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NONSTATIONARY DEFORMATION OF LONGITUDINALLY AND TRANSVERSELY REINFORCED CYLINDRICAL SHELLS ON AN ELASTIC FOUNDATION

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The problem of the forced vibrations of a discretely reinforced cylindrical shell on an elastic foundation under distributed impulsive loading is stated. The dynamic behavior of the inhomogeneous cylindrical shell is analyzed using the Timoshenko-type theory of shells. The problem is solved with the finite-difference method. Numerical results are analyzed

Keywords: discretely reinforced cylindrical shell, Timoshenko-type theory of shells and rods, elastic foundation, forced vibrations, numerical method

Introduction. The interaction of elastic structures with the environment is studied using two basic approaches to the formulation and solution of associated problems [1, 5]: (i) use of the three-dimensional equations of continuum mechanics and (ii) use of some integral kinematic and mechanical parameters to describe the effect on an elastic structure (Winkler, Pasternak, etc. foundations) [1, 3]. The use of the former approach involves some algorithmic and computational difficulties [1–3, 5]. The latter approach is to model the effect of the environment by an elastic foundation, which simplifies the formulation and solution of original problems. The dynamic behavior of a reinforced shell on an elastic foundation can be studied by solving two problems: (a) influence of the elastic foundation on plates and shells without reinforcement [4, 6, 10–12, 15–18, 21, 22] and (b) influence of reinforcement on the inhomogeneous structure [3, 7–9, 14, 17]. The influence of both elastic medium and reinforcement on the stress–strain state of inhomogeneous structures is addressed in [2, 3, 13, 16, 19, 20].

Here we will solve the problem of the forced vibrations of a discretely reinforced cylindrical shell on a Pasternak elastic two-parameter foundation under a distributed impulsive load. The dynamic behavior of the reinforced inhomogeneous shell will be analyzed using a geometrically linear theory of shells and rods and the Timoshenko hypotheses. The problem posed will be solved with the finite-difference method [3]. Numerical results will be obtained depending on the geometrical and mechanical parameters of the structure and the elastic foundation.

1. Problem Formulation. Consider an inhomogeneous elastic structure with discrete inclusions that is a shell reinforced with stringers and rings. Let us determine the stress–strain state of the shell and ribs using a geometrically linear theory of shells and rods and the Timoshenko hypotheses [3]. The strain state of the mid-surface of the shell is determined from the components of the generalized displacement vector $\overline{U} = (u_1, u_2, u_3, \varphi_1, \varphi_2)^T$. The strain state at the center of gravity of the cross section of a rib aligned with the x-axis is determined by the generalized displacement vector $\overline{U}_i = (u_{1i}, u_{2i}, u_{3i}, \varphi_{1i}, \varphi_{2i})^T$, and the rib aligned with the y-axis by the vector $\overline{U}_j = (u_{1j}, u_{2j}, u_{3j}, \varphi_{1j}, \varphi_{2j})^T$. We also assume that the discrete ribs are perfectly bonded to the shell.

We will use a general coordinate system to describe the mid-surface of the shell with thickness h, the coordinate z increasing along the outward normal to the initial surface.

The interface conditions relate the components of the displacement vector of the cross-sectional center of gravity of the ith rib aligned with the x-axis and the components of the generalized displacement vector of the mid-surface [3]:

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$$u_{1i}(x)=u_1(x,y_i)+h_{ci}\varphi_1(x,y_i),$$
$$u_{2i}(x)=u_2(x,y_i)+h_{ci}\varphi_2(x,y_i),$$
$$u_{3i}(x)=u_3(x,y_i),$$
$$\varphi_{1i}(x)=\varphi_1(x,y_i),\qquad \varphi_{2i}(x)=\varphi_2(x,y_i), \tag{1.1}$$

where h_{ci} is the distance from the initial mid-surface to the line of the cross-sectional center of gravity of the ith rib.

The interface conditions between the jth rib aligned with the y-axis and the shell are similar.

To derive the equations of the vibrations of the discretely reinforced structure, we will use the Hamilton–Ostrogradskii variational principle:

$$\int_{t_1}^{t_2}\left[\delta(\Pi-T)-\delta A\right]dt=0, \tag{1.2}$$

where Π is the potential energy of the system, including the elastic foundation; T is the kinetic energy of the system; A is the work done by external forces.

The potential energy is expressed as

$$\Pi=\Pi_0+\sum_{i=1}^{I}\Pi_i+\sum_{j=1}^{J}\Pi_j+\Pi_{\text{foun}},$$

where Π_0, Π_i, and Π_j are the potential energies of the shell, ith rib, and jth rib, respectively; Π_{foun} is the potential energy of the elastic foundation (Pasternak's model).

The expressions for $\delta\Pi$ and δT are the following:

$$\delta\Pi_0=\iint_S\int_z[\sigma_{11}^z\delta\varepsilon_{11}^z+\sigma_{22}^z\delta\varepsilon_{22}^z+\sigma_{12}^z\delta\varepsilon_{12}^z+\sigma_{13}^z\delta\varepsilon_{13}^z+\sigma_{23}^z\delta\varepsilon_{23}^z]dS,$$

$$\delta\Pi_i=\int_x\sum_{i=1}^{I}\int_{S_i}[\sigma_{11}^{yz}\delta\varepsilon_{11}^{yz}+\sigma_{12}^{yz}\delta\varepsilon_{12}^{yz}+\sigma_{13}^{yz}\delta\varepsilon_{13}^{yz}]dS_i\,dx,$$

$$\delta\Pi_j=\int_y\sum_{j=1}^{J}\int_{S_j}[\sigma_{22}^{xz}\delta\varepsilon_{22}^{xz}+\sigma_{21}^{xz}\delta\varepsilon_{21}^{xz}+\sigma_{23}^{xz}\delta\varepsilon_{23}^{xz}]dS_j\,dy,$$

$$\delta\Pi_{\text{foun}}=\iint_S\left[C_1u_3\delta u_3+C_2\frac{\partial u_3}{\partial x}\delta\left(\frac{\partial u_3}{\partial x}\right)+C_2\frac{\partial u_3}{\partial y}\delta\left(\frac{\partial u_3}{\partial y}\right)\right]dS,$$

$$\delta T=-\int_{t_1}^{t_2}\left\{\iint_S\sum\rho\int_z\left(\frac{\partial^2u_1^z}{\partial t^2}\delta u_1^z+\frac{\partial^2u_2^z}{\partial t^2}\delta u_2^z+\frac{\partial^2u_3^z}{\partial t^2}\delta u_3^z\right)dzdS\right.$$

$$+\int_x\sum_{i=1}^{I}\rho_i\int_{S_i}\left(\frac{\partial^2u_{1i}^{yz}}{\partial t^2}\delta u_{1i}^{yz}+\frac{\partial^2u_{2i}^{yz}}{\partial t^2}\delta u_{2i}^{yz}+\frac{\partial^2u_{3i}^{yz}}{\partial t^2}\delta u_{3i}^{yz}\right)dS_i\,dx$$

$$\left.+\int_y\sum_{j=1}^{J}\rho_j\int_{S_j}\left(\frac{\partial^2u_{1j}^{yz}}{\partial t^2}\delta u_{1j}^{yz}+\frac{\partial^2u_{2j}^{yz}}{\partial t^2}\delta u_{2j}^{yz}+\frac{\partial^2u_{3j}^{yz}}{\partial t^2}\delta u_{3j}^{yz}\right)dS_j\,dy\right\}dt, \tag{1.3}$$

where S, S_i, S_j are, respectively, the domains of integration over the mid-surface of the shell and the cross-sections of the *i*th and *j*th ribs; ρ, ρ_i, ρ_j are the densities of the materials of the shell and the ribs; C_1 is the modulus of subgrade reaction characterizing the resistance of the elastic foundation to tension/compression along the *z*-axis; C_2 is the modulus of subgrade reaction characterizing the resistance of the elastic foundation to transverse shear.

To derive the equations of the vibrations of the reinforced cylindrical shell on an elastic foundation, we will use the interface conditions (1.1) and the integral shell–rib interface conditions, according to [3].

Performing standard transformations of functional (1.2) and taking into account the integral stresses of the shell and ribs, we obtain three groups of equations:

the equations of the vibrations of the cylindrical shell

$$\frac{\partial T_{11}}{\partial x}+\frac{\partial S}{\partial y}+P_1=\rho h\frac{\partial^2 u_1}{\partial t^2},\qquad \frac{\partial S}{\partial x}+\frac{\partial T_{22}}{\partial y}+\frac{T_{23}}{R}+P_2=\rho h\frac{\partial^2 u_2}{\partial t^2},$$

$$C_2\left(\frac{\partial^2 u_3}{\partial x^2}+\frac{\partial^2 u_3}{\partial y^2}\right)+\frac{\partial T_{13}}{\partial x}+\frac{\partial T_{23}}{\partial y}-\frac{T_{22}}{R}-C_1u_3+P_3=\rho h\frac{\partial^2 u_3}{\partial t^2},$$

$$\frac{\partial M_{11}}{\partial x}+\frac{\partial H}{\partial y}-T_{13}=\rho\frac{h^3}{12}\frac{\partial^2\varphi_1}{\partial t^2},\qquad \frac{\partial H}{\partial x}+\frac{\partial M_{22}}{\partial y}-T_{23}=\rho\frac{h^3}{12}\frac{\partial^2\varphi_2}{\partial t^2}; \tag{1.4}$$

the equations of the vibrations of the *i*th rib along the line of its cross-sectional center of gravity

$$\frac{\partial T_{11i}}{\partial x}+[S]=\rho_iF_i\left(\frac{\partial^2 u_1}{\partial t^2}\pm h_{ci}\frac{\partial^2\varphi_1}{\partial t^2}\right),$$

$$\frac{\partial T_{12i}}{\partial x}+[T_{22}]=\rho_iF_i\left(\frac{\partial^2 u_2}{\partial t^2}\pm h_{ci}\frac{\partial^2\varphi_2}{\partial t^2}\right),\qquad \frac{\partial \overline{T}_{13i}}{\partial x}+[T_{23}]=\rho_iF_i\frac{\partial^2 u_3}{\partial t^2},$$

$$\frac{\partial M_{11i}}{\partial x}\pm h_{ci}\frac{\partial T_{11i}}{\partial x}-T_{13i}+[H]=\rho_iF_i\left[\pm h_{ci}\frac{\partial^2 u_1}{\partial t^2}+\left(h_{ci}^2+\frac{I_{1i}}{F_i}\right)\frac{\partial^2\varphi_1}{\partial t^2}\right],$$

$$\frac{\partial M_{12i}}{\partial x}\pm h_{ci}\frac{\partial \overline{T}_{12i}}{\partial x}+[M_{22}]=\rho_iF_i\left[\pm h_{ci}\frac{\partial^2 u_1}{\partial t^2}+\left(h_{ci}^2+\frac{I_{\mathrm{tw}i}}{F_i}\right)\frac{\partial^2\varphi_2}{\partial t^2}\right]; \tag{1.5}$$

the equations of the vibrations of the *j*th rib along the line of its cross-sectional center of gravity

$$\frac{\partial T_{21i}}{\partial y}+[T_{11}]=\rho_jF_j\left(\frac{\partial^2 u_1}{\partial t^2}\pm h_{cj}\frac{\partial^2\varphi_1}{\partial t^2}\right),$$

$$\frac{\partial T_{22j}}{\partial y}+\frac{T_{23j}}{R_j}+[S]=\rho_jF_j\left(\frac{\partial^2 u_2}{\partial t^2}\pm h_{cj}\frac{\partial^2\varphi_2}{\partial t^2}\right),$$

$$\frac{\partial T_{23j}}{\partial y}-\frac{T_{22j}}{R_j}+[T_{13}]=\rho_jF_j\frac{\partial^2 u_3}{\partial t^2},$$

$$\frac{\partial M_{21j}}{\partial y}\pm h_{cj}\frac{\partial T_{21j}}{\partial y}+[M_{11}]=\rho_iF_i\left[\pm h_{cj}\frac{\partial^2 u_1}{\partial t^2}+\left(h_{cj}^2+\frac{I_{\mathrm{tw}j}}{F_j}\right)\frac{\partial^2\varphi_1}{\partial t^2}\right],$$

$$\frac{\partial M_{22j}}{\partial y} \pm h_{cj}\frac{\partial T_{22j}}{\partial y} - T_{23j} + [H] = \rho_j F_j \left[\pm h_{cj}\frac{\partial^2 u_1}{\partial t^2} + \left(h_{cj}^2 + \frac{I_{2j}}{F_j} \right) \frac{\partial^2 \varphi_2}{\partial t^2} \right], \tag{1.6}$$

where the quantities in square brackets represent the forces/moments acting on the ith (or jth) rib aligned with the OX-axis (or the OY-axis): $[\Phi]_i = \Phi_i^+ - \Phi_i^-$, $[\Phi]_j = \Phi_j^+ - \Phi_j^-$.

The forces/moments and the strains of the shell in (1.4) are related by

$$T_{11} = B_{11}(\varepsilon_{11} + \nu_{21}\varepsilon_{22}), \quad T_{22} = B_{22}(\varepsilon_{22} + \nu_{12}\varepsilon_{11}),$$

$$S = B_{12}\varepsilon_{12}, \quad T_{13} = B_{13}\varepsilon_{13}, \quad T_{23} = B_{23}\varepsilon_{23},$$

$$M_{11} = D_{11}(\kappa_{11} + \nu_{21}\kappa_{22}), \quad M_{22} = D_{22}(\kappa_{22} + \nu_{12}\kappa_{11}), \quad H = D_{13}\kappa_{12} \tag{1.7}$$

$$[B_{11} = \frac{E_1 h}{1-\nu_{12}\nu_{21}}, \quad B_{22} = \frac{E_2 h}{1-\nu_{12}\nu_{21}}, \quad B_{12} = G_{12}h, \quad B_{13} = G_{13}hk^2, \quad B_{23} = G_{23}hk^2,$$

$$D_{11} = \frac{E_1 h^3}{12(1-\nu_{12}\nu_{21})}, \quad D_{22} = \frac{E_2 h^2}{12(1-\nu_{12}\nu_{21})}, \quad D_{12} = G_{12}h^3/12,$$

k^2 is the integral shear coefficient in the Timoshenko-type theory of plates and shells.

The strains and the components of the generalized displacement vector are related by

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial y} + \frac{u_3}{R}, \quad \varepsilon_{12} = \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}, \quad \varepsilon_{13} = \frac{\partial u_3}{\partial x} + \varphi_1,$$

$$\varepsilon_{23} = \frac{\partial u_3}{\partial y} + \varphi_2 - \frac{u_2}{R}, \quad \kappa_{11} = \frac{\partial \varphi_1}{\partial x}, \quad \kappa_{22} = \frac{\partial \varphi_2}{\partial y}, \quad \kappa_{12} = \frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi_1}{\partial y},$$

$$\theta_2 = \frac{\partial u_3}{\partial x}, \quad \theta_2 = \frac{\partial u_3}{\partial y} - \frac{u_2}{R}. \tag{1.8}$$

The forces/moments and the strains of the ith rib aligned with the OX-axis are related by

$$T_{11i} = E_i F_i \varepsilon_{11i}, \quad T_{12i} = G_i F_i \varepsilon_{12i}, \quad T_{13i} = G_i F_i k^2 \varepsilon_{13i},$$

$$M_{11i} = E_i I_{1i} \kappa_{11i}, \quad M_{12i} = G_i I_{\text{tw}\,i} \kappa_{12i} \tag{1.9}$$

$$\left(\varepsilon_{11i} = \frac{\partial u_1}{\partial x} \pm h_{ci}\frac{\partial \varphi_1}{\partial x}, \quad \varepsilon_{12i} = \theta_{2i}, \quad \varepsilon_{13i} = \varphi_1 + \theta_{1i}, \right.$$

$$\left. \theta_{2i} = \frac{\partial u_2}{\partial x} \pm h_{ci}\frac{\partial \varphi_2}{\partial x}, \quad \theta_{1i} = \frac{\partial u_3}{\partial x}, \quad \kappa_{11i} = \frac{\partial \varphi_1}{\partial x}, \quad \kappa_{12i} = \frac{\partial \varphi_2}{\partial x} \right). \tag{1.10}$$

The forces/moments and the strains of the jth rib aligned with the OY-axis are related by

$$T_{22j} = E_j F_j \varepsilon_{22j}, \quad T_{21j} = G_j F_j \varepsilon_{21j}, \quad T_{23j} = G_j F_j k_j^2 \varepsilon_{23j},$$

$$M_{22j} = E_j I_{2j} \kappa_{22j}, \quad M_{21j} = G_j I_{\text{tw}\,j} \kappa_{21j} \tag{1.11}$$

$$\left[\varepsilon_{22i} = \frac{\partial u_2}{\partial y} \pm h_{cj}\frac{\partial \varphi_2}{\partial y} + \frac{u_3}{R_j}, \quad \varepsilon_{21j} = \theta_{1j}, \quad \varepsilon_{23j} = \varphi_2 + \theta_{2j}, \right.$$

$$\theta_{1j}=\frac{\partial u_1}{\partial y}\pm h_{cj}\frac{\partial \varphi_1}{\partial y},\quad \theta_{2j}=\frac{\partial u_3}{\partial y}-\frac{1}{R}(u_2\pm h_{cj}\varphi_2),\quad \kappa_{22j}=\frac{\partial \varphi_2}{\partial y},\ \kappa_{21j}=\frac{\partial \varphi_1}{\partial y}\Bigg]. \tag{1.12}$$

The vibration equations (1.4)–(1.12) are supplemented with appropriate boundary and initial conditions.

If one of the ends ($x=0$ or $x=L$) is clamped, we have

$$u_1=u_2=u_3=0,\quad \varphi_1=\varphi_2=0. \tag{1.13}$$

2. Numerical Algorithm. Consider a domain D defined by $D=\{0\le x\le L,\ 0\le y\le 2\pi,\ 0\le t\le T\}$. The domains D is covered by a difference mesh $\Omega=\Omega_{\Delta x\Delta y}\cdot\Omega_\tau$, where

$$\Omega_{\Delta x\Delta y}=\{(x_k,y_i)=(k\Delta x,l\Delta y),\ \Delta x=L/K,\ \Delta y=2\pi R/M,\ k=\overline{0,K},\ l=\overline{0,M}\},$$

$$\Omega_\tau=\{t_n=n\tau,\ \tau=T/N,\ n=\overline{0,N}\}.$$

Along with the main mesh, we will use auxiliary difference meshes at discrete points $(x_{k\pm1/2},y_i)$, $(x_k,y_{l\pm1/2})$, $(x_{k\pm1/2},y_{l\pm1/2})$, where $x_{k\pm1/2}=(k\pm1/2)\Delta x$, $y_{l\pm1/2}=(l\pm1/2)\Delta y$.

The difference mesh is such that the points of discontinuity coincide with the integer nodes. Using the auxiliary difference meshes and the integro-interpolation method, we set up the difference equations in the domain $\Omega_1=\{x_{k-1/2}\le x\le x_{k+1/2},y_{l-1/2}\le y\le y_{l+1/2}\}$ for $t_{n-1/2}\le t\le t_{n+1/2}$:

$$\int_t\iint_{\Omega_1}\left[\frac{\partial T_{11}}{\partial x}+\frac{\partial S}{\partial y}+P_1\right]d\Omega_1dt=\int_t\iint_{\Omega_1}\left[\rho h\frac{\partial^2 u_1}{\partial t^2}\right]d\Omega_1dt,$$

$$\int_t\iint_{\Omega_1}\left[\frac{\partial S}{\partial x}+\frac{\partial T_{22}}{\partial y}+\frac{T_{23}}{R}+P_2\right]d\Omega_1dt=\int_t\iint_{\Omega_1}\left[\rho h\frac{\partial^2 u_2}{\partial t^2}\right]d\Omega_1dt,$$

$$\int_t\iint_{\Omega_1}\left[C_2\left(\frac{\partial^2 u_3}{\partial x^2}+\frac{\partial^2 u_3}{\partial y^2}\right)+\frac{\partial T_{13}}{\partial x}+\frac{\partial T_{23}}{\partial y}-\frac{T_{22}}{R}-C_1u_3+P_3\right]d\Omega_1dt=\int_t\iint_{\Omega_1}\left[\rho h\frac{\partial^2 u_3}{\partial t^2}\right]d\Omega_1dt,$$

$$\int_t\iint_{\Omega_1}\left[\frac{\partial M_{11}}{\partial x}+\frac{\partial H}{\partial y}-T_{13}\right]d\Omega_1dt=\int_t\iint_{\Omega_1}\left[\rho\frac{h^3}{12}\frac{\partial^2 \varphi_1}{\partial t^2}\right]d\Omega_1dt,$$

$$\int_t\iint_{\Omega_1}\left[\frac{\partial H}{\partial x}+\frac{\partial M_{22}}{\partial y}-T_{23}\right]d\Omega_1dt=\int_t\iint_{\Omega_1}\left[\rho\frac{h^3}{12}\frac{\partial^2 \varphi_2}{\partial t^2}\right]d\Omega_1dt. \tag{2.1}$$

After standard transformations in (2.1), we obtain difference equations that approximate the original equations (1.4) in the smooth domain:

$$\frac{T_{11k+1/2,l}-T_{11k-1/2,l}}{\Delta x}+\frac{S_{k,l+1/2}-S_{k,l-1/2}}{\Delta y}+P^n_{1k,l}=\rho h(u^n_{1k,l})_{\bar{t}t},$$

$$\frac{S_{k+1/2,l}-S_{k-1/2,l}}{\Delta x}+\frac{T_{22k,l+1/2}-T_{22k,l-1/2}}{\Delta y}+\frac{T_{23k,l+1/2}+T_{23k,l-1/2}}{2R}+P^n_{2k,l}=\rho h(u^n_{2k,l})_{\bar{t}t},$$

$$C_2\left[\frac{u^n_{3k+1,l}-2u^n_{3k,l}+u^n_{3k-1,l}}{(\Delta x)^2}+\frac{u^n_{3k,l+1}-2u^n_{3k,l}+u^n_{3k,l-1}}{(\Delta y)^2}\right]+\frac{T_{13k+1/2,l}-T_{13k-1/2,l}}{\Delta x}$$

$$+\frac{T_{23k,l+1/2}-T_{23k,l-1/2}}{\Delta y}-\frac{T_{22k,l+1/2}+T_{22k,l-1/2}}{2R}-C_1 u_{3k,l}+P^n_{3k,l}=\rho h(u^n_{3k,l})_{\bar{t}t},$$

$$\frac{M_{11k+1/2,l}-M_{11k-1/2,l}}{\Delta x}+\frac{H_{k,l+1/2}-H_{k,l-1/2}}{\Delta y}-\frac{T_{13k+1/2,l}+T_{13k-1/2,l}}{2}+m^n_{1k,l}=\rho\frac{h^3}{12}(\varphi^n_{1k,l})_{\bar{t}t},$$

$$\frac{H_{k+1/2,l}-H_{k-1/2,l}}{\Delta x}+\frac{M_{22k,l+1/2}-M_{22k,l-1/2}}{\Delta y}-\frac{T_{23k,l+1/2}+T_{23k,l-1/2}}{2}+m^n_{2k,l}=\rho\frac{h^3}{12}(\varphi^n_{2k,l})_{\bar{t}t}, \tag{2.2}$$

where difference derivatives are denoted as in [3].

Thus, in the difference equations, the components of the generalized displacement vector are referred to the integer nodes of the difference mesh, while the forces/moments to the difference mesh with fractional indices $(x_{k\pm1/2}, y_i)$ or $(x_k, y_{i\pm1/2})$. To match the forces/moments in (2.2), we integrate Eqs. (1.7) over the domains $\Omega_2=\{x_{k-1}\le x\le x_k, y_{l-1/2}\le y\le y_{l+1/2}\}$ and $\Omega_3=\{x_{k-1/2}\le x\le x_{k+1/2}, y_{l-1}\le y\le y_l\}$ for $t_{n-1/2}\le t\le t_{n+1/2}$.

In the domain Ω_2, we have

$$\int_t\iint_{\Omega_2}[T_{11}]\,d\Omega_2\,dt=\int_t\iint_{\Omega_2}[B_{11}(\varepsilon_{11}+\nu_{21}\varepsilon_{22})]\,d\Omega_2\,dt,\qquad \int_t\iint_{\Omega_2}[S]\,d\Omega_2\,dt=\int_t\iint_{\Omega_2}[B_{12}\varepsilon_{12}]\,d\Omega_2\,dt,$$

$$\int_t\iint_{\Omega_2}[\bar{T}_{13}]\,d\Omega_2\,dt=\int_t\iint_{\Omega_2}[T_{13}]\,d\Omega_2\,dt,\qquad \int_t\iint_{\Omega_2}[T_{13}]\,d\Omega_2\,dt=\int_t\iint_{\Omega_2}[B_{13}\varepsilon_{13}]\,d\Omega_2\,dt,$$

$$\int_t\iint_{\Omega_2}[M_{11}]\,d\Omega_2\,dt=\int_t\iint_{\Omega_2}[D_{11}(\kappa_{11}+\nu_{21}\kappa_{22})]\,d\Omega_2\,dt,\qquad \int_t\iint_{\Omega_2}[H]\,d\Omega_2\,dt=\int_t\iint_{\Omega_2}[D_{12}\kappa_{12}]\,d\Omega_2\,dt. \tag{2.3}$$

In the domain Ω_3, we have

$$\int_t\iint_{\Omega_3}[S]\,d\Omega_3\,dt=\int_t\iint_{\Omega_3}[B_{12}\varepsilon_{12})]\,d\Omega_3\,dt,\qquad \int_t\iint_{\Omega_3}[T_{22}]\,d\Omega_3\,dt=\int_t\iint_{\Omega_3}[B_{22}(\varepsilon_{22}+\nu_{12}\varepsilon_{11})]\,d\Omega_3\,dt,$$

$$\int_t\iint_{\Omega_3}[T_{23}]\,d\Omega_3\,dt=\int_t\iint_{\Omega_3}[T_{23}]\,d\Omega_3\,dt,\qquad \int_t\iint_{\Omega_3}[H]\,d\Omega_3\,dt=\int_t\iint_{\Omega_3}[D_{12}\kappa_{12}]\,d\Omega_3\,dt,$$

$$\int_t\iint_{\Omega_3}[M_{22}]\,d\Omega_3\,dt=\int_t\iint_{\Omega_3}[D_{22}(\kappa_{22}+\nu_{12}\kappa_{11})]\,d\Omega_3\,dt. \tag{2.4}$$

After standard transformations in (2.3) and (2.4), we obtain the following difference equations that relate the forces/moments and the strains:

$$T_{11k\pm1/2,l}=B_{11}(\varepsilon^n_{11k\pm1/2,l}+\nu_{21}\varepsilon^n_{22k\pm1/2,l}),\quad S_{k\pm1/2,l}=B_{12}\varepsilon^n_{12k\pm1/2,l},\quad T_{13k\pm1/2,l}=T^n_{13k\pm1/2,l},$$

$$T_{13k\pm1/2,l}=B_{13}\varepsilon^n_{13k\pm1/2,l},\quad M_{11k\pm1/2,l}=D_{11}(\kappa^n_{11k\pm1/2,l}+\nu_{21}\kappa^n_{22k\pm1/2,l}),\quad H_{k\pm1/2,l}=D_{13}\kappa^n_{12k\pm1/2,l}, \tag{2.5}$$

$$T_{22k,l\pm1/2}=B_{22}(\varepsilon^n_{22k,l\pm1/2}+\nu_{12}\varepsilon^n_{11k,l\pm1/2}),\quad T_{23k,l\pm1/2}=T^n_{23k,l\pm1/2},\quad T_{23k,l\pm1/2}=B_{13}\varepsilon^n_{13k,l\pm1/2},$$

$$M_{22k,l\pm1/2}=D_{22}(\kappa^n_{22k,l\pm1/2}+\nu_{12}\kappa^n_{11k,l\pm1/2}),\quad H_{k,l\pm1/2}=D_{12}\kappa^n_{12k,l\pm1/2}, \tag{2.6}$$

where the quantities with indices $(k+1/2,l)$ and $(k,l+1/2)$ are obtained by integrating Eqs. (1.7) over the domains $\{x_{k1}\le x\le x_{k+1}, y_{l-1/2}\le y\le y_{l+1/2}\}$ and $\{x_{k-1/2}\le x\le x_{k+1/2}, y_l\le y\le y_{l+1}\}$ for $t_{n-1/2}\le t\le t_{n+1/2}$.

To match the strains in (2.5) and (2.6), we integrate Eqs. (1.8) over the domains Ω_2 and Ω_3, respectively, for $t_{n-1/2}\le t\le t_{n+1/2}$. In the domain Ω_2, we have

$$\int_t \iint_{\Omega_2} [\varepsilon_{11}]\, d\Omega_2\, dt = \int_t \iint_{\Omega_2} \left[\frac{\partial u_1}{\partial x}\right] d\Omega_2\, dt,$$

$$\int_t \iint_{\Omega_2} [\varepsilon_{22}]\, d\Omega_2\, dt = \int_t \iint_{\Omega_2} \left[\frac{\partial u_1}{\partial y} + \frac{w_3}{R}\right] d\Omega_2\, dt,$$

$$\int_t \iint_{\Omega_2} [\varepsilon_{12}]\, d\Omega_2\, dt = \int_t \iint_{\Omega_2} \left[\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right] d\Omega_2\, dt, \tag{2.7}$$

$$\ldots$$

The relations for the domain Ω_3 are similar.

After standard transformations in (2.7), we obtain the following difference equations relating the strains and the components of the generalized displacement vector:

$$\varepsilon^n_{11k+1/2,l} = \frac{u^n_{1k+1,l} - u^n_{1k,l}}{\Delta x}, \quad \varepsilon^n_{11k-1/2,l} = \frac{u^n_{1k,l} - u^n_{1k-1,l}}{\Delta x},$$

$$\varepsilon^n_{22k+1/2,l} = \frac{u^n_{2k+1/2,l+1/2} - u^n_{2k+1/2,l-1/2}}{\Delta y} + \frac{u^n_{3k+1,l} + u^n_{3k,l}}{2R},$$

$$\varepsilon^n_{22k-1/2,l} = \frac{u^n_{2k-1/2,l+1/2} - u^n_{2k-1/2,l-1/2}}{\Delta y} + \frac{u^n_{3k,l} + u^n_{3k-1,l}}{2R},$$

$$\varepsilon^n_{12k,l+1/2} = \frac{u^n_{1k,l+1} - u^n_{1k,l}}{\Delta y} + \frac{u^n_{2k+1/2,l+1/2} - u^n_{2k-1/2,l+1/2}}{\Delta x},$$

$$\varepsilon^n_{12k,\,l-1/2} = \frac{u^n_{1k,l} - u^n_{1k,l-1}}{\Delta y} + \frac{u^n_{2k+1/2,l-1/2} - u^n_{2k-1/2,l-1/2}}{\Delta x}, \tag{2.8}$$

$$\ldots$$

Since the lines of discontinuities pass through the integer points of the difference mesh, the difference algorithm for the ith rib is constructed as follows: like the difference mesh for the smooth domain, we introduce difference meshes in the domains $\Omega_{1i} = \{x_{i-1/2} \le x \le x_{i+1/2}\}, \Omega_{2i} = \{x_{i-1} \le x \le x_i\}, \Omega_{3i} = \{x_i \le x \le x_{i+1}\}$ for $t_{n-1/2} \le t \le t_{n+1/2}$.

Integrating the vibration equations (1.5) over the domain Ω_{1i} for $t_{n-1/2} \le t \le t_{n+1/2}$, we obtain

$$\int_t \int_{\Omega_{1i}} \left\{\frac{\partial T_{11i}}{\partial x} + [S]\right\} d\Omega_{1i}\, dt = \rho_i F_i \int_t \int_{\Omega_{1i}} \left[\frac{\partial^2 u_1}{\partial t^2} \pm h_{\text{c}i} \frac{\partial^2 \varphi_1}{\partial t^2}\right] d\Omega_{1i}\, dt,$$

$$\int_t \int_{\Omega_{1i}} \left\{\frac{\partial T_{12i}}{\partial x} + [T_{22}]\right\} d\Omega_{1i}\, dt = \rho_i F_i \int_t \int_{\Omega_{1i}} \left[\frac{\partial^2 u_2}{\partial t^2} \pm h_{\text{c}i} \frac{\partial^2 \varphi_2}{\partial t^2}\right] d\Omega_{1i}\, dt,$$

$$\int_t \int_{\Omega_{1i}} \left\{\frac{\partial T_{13i}}{\partial x} + [T_{23}]\right\} d\Omega_{1i}\, dt = \rho_i F_i \int_t \int_{\Omega_{1i}} \left[\frac{\partial^2 u_3}{\partial t^2}\right] d\Omega_{1i}\, dt,$$

$$\int_t \int_{\Omega_{1i}} \left\{\frac{\partial M_{11i}}{\partial x} \pm h_{\text{c}i} \frac{\partial T_{11i}}{\partial x} - T_{13} + [H]\right\} d\Omega_{1i}\, dt = \rho_i F_i \int_t \int_{\Omega_{1i}} \left\{\pm h_{\text{c}i} \frac{\partial^2 u_1}{\partial t^2} + \left[h^2_{\text{c}i} + \frac{I_{1i}}{F_i}\right] \frac{\partial^2 \varphi_1}{\partial t^2}\right\} d\Omega_{1i}\, dt,$$

$$\int_t \int_{\Omega_{1i}} \left\{ \frac{\partial M_{12i}}{\partial x} \pm h_{ci} \frac{\partial T_{12i}}{\partial x} + [M_{22}] \right\} d\Omega_{1i} dt = \rho_i F_i \int_t \int_{\Omega_{1i}} \left\{ \pm h_{ci} \frac{\partial^2 u_2}{\partial t^2} + \left[h_{ci}^2 + \frac{I_{\mathrm{tw}i}}{F_i} \right] \frac{\partial^2 \varphi_2}{\partial t^2} \right\} d\Omega_{1i} dt. \tag{2.9}$$

Standard transformations in (2.9) yield the following difference equations for Eqs. (1.5):

$$\frac{T^n_{11ik+1/2} - T^n_{11ik-1/2}}{\Delta x} + [S]^n_i = \rho_i F_i \Big[(u^n_{1k,l})_{\bar{t}t} \pm h_{\mathrm{c}i} (\varphi^n_{1k,l})_{\bar{t}t} \Big],$$

$$\frac{T^n_{12ik+1/2} - T^n_{12ik-1/2}}{\Delta x} + [T_{22}]^n_i = \rho_i F_i \Big[(u^n_{2k,l})_{\bar{t}t} \pm h_{\mathrm{c}i} (\varphi^n_{2k,l})_{\bar{t}t} \Big],$$

$$\frac{T^n_{13ik+1/2} - T^n_{13ik-1/2}}{\Delta x} + [T_{23}]^n_i = \rho_i F_i (u^n_{3k,l})_{\bar{t}t},$$

$$\frac{M^n_{11ik+1/2} - M^n_{11ik-1/2}}{\Delta x} \pm h_{ci} \frac{T^n_{11ik+1/2} - T^n_{11ik-1/2}}{\Delta x} - \frac{T^n_{13ik+1/2} - T^n_{13ik-1/2}}{\Delta x} + [S]^n_i$$

$$= \rho_i F_i \left[\pm h_{\mathrm{c}i} (u^n_{1k,l})_{\bar{t}t} + \left[h_{\mathrm{c}i}^2 + \frac{I_{1i}}{F_i} \right] (\varphi^n_{1k,l})_{\bar{t}t} \right],$$

$$\frac{M^n_{21ik+1/2} - M^n_{21ik-1/2}}{\Delta x} \pm h_{ci} \frac{T^n_{12ik+1/2} - T^n_{12ik-1/2}}{\Delta x} + [M_{22}]^n_i$$

$$= \rho_i F_i \left[\pm h_{\mathrm{c}i} (u^n_{2k,l})_{\bar{t}t} + \left[h_{\mathrm{c}i}^2 + \frac{I_{\mathrm{tw}i}}{F_i} \right] (\varphi^n_{2k,l})_{\bar{t}t} \right]. \tag{2.10}$$

To match the difference forces/moments, we integrate Eqs. (1.9) over, respectively, the domains Ω_{2i} and Ω_{3i} for $t_{n-1/2} \le t \le t_{n+1/2}$.

$$\int_t \int_{\Omega_{2i}} [T_{11i}] d\Omega_{2i} dt = \int_t \int_{\Omega_{2i}} [E_i F_i \varepsilon_{11i}] d\Omega_{2i} dt, \quad \int_t \int_{\Omega_{2i}} [T_{12i}] d\Omega_{2i} dt = \int_t \int_{\Omega_{2i}} [G_i F_i \varepsilon_{12i}] d\Omega_{2i} dt,$$

$$\int_t \int_{\Omega_{2i}} [M_{11i}] d\Omega_{2i} dt = \int_t \int_{\Omega_{2i}} [E_i I_{1i} \kappa_{11i}] d\Omega_{2i} dt, \quad \int_t \int_{\Omega_{2i}} [M_{12i}] d\Omega_{2i} dt = \int_t \int_{\Omega_{2i}} [G_i I_{\mathrm{tw}i} \kappa_{12i}] d\Omega_{2i} dt. \tag{2.11}$$

Standard transformations in (2.11) yield

$$T^n_{11ik\pm1/2} = E_i F_i \varepsilon^n_{11ik\pm1/2}, \quad T^n_{12ik\pm1/2} = G_i F_i \varepsilon^n_{12ik\pm1/2}, \quad T^n_{13ik\pm1/2} = k_1^2 G_i F_i \varepsilon^n_{13ik\pm1/2},$$

$$M^n_{11ik\pm1/2} = E_i I_{1i} \kappa^n_{11ik\pm1/2}, \quad M^n_{12ik\pm1/2} = G_i I_{kri} \kappa^n_{12ik\pm1/2},$$

…

The difference algorithm for the jth rib is constructed in a similar way [3].

3. Numerical Example. Let us analyze the dynamic behavior of a rib-reinforced cylindrical shell on a Winkler foundation ($C_2 = 0$ in Eqs. (1.4)) under an internal distributed impulsive load. The ends $x = 0$ and $x = L$ of the shell are clamped. The rings are located in the sections $x_i = 0.25Li\,(i = \overline{1,3})$. The stringers are located in the sections $y_j = \pi R(j-1)/2\,(j = \overline{1,4})$ (the shell is reinforced with three rings and four stringers). The distributed impulsive load $P_3(s_1, s_2, t)$ is defined by

$$P_3(s_1, s_2, t) = A \cdot \sin \frac{\pi t}{T} [\eta(t) - \eta(t - T)],$$

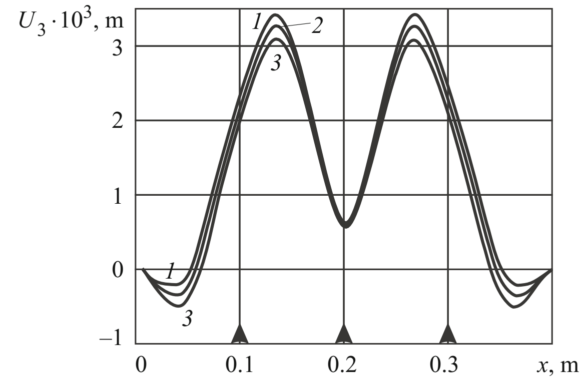

Fig. 1

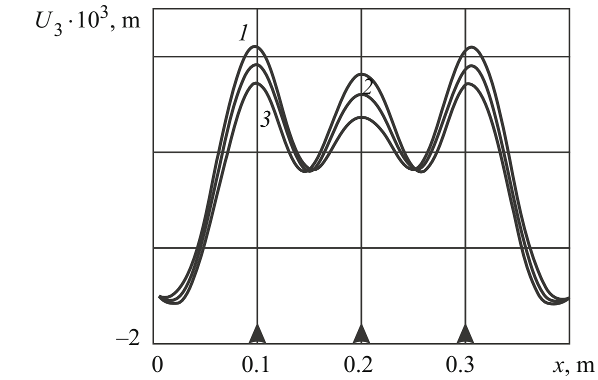

Fig. 2

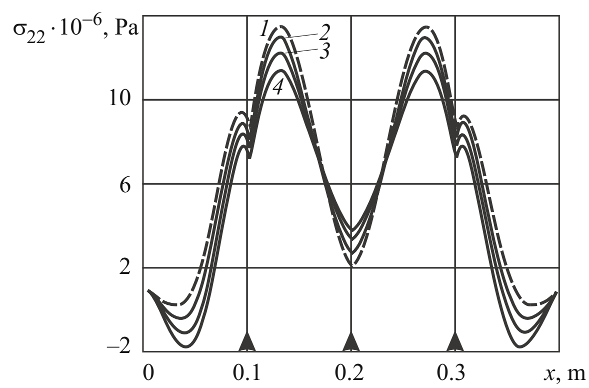

Fig. 3

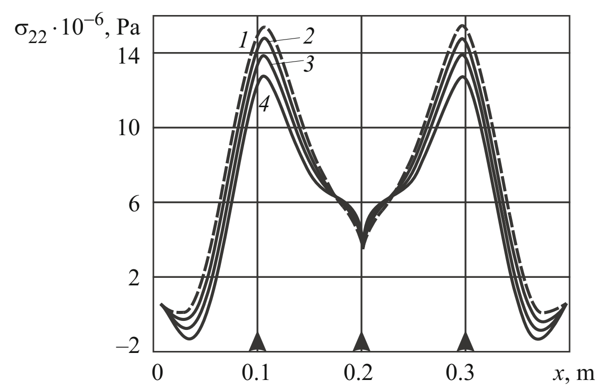

Fig. 4

where A and T are the amplitude and period. It is assumed that $E_1 / A = 7 \cdot 10^4$, $T = 2.5R / c$, $c = \{E_1 / [\rho(1-\nu_1\nu_2)]\}^{1/2}$.

The geometrical and mechanical parameters of the shell: $E_1 = E_2 = 7 \cdot 10^{10}$ Pa, $\nu_1 = \nu_2 = 0.3$, $R / h = 10$, $L / R = 4$. For the ribs, we have $E_i = E_j = E$, $F_i = F_j = a_j h_j$, $a_i = a_j = h$, $h_i = h_j = 2h$. The following values of the Winkler coefficient are used:

(i) $C_1 = 1 \cdot 10^9$ N/m^3,

(ii) $C_1 = 2 \cdot 10^9$ N/m^3,

(iii) $C_1 = 3 \cdot 10^9$ N/m^3.

The simulation period is $0 < t \le 40T$.

Figures 1 and 2 show the variation in the deflection u_3 along the length of the structure. Figure 1 shows the deflection u_3 as a function of the coordinate x in the section $y = \pi R / 4$ (between ribs) at the instant $t = 8.5T$ (at which u_3 becomes maximum for a case $C_1 = 1 \cdot 10^9$ N/m^3) for $C_1 = [(1 \cdot 10^9); (2 \cdot 10^9); (3 \cdot 10^9)]$ N/m^3 (curves *1–3*).

Figure 2 shows the same curves for the section $y = 0$.

Figures 3–6 show the strain ε_{22} and stress σ_{22} as functions of the coordinate x in the sections $y = 0$ (on a rib; Figs. 3 and 5) and $y = \pi R / 4$ (between ribs; Figs. 4 and 6) at the instant $t = 8.5T$ for $C_1 = \{0; 1 \cdot 10^9; 2 \cdot 10^9; 3 \cdot 10^9\}$ N/m^3 (curves *1–4*).

The effect of the ribs with respect to the coordinate x can be seen visually. The strain ε_{22} and stress σ_{22} undergo spatial discontinuities where the rings are located. These dependences are more pronounced between ribs. The elastic foundation with

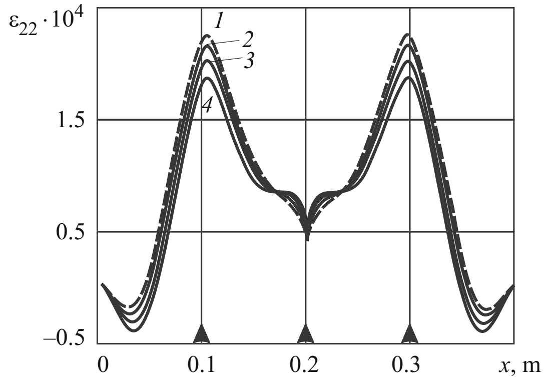

Fig. 5

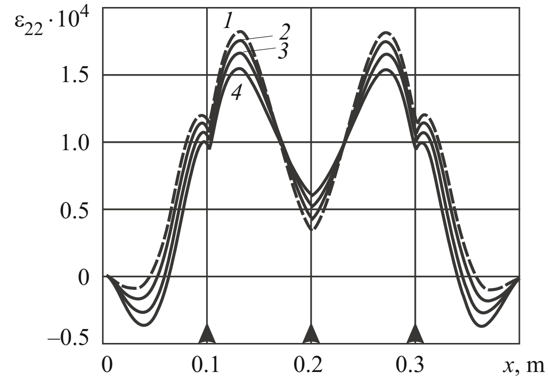

Fig. 6

the given coefficients C_1 causes minor changes in the stress–strain state of the inhomogeneous shell (the maximum difference is 14–16% for the strain ε_{22} and 25% for the stress σ_{22}).

The figures allow us to analyze the stress–strain state of an inhomogeneous elastic structure depending on its mechanical and geometrical parameters of the elastic foundation.

Conclusions. We have formulated the problem of the forced vibrations of a discretely reinforced cylindrical shell on an elastic foundation under a distributed load. The dynamic behavior of the inhomogeneous cylindrical shell has been analyzed using the Timoshenko-type theory of shells and rods. The problem posed has been solved with the finite-difference method. Numerical results have been presented and analyzed.

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