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FREE AXISYMMETRIC VIBRATIONS OF CYLINDRICAL SHELLS MADE OF FUNCTIONALLY GRADED MATERIALS

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The free axisymmetric vibrations of cylindrical shells made of a functionally graded material and subject to various boundary conditions are studied using the three-dimensional theory of elasticity and the Timoshenko–Mindlin refined theory of shells. The possibility of applying the refined theory to shells made of a material with properties gradually varying over the thickness is examined. The influence of different laws of variation in mechanical properties on the dynamic characteristics of cylindrical shells during axisymmetric vibrations is studied. The numerical results presented in the form of tables and graphs are analyzed

Keywords: cylindrical shell, free axisymmetric vibrations, functionally graded materials, three-dimensional theory of elasticity, Timoshenko–Mindlin shell theory

Introduction. Advanced materials technologies developed recently made it possible to create new materials with predictable properties such as functionally graded materials (FGMs). Materials with mechanical properties gradually varying in some direction (no layers and interfaces) can be made as a composition of two metals, or metal and ceramics, or two polymers. The physical properties of such materials can be controlled by assigning a desired law of variation in the elastic modulus in some direction. Such a material is often modeled by an isotropic material continuously inhomogeneous in the direction of variation of the elastic properties [2, 4, 6, 9–12].

General problems of elasticity for bodies made of hypothetic FGMs are addressed in [7, 8]. The vibrations of thick-walled cylinders made of polymeric composite FGMs were studied in [2] using the three-dimensional theory of elasticity. However, applying the three-dimensional theory to dynamic problems for bodies made of FGMs involves difficulties in most cases. The free vibrations of cylindrical bodies made of FGMs were studied in [6, 9–12] using different theories of shells. Emphasis was on the change in the dynamic characteristics depending on the law of variation in the elastic properties. Of interest is the use of shell theories to study the free vibrations of hollow cylindrical bodies made of materials essentially inhomogeneous over the thickness.

In what follows, we will examine the possibility of applying the Timoshenko–Mindlin shell theory to problems of the free vibrations of cylindrical shells made of materials with mechanical characteristics gradually varying over the thickness and the influence of laws of variation in the mechanical properties on the dynamic characteristics of these shells during radial-and-longitudinal axisymmetric vibrations. We will use the spline-collocation method in combination with incremental search and discrete orthogonalization [2, 3, 5].

1. Problem Statement. Basic Equations.

1.1. Timoshenko–Mindlin Shell Theory. Consider circular cylindrical shells made of FGMs with elastic properties gradually varying in the direction perpendicular to the shell midsurface. The shells undergo free vibrations. We will employ the Timoshenko model based on the straight-normal hypothesis (normals to the coordinate surface remain straight and unstretched after deformation, but do not remain normal). According to the accepted hypothesis, the small displacements of particles of the

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shell can be represented in the coordinate system γ, θ, z fixed to the mid-surface (γ is the normal coordinate to the mid-surface, $-h/2 \le \gamma \le h/2$, $0 \le \theta \le 2\pi$, $0 \le z \le L$) as follows:

$$u_\gamma(\gamma,\theta,z,t) = w(\theta,z,t), \qquad u_\theta(\gamma,\theta,z,t) = v(\theta,z,t) + \gamma\psi_\theta(\theta,z,t),$$
$$u_z(\gamma,\theta,z,t) = u(\theta,z,t) + \gamma\psi_z(\theta,z,t), \tag{1}$$

where $u(\theta,z,t)$, $v(\theta,z,t)$, $w(\theta,z,t)$ are the displacements of the coordinate surface; $\psi_\theta(\theta,z,t)$, $\psi_z(\theta,z,t)$ are functions describing the independent complete rotation of the normal.

The expressions for strains below follow from (1):

$$e_\theta(\gamma,\theta,z,t) = \varepsilon_\theta(\theta,z,t) + \gamma\kappa_\theta(\theta,z,t),$$
$$e_z(\gamma,\theta,z,t) = \varepsilon_z(\theta,z,t) + \gamma\kappa_z(\theta,z,t),$$
$$e_{\theta z}(r,\theta,z,t) = \varepsilon_{\theta z}(\theta,z,t) + 2\gamma\kappa_{\theta z}(\theta,z,t),$$
$$e_{\gamma\theta}(r,\theta,z,t) = \gamma_\theta(\theta,z,t), \qquad e_{\gamma z}(r,\theta,z,t) = \gamma_z(\theta,z,t), \tag{2}$$

where $\varepsilon_\theta, \varepsilon_z, \varepsilon_{\theta z}$ are the tangential strains of the coordinate surface; $\kappa_\theta, \kappa_z, \kappa_{\theta z}$ are the bending strain components; γ_θ and γ_z are the angles of rotation of the normal caused by transverse shear.

The strains and displacements of the midsurface of the shell are related by

$$\varepsilon_\theta = \frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{1}{R}w, \qquad \varepsilon_z = \frac{\partial u}{\partial z}, \qquad \varepsilon_{\theta z} = \frac{1}{R}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial z}, \qquad \kappa_z = \frac{\partial\psi_z}{\partial z},$$
$$\kappa_\theta = \frac{1}{R}\frac{\partial\psi_\theta}{\partial\theta} - \frac{1}{R}\left(\frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{1}{R}w\right), \qquad 2\kappa_{\theta z} = \frac{1}{R}\frac{\partial\psi_z}{\partial\theta} + \frac{\partial\psi_\theta}{\partial z} - \frac{1}{R}\frac{\partial u}{\partial\theta},$$
$$\gamma_\theta = \psi_\theta + \frac{1}{R}\frac{\partial w}{\partial\theta} - \frac{1}{R}v, \qquad \gamma_z = \psi_z + \frac{\partial w}{\partial z}. \tag{3}$$

The motion of an element of the coordinate surface is described by the equation

$$\frac{\partial N_z}{\partial z} + \frac{1}{R}\frac{\partial N_{\theta z}}{\partial\theta} = I_0\frac{\partial^2 u}{\partial t^2} + I_1\frac{\partial\psi_z}{\partial t^2}, \qquad \frac{1}{R}\frac{\partial N_\theta}{\partial\theta} + \frac{\partial N_{z\theta}}{\partial z} + \frac{1}{R}Q_\theta = I_0\frac{\partial v}{\partial t^2} + I_1\frac{\partial^2\psi_\theta}{\partial t^2},$$
$$\frac{\partial Q_z}{\partial z} + \frac{1}{R}\frac{\partial Q_\theta}{\partial\theta} - \frac{1}{R}N_\theta = I_0\frac{\partial^2 w}{\partial t^2}, \qquad \frac{\partial M_z}{\partial z} + \frac{1}{R}\frac{\partial M_{\theta z}}{\partial\theta} - Q_z = I_1\frac{\partial^2 u}{\partial t^2} + I_2\frac{\partial\psi_z}{\partial t^2},$$
$$\frac{1}{R}\frac{\partial M_\theta}{\partial\theta} + \frac{\partial M_{z\theta}}{\partial z} - Q_\theta = I_1\frac{\partial v}{\partial t^2} + I_2\frac{\partial^2\psi_\theta}{\partial t^2}, \tag{4}$$

and $N_{z\theta} - M_{\theta z}R^{-1} - N_{\theta z} = 0$. Here $N_z, N_\theta, N_{\theta z}, N_{z\theta}$ are the tangential forces; Q_θ, Q_z are the transverse forces; M_θ, M_z, $M_{\theta z}$, $M_{z\theta}$ are the bending and twisting moments; $\rho(\gamma)$ is the density of the shell's material; I_0, I_1, I_2 are the inertial terms calculated taking into account the gradient of the elastic properties:

$$I_0 = \int_{-h/2}^{h/2}\rho(\gamma)d\gamma, \qquad I_1 = \int_{-h/2}^{h/2}\rho(\gamma)\gamma d\gamma, \qquad I_2 = \int_{-h/2}^{h/2}\rho(\gamma)\gamma^2 d\gamma. \tag{5}$$

The force–deformation constitutive relations for cylindrical shells made of an FGM without symmetry of elastic properties about the midsurface have the form

$$N_z = C_{11}\varepsilon_z + C_{12}\varepsilon_\theta + K_{11}\kappa_z + K_{12}\kappa_\theta, \qquad N_\theta = C_{12}\varepsilon_z + C_{22}\varepsilon_\theta + K_{12}\kappa_z + K_{22}\kappa_\theta,$$

$$N_{z\theta}=C_{66}\varepsilon_{\theta z}+2D_{66}R^{-1}\kappa_{\theta z},\quad M_z=K_{11}\kappa_z+K_{12}\kappa_\theta+D_{11}\kappa_z+D_{12}\kappa_\theta,$$

$$M_\theta=K_{12}\kappa_z+K_{12}\kappa_\theta+D_{12}\kappa_z+D_{22}\kappa_\theta,\quad M_{\theta z}=M_{z\theta}=2D_{66}\kappa_{\theta z},$$

$$Q_\theta=K_2\gamma_\theta,\quad Q_z=K_1\gamma_z,\quad N_{\theta z}=C_{66}\varepsilon_{\theta z},\tag{6}$$

where the stiffnesses referred to the coordinate surface are defined by

$$C_{11}=\int_{-h/2}^{h/2}B_{11}(\gamma)d\gamma,\quad C_{12}=\int_{-h/2}^{h/2}B_{12}(\gamma)d\gamma,\quad C_{22}=\int_{-h/2}^{h/2}B_{22}(\gamma)d\gamma,\quad C_{66}=\int_{-h/2}^{h/2}B_{66}(\gamma)d\gamma,$$

$$K_{11}=\int_{-h/2}^{h/2}B_{11}(\gamma)\gamma d\gamma,\quad K_{12}=\int_{-h/2}^{h/2}B_{12}(\gamma)\gamma d\gamma,\quad K_{22}=\int_{-h/2}^{h/2}B_{22}(\gamma)\gamma d\gamma,\quad K_1=\int_{-h/2}^{h/2}G(\gamma)d\gamma,$$

$$D_{11}=\int_{-h/2}^{h/2}B_{11}(\gamma)\gamma^2 d\gamma,\quad D_{12}=\int_{-h/2}^{h/2}B_{12}(\gamma)\gamma^2 d\gamma,\quad D_{22}=\int_{-h/2}^{h/2}B_{22}(\gamma)\gamma^2 d\gamma,$$

$$D_{66}=\int_{-h/2}^{h/2}B_{66}(\gamma)\gamma^2 d\gamma,\quad B_{11}(\gamma)=B_{22}(\gamma)=E(\gamma)/(1-\nu^2(\gamma)),$$

$$B_{12}(\gamma)=\nu(\gamma)E(\gamma)/(1-\nu^2(\gamma)),\quad B_{66}=\frac{E}{2(1+\nu)},\tag{7}$$

where E, G, and ν are the elastic and shear moduli and Poisson's ratio, respectively, which are functions of the coordinate γ for a material with properties gradually varying over the thickness.

Consider the following boundary conditions at the ends $z=0$ and $z=L$:

(i) the ends are clamped: $u=v=w=0$, $\psi_\theta=\psi_z=0$;

(ii) the ends are hinged and free in the longitudinal direction: $\partial u/\partial z=0$, $v=w=0$, $\partial\psi_z/\partial z=\psi_\theta=0$;

(iii) the ends are free: $N_z=0$, $M_z=0$, $Q_z=0$.

If free vibrations are axisymmetric (all functions in Eqs. (3), (4), (6) are independent of θ, while their derivatives with respect to θ are equal to zero, i.e., $\partial f/\partial\theta=0$), system (4) breaks up into two independent systems, one describing radial-and-longitudinal vibrations and the other representing torsional vibrations. The radial-and-longitudinal vibrations are now described by the equations

$$\frac{\partial N_z}{\partial z}=I_0\frac{\partial^2 u_z}{\partial t^2}+I_1\frac{\partial^2\psi_z}{\partial t^2},\quad \frac{\partial Q_z}{\partial z}-\frac{1}{R}N_\theta=I_0\frac{\partial^2 w}{\partial t^2},$$

$$\frac{\partial M_z}{\partial z}-Q_z=I_1\frac{\partial^2 u_z}{\partial t^2}+I_2\frac{\partial^2\psi_z}{\partial t^2}.\tag{8}$$

The equations relating the tangential and bending strains of the midsurface to its displacements and the angle of rotation of the normal due to transverse shears are become simpler as well:

$$\varepsilon_\theta=\frac{1}{R}w,\quad \varepsilon_z=\frac{\partial u}{\partial z},\quad \kappa_z=\frac{\partial\psi_z}{\partial z},\quad \kappa_\theta=\frac{1}{R^2}w,\quad \gamma_z=\psi_z+\frac{\partial w}{\partial z}.\tag{9}$$

The force–deformation relations become

$$N_z=C_{11}\varepsilon_z+C_{12}\varepsilon_\theta+K_{11}\kappa_z+K_{12}\kappa_\theta,\quad N_\theta=C_{12}\varepsilon_z+C_{22}\varepsilon_\theta+K_{12}\kappa_z+K_{22}\kappa_\theta,$$

$$M_z=K_{11}\kappa_z+K_{12}\kappa_\theta+D_{11}\kappa_z+D_{12}\kappa_\theta,\quad M_z=K_{11}\kappa_z+K_{12}\kappa_\theta+D_{11}\kappa_z+D_{12}\kappa_\theta,$$

$$Q_z = K_1 \gamma_z. \tag{10}$$

Assume that all particles of the cylindrical shell undergo harmonic vibrations with circular frequency ω:

$$\{u(z,t), w(z,t), \psi_z(z,t)\} = \{\tilde{u}(z),\ \tilde{w}(z),\ \tilde{\psi}_z(z)\}e^{i\omega t} \tag{11}$$

(hereafter the sign ~ is omitted).

Using (11), we will write the system of equations (8)–(10) for displacements:

$$C_{11}\frac{d^2u}{dz^2} + K_{11}\frac{d^2\psi_z}{dz^2} = -I_0\omega^2 u - C_{12}\frac{1}{R}\frac{dw}{dz} + K_{12}\frac{1}{R^2}\frac{dw}{dz} - I_1\omega^2\psi_z,$$

$$K_{11}\frac{d^2u}{dz^2} + D_{11}\frac{d^2\psi_z}{dz^2} = -I_1\omega^2 u - K_{12}\frac{1}{R}\frac{dw}{dz} + D_{12}\frac{1}{R^2}\frac{dw}{dz} + K_1\frac{dw}{dz} + K_1\psi_z - I_1\omega^2\psi_z,$$

$$\frac{d^2\psi_z}{dz^2} = \frac{C_{12}}{K_1}\frac{1}{R}\frac{du}{dz} + \frac{C_{22}}{K_1}\frac{1}{R^2}w + \frac{K_{22}}{K_1}\frac{1}{R^2}w - \frac{I_0}{K_1}\omega^2 w - \frac{d\psi_z}{dz} + \frac{K_{12}}{K_1}\frac{1}{R}\frac{d\psi_z}{dz}. \tag{12}$$

After some transformations, this system takes the form

$$\frac{d^2u}{dz^2} = \frac{-D_{11}I_0 + K_{11}I_1}{\Delta}\omega^2 u + \frac{-D_{11}I_1\omega^2 + K_{11}I_2\omega^2 - K_{11}K_1}{\Delta}\psi_z$$

$$+\frac{D_{11}K_{12} - D_{12}K_{11}}{\Delta R^2}\frac{dw}{dz} + \frac{K_{11}K_{12} - D_{11}C_{12}}{\Delta R^2}\frac{dw}{dz} - \frac{K_{11}K_1}{\Delta}\frac{dw}{dz},$$

$$\frac{d\psi_z^2}{dz} = -\frac{C_{11}I_0\omega^2}{\Delta}u - \frac{C_{11}I_2\omega^2}{\Delta}\psi_z + \frac{K_1C_{11}}{\Delta R^2}\psi_z + \frac{-C_{11}K_{12}}{\Delta}\frac{dw}{dz} + \frac{D_{11}C_{11}}{\Delta R^2}\frac{dw}{dz} + \frac{C_{11}K_1}{\Delta R^2}\frac{dw}{dz},$$

$$\frac{d^2w}{dz^2} = \frac{C_{22}}{K_{11}}\frac{1}{R^2}w - \frac{I_0}{K_1}\omega^2 w + \frac{K_{22}}{K_1}\frac{1}{R^2}w + \frac{C_{12}}{K_{11}}\frac{1}{R}\frac{du}{dz} - \frac{d\psi_z}{dz} - \frac{K_{12}}{K_1}\frac{1}{R}\frac{d\psi_z}{dz} + \frac{D_{11}C_{11}}{\Delta R^2}\frac{dw}{dz}. \tag{13}$$

Thus, the eigenvalue problem has been reduced to the system of ordinary differential equations (13) subject to boundary conditions at $z = 0$ and $z = L$.

1.2. Three-Dimensional Theory of Elasticity. Consider a thick-walled cylindrical shell of constant thickness $2H$, length L, inner radius $R - H$, and outer radius $R + H$ (R is the mid-radius). The shell is made of an FGM with elastic properties varying in the direction perpendicular to the midsurface. If the radial-and-longitudinal vibrations are axisymmetric, then the components of the displacement vector and stress and strain tensors are independent of the circumferential coordinate θ of the cylindrical coordinate system r, θ, z chosen. The equations of motion become simpler:

$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho(r)\frac{\partial^2 u_r}{\partial t^2}, \qquad \frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = \rho(r)\frac{\partial^2 u_z}{\partial t^2}. \tag{14}$$

The kinematic equations in the axisymmetric case are

$$e_r = \frac{\partial u_r}{\partial r}, \qquad e_\theta = \frac{1}{r}u_r, \qquad e_z = \frac{\partial u_z}{\partial z}, \qquad 2e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}. \tag{15}$$

The system of equations (14), (15) is supplemented with the generalized Hooke's law for an orthotropic elastic material:

$$\sigma_r = \lambda_{11}e_r + \lambda_{12}e_\theta + \lambda_{13}e_z, \qquad \sigma_\theta = \lambda_{12}e_r + \lambda_{22}e_\theta + \lambda_{23}e_z,$$

$$\sigma_z = \lambda_{13}e_r + \lambda_{23}e_\theta + \lambda_{33}e_z, \qquad \sigma_{rz} = 2\lambda_{55}e_{rz}. \tag{16}$$

The elements of the stiffness matrix $\lambda_{ij} = \lambda_{ij}(r)$ and the density $\rho(r)$ of the shell's material are continuous and differentiable functions of the radial coordinate r. Here t is the time coordinate; $u_r(r,z,t)$ and $u_z(r,z,t)$ are the projections of the displacement vector onto tangents to the coordinate lines r and z, respectively; $e_r\ (r,z,t)$, $e_\theta\ (r,z,t)$, $e_z\ (r,z,t)$ are the relative linear strains along the coordinate lines; $e_{rz}(r,z,t)$ are the shear strains; $\sigma_r(r,z,t)$, $\sigma_\theta(r,z,t)$, $\sigma_z(r,z,t)$ are the normal stresses; $\sigma_{rz}(r,z,t)$ are the tangential stresses.

The elements λ_{ij} of the stiffness matrix can be expressed in terms of the elements c_{ij} of the compliance matrix as

$$\lambda_{11} = \lambda_{22} = \lambda_{33} = (c_{22}c_{33} - c_{23}^2)/\Delta, \qquad \lambda_{12} = \lambda_{13} = \lambda_{23} = (c_{13}c_{23} - c_{12}c_{33})/\Delta,$$

$$\lambda_{55} = 1/c_{55}, \qquad \Delta = c_{11}(c_{22}c_{33} - c_{23}^2) - c_{12}(c_{12}c_{33} - c_{13}c_{23}) + c_{13}(c_{12}c_{23} - c_{13}c_{22}).$$

In turn, the elements of the compliance matrix can be expressed in terms of the technical constants:

$$c_{11} = c_{22} = c_{33} = \frac{1}{E}, \qquad c_{12} = c_{13} = c_{23} = -\frac{\nu}{E}, \qquad c_{55} = -\frac{1}{G}, \tag{17}$$

where $E(r)$ is Young's modulus; $G(r)$ is the shear modulus; $\nu(r)$ is Poisson's ratio of a FGM.

The inside and outside lateral surfaces $r = R - H$ and $r = R + H$ of the shell are free from stresses:

$$\sigma_r(R \pm H/2, z, t) = 0, \qquad \sigma_{rz}(R \pm H/2, z, t) = 0. \tag{18}$$

The following boundary conditions can be specified at the ends $z = 0$ and $z = L$:

(i) $$\sigma_r = 0, \quad u_r = 0 \quad \text{or} \quad \frac{\partial u_z}{\partial z} = 0, \quad u_r = 0, \tag{19}$$

(ii) $$u_z = 0, \quad \sigma_{rz} = 0 \quad \text{or} \quad u_z = 0, \quad \frac{\partial u_r}{\partial z} = 0, \tag{20}$$

(iii) $$u_r = 0, \quad u_z = 0. \tag{21}$$

Assume that all particles of the shell undergo harmonic vibrations with frequency ω, i.e., $\{u_r(r,z,t),\ u_z(r,z,t)\} = \{\tilde{u}_r(r,z),\ \tilde{u}_z(r,z)\}e^{i\omega t}$ (hereafter the sign ~ is omitted).

The governing equations for displacements have the form

$$\frac{\partial^2 u_r}{\partial r^2} = \left(-\frac{1}{\lambda_{11}}\frac{\partial \lambda_{12}}{\partial r}\frac{1}{r} + \frac{\lambda_{22}}{\lambda_{11}}\frac{1}{r^2} - \frac{1}{\lambda_{11}}\rho\omega^2\right)u_r - \frac{1}{\lambda_{11}}\frac{\partial \lambda_{55}}{\partial z}\frac{\partial u_r}{\partial z} - \frac{\lambda_{55}}{\lambda_{11}}\frac{\partial^2 u_r}{\partial z^2}$$

$$-\left(\frac{1}{\lambda_{11}}\frac{\partial \lambda_{11}}{\partial r} + \frac{1}{r}\right)\frac{\partial u_r}{\partial r} - \left(\frac{1}{\lambda_{11}}\frac{\partial \lambda_{13}}{\partial r} - \frac{\lambda_{23} - \lambda_{13}}{\lambda_{11}}\frac{1}{r}\right)\frac{\partial u_z}{\partial z}$$

$$-\frac{1}{\lambda_{11}}\frac{\partial \lambda_{55}}{\partial z}\frac{\partial u_z}{\partial r} - \frac{\lambda_{13} + \lambda_{55}}{\lambda_{11}}\frac{\partial^2 u_z}{\partial z \partial r},$$

$$\frac{\partial^2 u_z}{\partial r^2} = -\frac{1}{\lambda_{55}}\frac{\partial \lambda_{23}}{\partial z}\frac{u_r}{r} - \left(\frac{1}{\lambda_{55}}\frac{\partial \lambda_{55}}{\partial r} + \frac{\lambda_{23}}{\lambda_{55}}\frac{1}{r} + \frac{1}{r}\right)\frac{\partial u_r}{\partial z} - \left(1 + \frac{\lambda_{13}}{\lambda_{55}}\right)\frac{\partial^2 u_r}{\partial r \partial z}$$

$$-\frac{1}{\lambda_{55}}\frac{\partial \lambda_{13}}{\partial z}\frac{\partial u_r}{\partial r} - \frac{1}{\lambda_{55}}\rho\omega^2 u_z - \frac{1}{\lambda_{55}}\frac{\partial \lambda_{33}}{\partial z}\frac{\partial u_z}{\partial z} - \frac{\lambda_{33}}{\lambda_{55}}\frac{\partial^2 u_z}{\partial z^2} - \left(\frac{1}{r} + \frac{1}{\lambda_{55}}\frac{\partial \lambda_{55}}{\partial r}\right)\frac{\partial u_z}{\partial r}. \tag{22}$$

The boundary conditions (18) on the inside and outside surfaces become

$$\lambda_{11}\frac{\partial u_r}{\partial r}+\lambda_{12}\frac{u_r}{r}+\lambda_{13}\frac{\partial u_z}{\partial z}=0,\qquad \lambda_{55}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)=0. \tag{23}$$

The system of ordinary differential equations (22) and the boundary conditions constitute an eigenvalue problem.

2. Problem-Solving Method. We will solve problem (13) by discrete orthogonalization and incremental search. Introducing functions $\bar{u}=\frac{\partial u}{\partial z}$, $\bar{\psi}_z=\frac{\partial \psi_z}{\partial z}$, $\bar{w}=\frac{\partial w}{\partial z}$, and $\bar{Y}=\{u,\bar{u},\psi_z,\bar{\psi}_z,w,\bar{w}\}^T$, which is a vector function of z, we represent system (13) as

$$\frac{d\bar{Y}}{dz}=A(z,\omega)\bar{Y}\qquad (0\le z\le L), \tag{24}$$

where $A(z,\omega)$ is a (6×6)-matrix. For this system of ordinary differential equations, the boundary conditions at $z=0$ and $z=L$ are as follows:

$$B_1\bar{Y}(0)=\bar{0},\qquad B_2\bar{Y}(L)=\bar{0}, \tag{25}$$

where B_1 and B_2 are (3×6)-matrices.

The boundary-value eigenvalue problem (22), (23) can be solved using discrete orthogonalization in combination with incremental search [1–3]. To calculate the inertial terms (5) and the integral stiffnesses (7), we will employ the Newton–Cotes formulas with automatic adjustment of the step size.

Problem (24) with the boundary conditions (25) can be solved by the spline-collocation method. To this end, we represent the unknown functions $u_r(r,z)$ and $u_x(r,z)$ as

$$u_r=\sum_{i=0}^{N}u_{ri}(r)\varphi_i^{(1)}(z),\qquad u_z=\sum_{i=0}^{N}u_{zi}(r)\varphi_i^{(2)}(z), \tag{26}$$

where $u_{ri}(r)$ and $u_{zi}(r)$ are unknown functions of r, $\varphi_i^{(j)}(z)\,(j=1,2,\ i=0,1,\ldots,N)$ are linear combinations (which allow for the boundary conditions at $z=0$ and $z=L$) of B-splines over the uniform mesh Δ: $0=z_0<z_1<\ldots<z_N=L$. System (22) includes no higher than 2nd-order derivatives of the unknown functions with respect to z; therefore, it is sufficient to use cubic splines.

Substituting (26) into Eqs. (22), we require them to hold at the given collocation points $\xi_k\in[0,L]$, $k=0,N$. The number of nodes in the mesh (including z_0) is even, i.e., $N=2n+1\,(n\ge 3)$, and the number of collocation points $\bar{N}=N+1$. We obtain a system of $4(N+1)$ linear differential equations for the functions $u_{ri},\tilde{u}_{ri},u_{zi},\tilde{u}_{zi}$ $(i=0,\ldots,N)$:

$$\frac{d\bar{Y}}{dr}=A(r,\omega)\bar{Y}\qquad (R-H\le r\le R+H), \tag{27}$$

where $\bar{Y}=\{u_{r_0},\ldots u_{rN},\tilde{u}_{r0},\ldots,\tilde{u}_{rN},u_{z0},\ldots,u_{zN},\tilde{u}_{zo},\ldots,\tilde{u}_{zN}\}^T$ is a vector function of r; $A(r,\omega)$ is a $4(N+1)\times 4(N+1)$-matrix.

The boundary conditions for this system are given by

$$B_1\bar{Y}(R-H)=\bar{0},\qquad B_2\bar{Y}(R+H)=\bar{0}, \tag{28}$$

where B_1 and B_2 are $2(N+1)\times 4(N+1)$-matrices.

The boundary-value eigenvalue problem (27), (28) can be solved using discrete orthogonalization in combination with incremental search [1–3, 5].

3. Analysis of the Numerical Results.

3.1. Consider a cylindrical shell made of a two-component FGM whose elastic modulus E, Poisson's ratio ν, and density ρ can be determined from the concentration of the components. Let the elastic properties vary with the thickness coordinate. The elastic modulus E, Poisson's ratio ν, and density ρ of this FGM are related to those of its components by the formulas

$$E=(E_2-E_1)V+E_1,\qquad \nu=(\nu_2-\nu_1)V+\nu_1,\qquad \rho=(\rho_2-\rho_1)V+\rho_1, \tag{29}$$

TABLE 1

Material	E, GPa	ν	ρ, kg/m^3
Aluminum	70	0.3	2707
SiC	427	0.17	3100

TABLE 2

m	Theory	$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$
0.5	Timoshenko–Mindlin	3.1605	3.5005	4.6404	5.2606	6.5207
	Three-dimensional	3.3005	3.6605	4.8407	5.5506	6.8507
2	Timoshenko–Mindlin	2.4805	2.7405	3.6005	4.2807	5.0604
	Three-dimensional	2.5305	2.7905	3.6605	4.4307	5.1504
5	Timoshenko–Mindlin	2.0506	2.3505	3.1604	3.6206	4.5004
	Three-dimensional	2.0853	2.3505	3.1406	3.7006	4.4304
10	Timoshenko–Mindlin	1.8671	2.1405	2.9605	3.2605	4.2204
	Three-dimensional	1.8605	2.1305	2.9195	3.2005	4.1205

where E_1, ν_1, ρ_1 and E_2, ν_2, ρ_2 are the mechanical characteristics of the first and second materials, respectively; V is the concentration of the second material depending on the thickness coordinate γ. Let the elastic properties of the FGM vary with the thickness coordinate as $V = (\gamma / h + 0.5)^m$.

The elastic constants of the components are summarized in Table 1.

Table 2 collects the first five frequencies $\overline{\omega}_i = \omega_i l_0 \sqrt{\rho_0 / G_0} \cdot 10^2$ of free vibrations of a cylindrical shell made of FGM determined using the three-dimensional and Timoshenko–Mindlin theories for different values of m. To avoid using dimensions, we choose $\rho_0 = 1\,\text{kg/m}^3$ and $G_0 = 1\,\text{GPa}$. The ends of the shell are hinged. The shell has the following geometrical parameters: length $L = 20 l_0$, radius $R = 10 l_0$, thickness $h = 2 l_0$.

Analyzing the data in Table 2, we conclude that the natural frequencies of the shell calculated using the two theories differ a little. The difference between the frequencies is maximum (≈5%) at $m = 0.5$. For the other values of m, this difference does not exceed 3%. The frequencies obtained with the refined shell theory are lower for $m = 0.5$ and $m = 2$ and higher for $m = 10$ than those obtained with the three-dimensional theory. For $m = 5$, the first frequency calculated with the refined theory is higher than that predicted by the three-dimensional theory, while the third, fourth, and fifth frequencies found with the Timoshenko–Mindlin theory are lower than those obtained with the three-dimensional theory.

Tables 3–5 summarize the first four frequencies $\overline{\omega}_i = \omega_i l_0 \sqrt{\rho_0 / G_0}$ of free vibrations of the shell with either hinged, or clamped, or free ends, respectively, calculated using the Timoshenko–Mindlin theory for different values of m.

The stiffness of the material increases with m, which results in lower frequencies.

3.2. Let us analyze the vibrations of thick-walled shells made of polymeric FGMs using the three-dimensional theory of elasticity. Polymeric FGMs must meet the following requirements: the behavior of materials in all gradient zones must be elastic rather than viscoelastic and the gradual variation of their properties must remain over a wide temperature range. It should be borne in mind that there are no perfectly elastic polymeric FGMs and the elastic behavior of polymers is a conventional notion

TABLE 3

$\overline{\omega}$	m				
	0.5	1	2	5	10
$\overline{\omega}_1$	0.03161	0.02850	0.02481	0.02052	0.01861
$\overline{\omega}_2$	0.03500	0.03141	0.02740	0.02351	0.02142
$\overline{\omega}_3$	0.04643	0.04152	0.03600	0.03162	0.02964
$\overline{\omega}_4$	0.05265	0.04854	0.04281	0.03623	0.03262

TABLE 4

$\overline{\omega}$	m				
	0.5	1	2	5	10
$\overline{\omega}_1$	0.03606	0.03042	0.02661	0.02263	0.02001
$\overline{\omega}_2$	0.03901	0.03481	0.03033	0.02601	0.02442
$\overline{\omega}_3$	0.05242	0.04760	0.04184	0.03600	0.03260
$\overline{\omega}_4$	0.05360	0.04762	0.04240	0.03683	0.03493

TABLE 5

$\overline{\omega}$	m				
	0.5	1	2	5	10
$\overline{\omega}_1$	0.03160	0.02862	0.02481	0.02081	0.01860
$\overline{\omega}_2$	0.03181	0.02881	0.02543	0.02123	0.01901
$\overline{\omega}_3$	0.03262	0.02950	0.02588	0.02182	0.01983
$\overline{\omega}_4$	0.03926	0.03584	0.03062	0.02664	0.02467

(behavior is considered elastic when a stress is relaxing very slowly). Since all attempts made so far to specify laws of variation in the properties of polymeric FGMs with allowance for the properties of their components have failed, the properties of FGMs are determined experimentally [2].

Table 6 collects the frequencies $\overline{\omega} = \omega H\sqrt{\rho_{av} / E_{av}}$ of free vibrations of shells made of polymeric FGMs with Young's modulus varying as $E(r) = ar^2 + br + c$.

The following cases are examined:

(i) decreasing Young's modulus ($E(R-H) = 243.0$ MPa, $E(R) = 150.0$ MPa, $E(R+H) = 110.0$ MPa, $a = 26.5$ MPa, $b =$ –278.5 MPa, $c = 839.5$ MPa);

(ii) increasing Young's modulus ($E(R-H) = 100.0$ MPa, $E(R) = 150.0$ MPa, $E(R+H) = 243.0$ MPa, $a = 6.5$ MPa, $b =$ –59.6 MPa, $c = 243$ MPa);

TABLE 6

$\bar{\omega}_i$	Hinged shell			Clamped shell		
	I	II	III	I	II	III
$\bar{\omega}_1$	0.3094 (1)	0.2919 (1)	0.3019 (1)	0.4068	0.3917	0.4007
$\bar{\omega}_2$	0.5981 (2)	0.5869 (2)	0.5969 (2)	0.6053	0.6053	0.6176
$\bar{\omega}_3$	0.6543 (1)	0.6856 (1)	0.6788 (1)	0.6932	0.7321	0.7222
$\bar{\omega}_4$	0.9556 (3)	0.9531 (3)	0.9619 (3)	0.9776	0.9730	0.9830
$\bar{\omega}_5$	1.1269 (2)	1.1475 (1)	1.1844 (1)	1.1503	1.1441	1.1754
$\bar{\omega}_6$	1.1700 (1)	1.1681 (2)	1.1863 (2)	1.1961	1.2429	1.2639

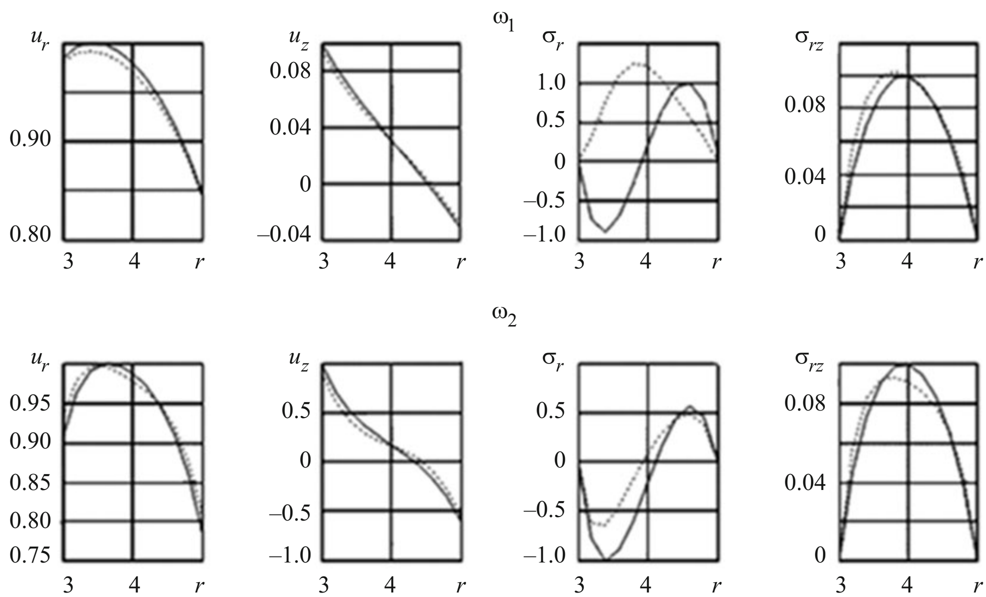

Fig. 1

(iii) thickness-average Young's modulus ($E_{av} = 158.33$ MPa).

Poisson's ratio is chosen $\nu = 0.4$ because Poisson's ratios of the polymeric components of the FGM are hardly different. The density of the FGM is assumed constant and equal to the density averaged over the thickness, ρ_{av}. The cylinder has the following geometrical parameters: length $L = 5$, inner radius $R_{in} = R - H = 3$, outer radius $R_{ex} = R + H = 5$, $H / R = 0.25$. In Table 6, the number of longitudinal half-waves is indicated in brackets after the frequency values.

It can be seen that the modes corresponding to the fifth and sixth frequencies are different for different laws of variation in Young's modulus across the thickness. The frequencies determined in increasing order differ by 1.5–5.6%.

Figure 1 shows the vibration modes for the first and second natural frequencies of the clamped thick-walled shell made of either homogeneous material (solid lines) or an FGM (dashed lines) with Young's modulus varying as in case (i) for $z = L/2$. The displacements and stresses are normalized to their maximum magnitudes for the homogeneous material.

The inhomogeneity of the material has the strongest effect on the distribution of the stress σ_r over the thickness of the shell. The number of half-waves across the thickness changes at the first frequency. If Young's modulus varies as in case (i), the maximums of the displacement u_r and stress σ_{rz} shift toward the inside surface of the shell. The displacement u_z in the cylinder made of the FGM changes less than that in the shell made of the homogeneous material at both the first and second natural frequencies. The position of the inflexion point on the midsurface remains the same for different materials.

Conclusions. We have solved the problem of the free axisymmetric vibrations of cylindrical shells made of a functionally graded material using the three-dimensional theory of elasticity and the Timoshenko–Mindlin refined theory of shells for different boundary conditions. The applicability of the refined theory to shells made of a material with properties gradually varying over the thickness and the influence of various laws of variation in the properties on the dynamic characteristics of cylindrical shells undergoing axisymmetric vibrations have been analyzed.

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