^*_c=\left(\frac{c_1}{K^*_1}+\frac{c_2}{K^*_{21}}\right)^{-1}+\frac{1}{3}m^*_c,\quad \overline{\mu}^*=\mu^*_c,\quad \mu^*_c=m^*_c. \tag{2.4}$$

It is assumed that the damaged fibers are stiffer than the damaged matrix.

The effective moduli of a unidirectional fibrous material, according to (1.14), (1.18), (1.19), (2.4), are expressed as

$$k^* = \left(\frac{c_1 k_1^*}{k_1^* + \bar{k}^*} + \frac{c_2 k_2^*}{k_2^* + \bar{k}^*} \right) \left(\frac{c_1}{k_1^* + \bar{k}^*} + \frac{c_2}{k_2^* + \bar{k}^*} \right)^{-1},$$

$$m^* = \left(\frac{c_1 \mu_1^*}{\mu_1^* + \bar{m}^*} + \frac{c_2 \mu_1^*}{\mu_2^* + \bar{m}^*} \right) \left(\frac{c_1}{\mu_1^* + \bar{m}^*} + \frac{c_2}{\mu_2^* + \bar{m}^*} \right)^{-1},$$

$$\lambda_{11}^* = k^* + m^*, \quad \lambda_{12}^* = k^* - m^*, \quad \lambda_{13}^* = \left(\frac{c_1 \lambda_1^*}{k_1^* + \bar{k}^*} + \frac{c_2 \lambda_2^*}{k_2^* + \bar{k}^*} \right)^2 \left(\frac{c_1}{k_1^* + \bar{k}^*} + \frac{c_2}{k_2^* + \bar{k}^*} \right)^{-1},$$

$$\lambda_{33}^* = c_1 (\lambda_1^* + 2\mu_1^*) + c_2 (\lambda_2^* + 2\mu_2^*) + \left(\frac{c_1 \lambda_1^*}{k_1^* + \bar{k}^*} + \frac{c_2 \lambda_2^*}{k_2^* + \bar{k}^*} \right)^2 \left(\frac{c_1}{k_1^* + \bar{k}^*} + \frac{c_2}{k_2^* + \bar{k}^*} \right)^{-1} - \left(\frac{c_1 \lambda_1^{*2}}{k_1^* + \bar{k}^*} + \frac{c_2 \lambda_2^{*2}}{k_2^* + \bar{k}^*} \right),$$

$$\mu^* = \left(\frac{c_1 \mu_1^*}{\mu_1^* + \bar{\mu}^*} + \frac{c_2 \mu_2^*}{\mu_2^* + \bar{\mu}^*} \right) \left(\frac{c_1}{\mu_1^* + \bar{\mu}^*} + \frac{c_2}{\mu_2^* + \bar{\mu}^*} \right)^{-1}. \tag{2.5}$$

From (2.2)–(2.4) we get

$$J_\varepsilon^\nu = \frac{1 - p_\nu}{1 + \left[\eta_\nu \hat{\mu}_\nu (J_\varepsilon^\nu) - 1 \right] p_\nu} J_\varepsilon^{*\nu},$$

$$J_\varepsilon^{*\nu} = \left\{ \frac{2}{3} (l_{\nu1}^{*2} + l_{\nu2}^{*2} - l_{\nu1}^* l_{\nu2}^*)(\langle \varepsilon_{11} \rangle + \langle \varepsilon_{22} \rangle)^2 - 2 (l_{\nu1}^* - l_{\nu2}^*)^2 \langle \varepsilon_{11} \rangle \langle \varepsilon_{22} \rangle \right.$$

$$- \frac{2}{3} (l_{\nu1}^* + l_{\nu2}^*)(1 - l_{\nu3}^*)(\langle \varepsilon_{11} \rangle + \langle \varepsilon_{22} \rangle) \langle \varepsilon_{33} \rangle + \frac{2}{3} (1 + 2 l_{\nu3}^{*2}) \langle \varepsilon_{33}^2 \rangle$$

$$\left. + 2 \left[(l_{\nu1}^* - l_{\nu2}^*)^2 \langle \varepsilon_{12}^2 \rangle + l_{\nu4}^{*2} (\langle \varepsilon_{13} \rangle^2 + \langle \varepsilon_{23}^2 \rangle) \right] \right\}^{1/2}. \tag{2.6}$$

Using the following strain-based failure criterion for a microvolume of the undamaged portion of the νth component:

$$J_\varepsilon^\nu = r_\nu \quad (\nu = 1, 2) \tag{2.7}$$

considering that the ultimate strength r_ν forms an ergodic random field, using the properties of a one-point distribution function $F_\nu(r_\nu)$, and modeling the destroyed microvolumes by random quasispherical pores, we arrive at the damage (porosity) balance equation for the νth component:

$$p_\nu = p_{\nu0} + (1 - p_{\nu0}) F_\nu (J_\varepsilon^\nu). \tag{2.8}$$

From (2.6) and (2.8) we obtain a system of nonlinear equations for J_ε^ν and p_ν:

$$J_\varepsilon^\nu \{ 1 + \left[\eta_\nu \hat{\mu}_\nu (J_\varepsilon^\nu) - 1 \right] p_\nu \} = (1 - p_\nu) \left\{ \frac{2}{3} (l_{\nu1}^{*2} + l_{\nu2}^{*2} - l_{\nu1}^* l_{\nu2}^*) \left(\langle \varepsilon_{11} \rangle + \langle \varepsilon_{22} \right)^2 \right.$$

$$- 2 (l_{\nu1}^* - l_{\nu2}^*)^2 \langle \varepsilon_{11} \rangle \langle \varepsilon_{22} \rangle - \frac{2}{3} (l_{\nu1}^* + l_{\nu2}^*)(1 - l_{\nu3}^*)(\langle \varepsilon_{11} \rangle + \langle \varepsilon_{22} \rangle) \langle \varepsilon_{33} \rangle$$

$$\left. + \frac{2}{3} (1 + 2 l_{\nu3}^{*2}) \langle \varepsilon_{33}^2 \rangle + 2 \left[(l_{\nu1}^* - l_{\nu2}^*)^2 \langle \varepsilon_{12} \rangle^2 + l_{\nu4}^{*2} (\langle \varepsilon_{13} \rangle^2 + \langle \varepsilon_{23}^2 \rangle) \right] \right\}^{1/2},$$

$$p_{\nu} = p_{\nu 0} + (1 - p_{\nu 0}) F_{\nu}(J_{\varepsilon}^{\nu}) \quad (\nu = 1, 2). \tag{2.9}$$

Let us consider cases of specifying various simple macrostrains. If the composite is subject to uniaxial cross-fiber tension

$$\langle \varepsilon_{11} \rangle \neq 0, \quad \langle \sigma_{22} \rangle = \langle \sigma_{33} \rangle = \langle \sigma_{12} \rangle = \langle \sigma_{13} \rangle = \langle \sigma_{23} \rangle = 0 \tag{2.10}$$

then

$$J_{\varepsilon}^{*\nu} = \left\{ \frac{2}{3} (l_{\nu 1}^{*2} + l_{\nu 2}^{*2} - l_{\nu 1}^{*} l_{\nu 2}^{*})(1 - \nu_{21}^{*})^2 + 2(l_{\nu 1}^{*} + l_{\nu 2}^{*})^2 \nu_{21}^{*} + \frac{2}{3} \Big[(l_{\nu 1}^{*} + l_{\nu 2}^{*})(1 - l_{\nu 3}^{*})(1 - \nu_{21}^{*}) \nu_{31}^{*}$$

$$+ (1 + 2 l_{\nu 3}^{*2}) \nu_{31}^{*2} \Big] \right\}^{1/2} \langle \varepsilon_{11} \rangle, \quad \langle \varepsilon_{22} \rangle = -\nu_{21}^{*} \langle \varepsilon_{11} \rangle, \quad \langle \varepsilon_{33} \rangle = -\nu_{31}^{*} \langle \varepsilon_{11} \rangle,$$

$$\nu_{21}^{*} = \nu_{12}^{*} = \frac{\lambda_{12}^{*} \lambda_{33}^{*} - \lambda_{13}^{*2}}{\lambda_{11}^{*} \lambda_{33}^{*} - \lambda_{13}^{*2}}, \quad \nu_{31}^{*} = \frac{(\lambda_{11}^{*} - \lambda_{12}^{*}) \lambda_{13}^{*}}{\lambda_{11}^{*} \lambda_{33}^{*} - \lambda_{13}^{*2}}, \quad \langle \sigma_{11} \rangle = (\lambda_{11}^{*} - \nu_{21}^{*} \lambda_{12}^{*} - \nu_{31}^{*} \lambda_{13}^{*}) \langle \varepsilon_{11} \rangle. \tag{2.11}$$

If the composite is subject to uniaxial along-fiber tension

$$\langle \varepsilon_{33} \rangle \neq 0, \quad \langle \sigma_{11} \rangle = \langle \sigma_{22} \rangle = \langle \sigma_{12} \rangle = \langle \sigma_{13} \rangle = \langle \sigma_{23} \rangle = 0 \tag{2.12}$$

then

$$J_{\varepsilon}^{*\nu} = \left[\frac{8}{3} (l_{\nu 1}^{*2} + l_{\nu 2}^{*2} - l_{\nu 1}^{*} l_{\nu 2}^{*}) \nu_{13}^{*2} - 2(l_{\nu 1}^{*} - l_{\nu 2}^{*})^2 \nu_{13}^{*2} + \frac{4}{3} (l_{\nu 1}^{*} + l_{\nu 2}^{*})(1 - l_{\nu 3}^{*}) \nu_{13}^{*2} + \frac{2}{3} (1 + 2 l_{\nu 3}^{*2}) \right]^{1/2} \langle \varepsilon_{33} \rangle,$$

$$\langle \varepsilon_{11} \rangle = \langle \varepsilon_{22} \rangle = -\nu_{13}^{*} \langle \varepsilon_{33} \rangle, \quad \nu_{13}^{*} = \frac{\lambda_{13}^{*}}{\lambda_{11}^{*} + \lambda_{12}^{*}}, \quad \langle \sigma_{33} \rangle = (\lambda_{33}^{*} - 2\nu_{13}^{*} \lambda_{13}^{*}) \langle \varepsilon_{33} \rangle. \tag{2.13}$$

In the case of shear deformation

$$\langle \varepsilon_{12} \rangle \neq 0, \quad \langle \sigma_{11} \rangle = \langle \sigma_{22} \rangle = \langle \sigma_{33} \rangle = \langle \sigma_{13} \rangle = \langle \sigma_{23} \rangle = 0 \tag{2.14}$$

and

$$\langle \varepsilon_{13} \rangle \neq 0, \quad \langle \sigma_{11} \rangle = \langle \sigma_{22} \rangle = \langle \sigma_{33} \rangle = \langle \sigma_{12} \rangle = \langle \sigma_{23} \rangle = 0, \tag{2.15}$$

then

$$J_{\varepsilon}^{*\nu} = \sqrt{2} (l_{\nu 1}^{*} - l_{\nu 2}^{*}) \langle \varepsilon_{12} \rangle, \quad \langle \sigma_{12} \rangle = 2m^{*} \langle \varepsilon_{12} \rangle, \tag{2.16}$$

$$J_{\varepsilon}^{*\nu} = \sqrt{2} l_{\nu 4}^{*} \langle \varepsilon_{13} \rangle, \quad \langle \sigma_{13} \rangle = 2\mu^{*} \langle \varepsilon_{13} \rangle. \tag{2.17}$$

3. Numerical Results. The coupled processes of deformation and damage were numerically analyzed for a unidirectional fiberglass-reinforced plastic composite with cured epoxy matrix whose stress–strain curve has a descending section with the following dimensionless characteristics:

$$\frac{K_2}{\mu_{20}} = 3.238, \quad \frac{k_2}{\mu_{20}} = 0.02207, \quad \frac{\mu_2'}{\mu_{20}} = -0.05. \tag{3.1}$$

Let the one-point distribution function of the ultimate microstrength of the matrix have the form of Weibull distribution:

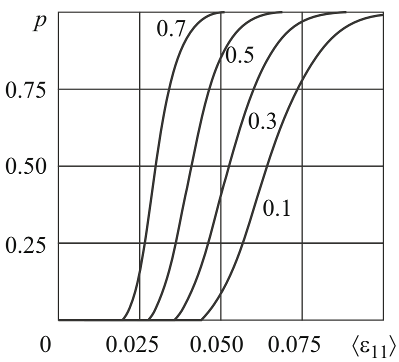

Fig. 1

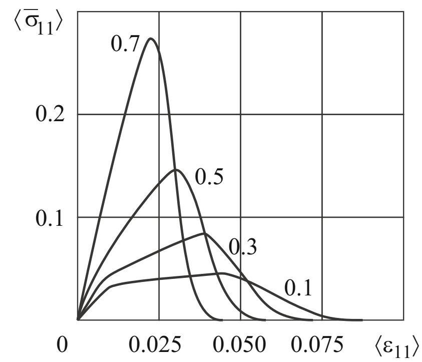

Fig. 2

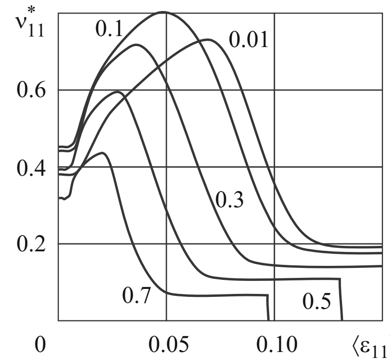

Fig. 3

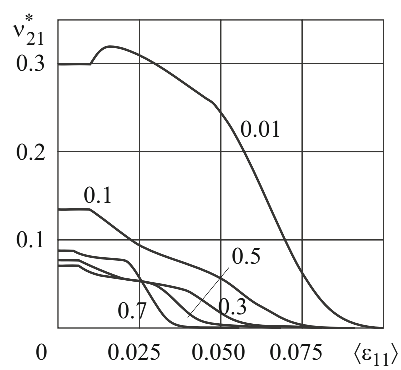

Fig. 4

$$F_2(r_2) = \begin{cases} 0, & r_2 < r_{20}, \\ 1 - \exp\left[-m_2 \left(r_2 - r_{20} \right)^{n_2} \right], & r_2 \geq r_{20}, \end{cases} \tag{3.2}$$

$$m_2 = 1000, \quad n_2 = 2, \quad r_{20} = 0.05, \quad p_{20} = 0.$$

The fibers are assumed to be linear elastic and have the following dimensionless characteristics:

$$\frac{K_1}{\mu_{20}} = 33.584, \quad \frac{\mu_1}{\mu_{20}} = 25.188, \tag{3.3}$$

the fibers being not damaged, i.e., $p_1 = p_{10} = 0$. In this case, the system of nonlinear equations (2.9) reduces to two equations:

$$J_\varepsilon^2 \left\{ 1 + \left[\eta_2 \hat{\mu}_2 (J_\varepsilon^2) - 1 \right] \right\} = (1 - p^2) \left\{ \frac{2}{3} (l_{21}^{*2} + l_{22}^{*2} - l_{21}^* l_{22}^*)(\langle \varepsilon_{11} \rangle + \langle \varepsilon_{22} \rangle)^2 \right.$$

$$- 2(l_{21}^* - l_{22}^*)^2 \langle \varepsilon_{11} \rangle \langle \varepsilon_{22} \rangle - \frac{2}{3} (l_{21}^* + l_{22}^*)(1 - l_{23}^*)(\langle \varepsilon_{11} \rangle + \langle \varepsilon_{22} \rangle) \langle \varepsilon_{33} \rangle + \frac{2}{3} (1 + 2 l_{23}^{*2}) \langle \varepsilon_{33} \rangle^2$$

$$\left. + 2 \left[(l_{21}^* - l_{22}^*)^2 \langle \varepsilon_{12} \rangle^2 + l_{24}^{*2} (\langle \varepsilon_{13} \rangle^2 + \langle \varepsilon_{23}^2 \rangle) \right] \right\}^{1/2}, \quad p_2 = F_2(J_\varepsilon^2). \tag{3.4}$$

The numerical solution of the nonlinear equations (3.4) with (2.2), (2.4), (2.5), (2.10)–(2.13) for the unidirectional fibrous composite is presented in Figs. 1–7. They show the porosity p_2, the macrostresses

$$\langle \bar{\sigma}_{11} \rangle = \frac{1}{\mu_{20}} \langle \sigma_{11} \rangle,$$

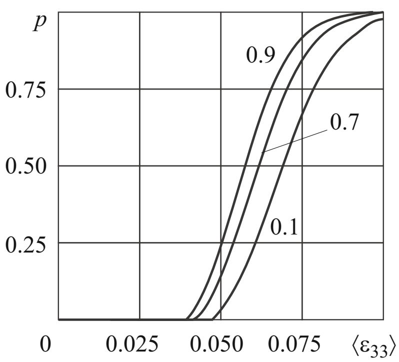

Fig. 5

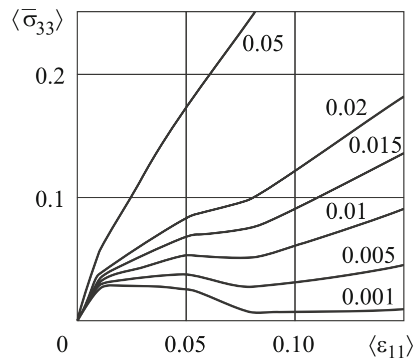

Fig. 6

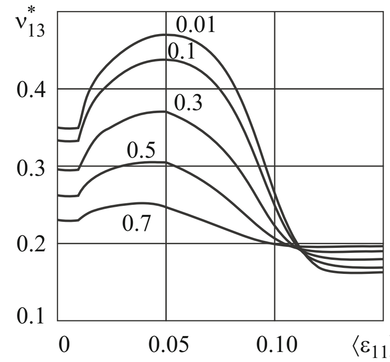

Fig. 7

$$\langle \bar{\sigma}_{33} \rangle = \frac{1}{\mu_{20}} \langle \sigma_{33} \rangle$$

and Poisson's ratios $\nu_{21}^*, \nu_{31}^*, \nu_{13}^*$ versus the macrostrains $\langle \varepsilon_{11} \rangle$ and $\langle \varepsilon_{33} \rangle$ for different values of the fiber volume fraction c_1.

It can be seen that the damage of the matrix (Fig. 1) subject to uniaxial cross-fiber tension (2.10) is more strongly dependent on the fiber volume fraction than in the case of uniaxial along-fiber tension (2.12) (Fig. 5).

Poisson's ratios depend on the fiber volume fraction and macrostrains in a more complicated manner (Figs. 3, 4, 7).

In the case of cross-fiber tension (Fig. 2), the stress–strain curves have ascending and descending sections, differing only quantitatively depending on c_1. The descending sections are due to the combination of the nonlinearity and the damage of the matrix. When tension is along the fibers, the effect of the nonlinearity and damage of the matrix on the stress–strain curves is significant only for $0.001 \le c_1 \le 0.05$ (Fig. 6). In the range $0 < c_1 \le 0.011$, the stress–strain curves have three sections: ascending, descending, and ascending.

Poisson's ratios ν_{21}^* and ν_{13}^* peak at certain values of the macrostrains $\langle \varepsilon_{11} \rangle \neq 0$ and $\langle \varepsilon_{33} \rangle \neq 0$, respectively, for all fiber volume fractions. Poisson's ratio ν_{31}^*, however, peaks at $\langle \varepsilon_{11} \rangle \neq 0$ only for $c_1 < 0.04$.

Conclusions. Structural models of the coupled processes of deformation and damage of homogeneous and composite materials based on the ideas about the stochastic micrononuniformity of strength are constructed by deriving the stochastic equations of the statics of an elastic body, the damage (porosity) balance equations, and failure criteria for microvolumes of the undamaged material. Using stress-based failure criteria allows us to construct models only for the ascending section of the nonlinear stress–strain curve. With strain-based failure criteria, we can construct models for the whole nonlinear stress–strain curve of undamaged microvolumes of homogeneous and composite materials. By studying the deformation and damage of a unidirectional fibrous composite with nonlinearly elastic matrix whose stress–strain curve have a descending section, we have established the qualitative and quantitative dependence of elastic properties and damage of the composite on the macrostrains for different volume fractions of fibers and plotted the corresponding macrostress–macrostrain curves.

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