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IMPROVEMENT OF AIRCRAFT'S CAPABILITY OF TRACKING THE REFERENCE TRAJECTORY

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The problem of improving aircraft's capability of tracking the reference trajectory is solved by feeding the reference signal with a certain advance to the control system. It is essential that this approach imposes no constraints on the derivatives of the reference signal. The efficiency of the algorithm proposed is demonstrated against an example of an aircraft tracking a reference trajectory that does not have derivatives at some points

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Introduction. Vehicle maneuvering tasks [2] require high-accuracy navigators [8, 9] and accurate tracking of the reference trajectory. In this connection, researchers are still interested in improving aircraft's capability of tracking the reference trajectory (see, e.g., [5, 11] and the references therein).

We will discuss the design of an aircraft's servo system based on the approaches from [6, 7]. Following the procedure from [7], we will decompose the original problem into a linear velocity control problem and an attitude control problem. To improve the aircraft's capability of tracking the reference trajectory, we will use the approach [6] in which the reference signal is fed with a certain advance to the servo system. The algorithm proposed will be demonstrated against an example of tracking a reference trajectory similar to that in [3, Fig. 11.6]. Note that this approach imposes no constraints on the differentiability of the reference signal.

1. Equations of Motion. To describe the motion of the aircraft, we will use equations similar to those in [5, Eq. (1)] and [11, Eq. 2]:

$$\dot{x} = V\cos\theta, \qquad \dot{y} = V\sin\theta, \qquad \dot{\theta} = V\psi, \tag{1.1}$$

where x and y are the coordinates of the aircraft; θ is the heading angle; V is the velocity. According to [10, formula 10.43], the variable ψ is related to the bank angle φ by

$$\psi = \frac{g}{V^2}\tan\varphi\,. \tag{1.2}$$

To limit the rate of variation in $\dot{\theta}$, we assume that the control u is related to $d\psi / dt$ by

$$\dot{\psi} = uV\cos\theta. \tag{1.3}$$

It is expedient to use Eqs. (1.1) and (1.3) to analyze the straight-line motion of the aircraft. If the aircraft circles, then it makes sense to use polar coordinates ($x = R\cos\gamma$, $y = R\sin\gamma$). Then Eq. (1.1) become:

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$$\varepsilon = \frac{\pi}{2} + \gamma - \theta, \qquad \dot{R} = V \sin \varepsilon, \qquad \dot{\gamma} = \frac{V}{R} \cos \varepsilon, \qquad \dot{\varepsilon} = \dot{\gamma} - V\psi. \tag{1.4}$$

Supplementing system (1.4) with the equation $\dot{\psi} = \dot{\gamma} u$, which is analogous to Eq. (1.3), we obtain

$$\dot{R} = V \sin \varepsilon, \qquad \dot{\gamma} = \frac{V}{R} \cos \varepsilon, \qquad \dot{\varepsilon} = \dot{\gamma} - V\psi, \qquad \dot{\psi} = \dot{\gamma} u. \tag{1.5}$$

2. Problem Decomposition. Equations (1.1), (1.3), (1.5) are such that we can decompose the original problem into the problems of controlling the angle θ and the velocity V.

Let us first address the control of θ. Assume that $V > 0$ $(\dot{x} > 0, |\theta| < \pi/2)$, $\dot{\gamma} > 0$. Then, we can choose x or γ as an independent variable in Eqs. (1.1), (1.3), (1.5), thus reducing the order of the system. Then the following equations are analogs of systems (1.1)–(1.5):

$$y' = \frac{dy}{dx} = \tan \theta, \qquad \theta' = \frac{d\theta}{dx} = \frac{\psi}{\cos \theta}, \qquad \frac{d\psi}{dx} = u, \tag{2.1}$$

$$R' = \frac{dR}{d\gamma} = R \tan \varepsilon, \qquad \varepsilon' = \frac{d\varepsilon}{d\gamma} = 1 - \frac{R\psi}{\cos \varepsilon}, \qquad \frac{d\psi}{d\gamma} = u. \tag{2.2}$$

Note that the prime denotes differentiation with respect to x in (2.1) and differentiation with respect to γ in (2.2) (this does not apply to the derivatives of ψ, which remain to be denoted by $d\psi / dx$ and $d\psi / d\gamma$).

Let $V > 0$. It is expedient to replace system (2.1) with one differential equation of the third order:

$$y''' = v \tag{2.3}$$

$$\left(v = \frac{u}{(\cos \theta)^3} + \frac{3 \sin \theta \psi^2}{(\cos \theta)^5} \right), \tag{2.4}$$

considering that

$$y' = \tan \theta,$$

$$y'' = \frac{\psi}{(\cos \theta)^3}. \tag{2.5}$$

Using $w^T = \begin{bmatrix} y & y' & y'' \end{bmatrix}$ as a phase vector (the superscript "T" denotes transposition), we rearrange (2.3) to

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.6}$$

According to (2.4), (2.5), the variables of motion (θ, ψ) and the control u are expressed in terms of the components of the vectors $w(y', y'')$ and v as follows:

$$\theta = \arctan y', \qquad \psi = y''(\cos \theta)^3, \qquad u = v(\cos \theta)^3 - \frac{3 \sin \theta \psi^2}{(\cos \theta)^2}.$$

Obviously, a similar approach can be applied to circular motion (Eqs. (2.2)). Choosing $w^T = \begin{bmatrix} R & R' & R'' \end{bmatrix}$ as a phase vector and considering that

$$\varepsilon = \frac{\pi}{2} - \theta + \gamma, \qquad R' = R \tan \varepsilon, \qquad R'' = R\left(\tan^2 \varepsilon + \frac{1}{\cos^2 \varepsilon}\left(1 - \frac{R\psi}{\cos \varepsilon} \right) \right), \tag{2.7}$$

we can replace (2.2) with the following equations of the third order (analog of (2.3), (2.4)):

$$R''' = v, \tag{2.8}$$

$$v = \frac{-R}{ce^5}\left(se\,ce^2(-6+ce^2) - 3R^2 se\,\psi^2 + 9Rce\,se\,\psi + uR\,ce^2\right), \tag{2.9}$$

where $se = \sin\varepsilon$, $ce = \cos\varepsilon$.

According to (2.7) and (2.9), the "physical" variables of motion and the control u are expressed in terms of R, R', R'', and v as follows:

$$\varepsilon = \arctan\left(\frac{R'}{R}\right), \qquad \theta = \frac{\pi}{2} + \gamma - \varepsilon, \qquad \psi = \left(\frac{ce}{R^2}(R''ce^2 - 2R + Rce^2)\right),$$

$$u = -\left(\frac{v}{R^2}ce^3 - \frac{6}{R}se + \frac{se\,ce^2}{R} + 9\frac{se\,\psi}{ce} - 3\frac{Rse\,\psi^2}{ce^2}\right). \tag{2.10}$$

3. Stabilization of Motion. Let us solve the stabilization problem when the motion is described by Eq. (2.1). The task is to find a law of variation in u with y, θ, ψ such that the zero solution of Eq. (2.1) is asymptotically stable.

From (2.3) and (2.4) we get

$$y''' = \frac{u}{(\cos\theta)^3} + \frac{3\sin\theta\,\psi^2}{(\cos\theta)^5}. \tag{3.1}$$

The control u is chosen so that Eq. (3.1) has the form

$$y''' = -ay'' - by' - dy, \tag{3.2}$$

where a, b, d are given constants. Then we have

$$u = -(a\psi + b(\cos\theta)^2\sin\theta + d(\cos\theta)^3 y) - \frac{3\sin\theta\psi^2}{(\cos\theta)^2}. \tag{3.3}$$

Thus, we have a nonlinear algorithm to stabilize the system. We need a procedure for finding the values of the constants in this algorithm. For example, the feedback law defined by (3.3) stabilizes the system if the constants a, b, d have such values that system (3.2) is asymptotically stable, i.e., $a, b, d > 0$ and $d < ab$. Naturally, these constants should be selected using some optimization procedure. This will be demonstrated below for Eq. (2.6).

According to (3.2), the constants a, b, d are determined by the feedback law $u = kw$, which (k is the feedback-gain matrix) can be found by, for example, solving the linear quadratic problem (see, e.g., [4]), i.e., optimizing the system according to the quadratic performance criterion

$$J = \int_0^\infty (w^T Q w + rv^2)dx, \tag{3.4}$$

or by the modal control method [1].

4. Design of a Servo System. Let us generalize the stabilization algorithm (3.2) to the case of tracking a reference trajectory. Let the reference path be nearly straight and let $\bar{y}(x)$ describe the reference trajectory of the aircraft. We rearrange the stabilization law (3.2) to the form

$$y''' = -ay'' - by' - d(y - \bar{y}(x)), \tag{4.1}$$

to allow the coordinate y to "track" the reference trajectory ($\bar{y}(x)$). However, such a (relatively simple) algorithm cannot ensure sufficient accuracy of tracking if $\bar{y}(x)$ is a relatively quickly varying function.

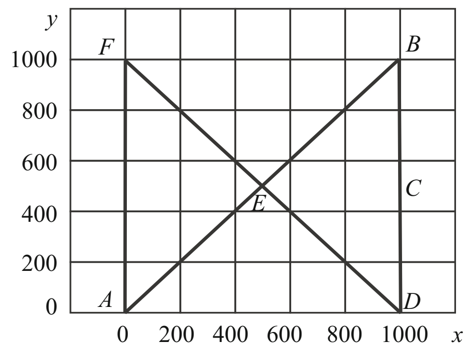

Fig. 1

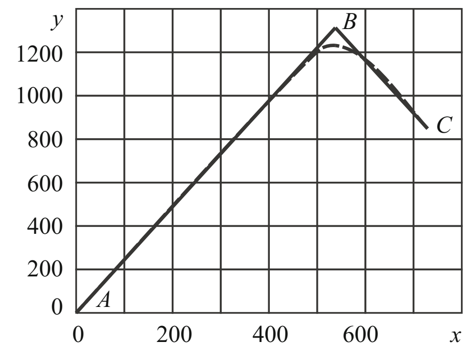

Fig. 2

Let us discuss the possibility of improving the accuracy of trajectory tracking. The transfer function $H(s)$ between the input ($\bar{y}(x)$) and the output ($y(x)$) of system (4.1) is given by

$$H(s) = \frac{d}{(s^3 + as^2 + bs + d)}. \tag{4.2}$$

At relatively low frequencies ($\omega << 1$, $s = i\omega$), the absolute value of the transfer function (4.2) is close to 1. In this frequency range, the error of reproducing the input ($y(x)$) by the output ($\bar{y}(x)$) is due to the phase lag. The phase lag ν of the system with transfer function (4.2) at frequency ω is given by

$$\tan\nu = \frac{\omega^2 - b}{d - a\omega^2}\,\omega.$$

If ω is small, then

$$\nu \cong -(b/d)\omega. \tag{4.3}$$

This phase lag can be counterbalanced by feeding $\bar{y}(x)$ to algorithm (4.1) with some advance Δ:

$$y''' = -ay'' - by' - d(y(x) - \bar{y}(x+\Delta)). \tag{4.4}$$

Indeed, the associated transfer function $H_1(s)$ is given by

$$H_1(s) = \frac{de^{\Delta s}}{s^3 + as^2 + bs + d} = H(s)e^{\Delta s}.$$

The phase lag of this system at low frequencies (analog of (4.3)) can be expressed as

$$\nu = -(b/d)\omega + \Delta\omega. \tag{4.5}$$

Choosing Δ such that $\nu = 0$ and using (4.5), we get

$$\Delta = b/d. \tag{4.6}$$

Thus, the feedback law (2.3) can be modified as follows:

$$u = -(a\psi + b(\cos\theta)^2 \sin\theta + d(\cos\theta)^3 (y - \bar{y}(x+\Delta))) - \frac{3\sin\theta\psi^2}{(\cos\theta)^2}. \tag{4.7}$$

Stabilization algorithms similar to (2.3) and (4.7) can be designed if a polar coordinate system (Eqs. (2.2)) is used to describe the motion of the system.

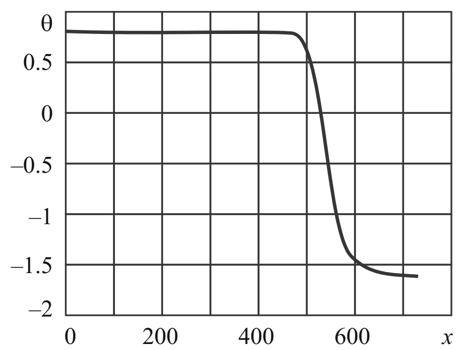

Fig. 3

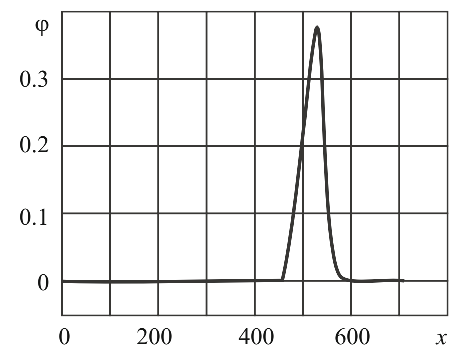

Fig. 4

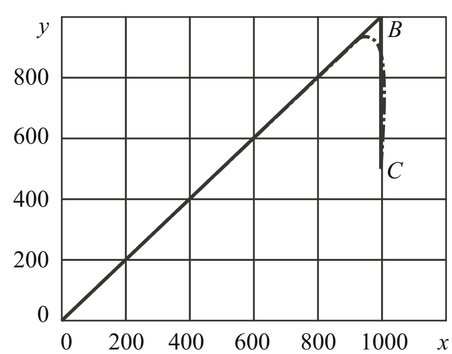

Fig. 5

It is also possible to use the algorithm when the condition $\dot{x} > 0$ fails. In some cases, the reference trajectory can be "cut" into sections on which the condition $\dot{x} > 0$ is satisfied, provided that the coordinates are transformed appropriately. Then the tracking problem can be solved in the new coordinate system (in which $\dot{x} > 0$). The solution found can then by transformed to the original coordinates (see the example below).

Example. Let us illustrate the algorithm by way of an example where an aircraft tracks the reference trajectory shown in Fig. 1 (two sides and two diagonals of a square with a side length of 1000 m (an analog of the trajectory shown in [3, Fig. 11.6])). The aircraft is programmed to track the reference signals from the sections *AB*, *BC*, *CD*, *DE*, *EF*, and *FA*. The equations of motion of the aircraft in Cartesian coordinates have the form (2.6). In designing the control system, the coefficients *a*, *b*, and *d* appearing in (3.2) are found by minimizing functional (3.4), where

$$Q = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q_{33} \end{bmatrix}, \quad r = 1, \quad q_{11} = 10^{-8}, \quad q_{33} = 0.1.$$

The following values of *a*, *b*, *d* correspond to the chosen *Q* and *r*: $a = 0.341$, $b = 8.26 \cdot 10^{-3}$, $d = 10^{-4}$, $\Delta = 82.6$ m, according to (4.6).

Let us analyze in detail the motion of the aircraft along the trajectory defined by the points *A*, *B*, and *C*. It is obvious that the condition $\dot{x} > 0$ fails on the section *BC* in the original coordinate system. In this connection, we will use a coordinate system turned by an angle $-\pi/8$ relative to the original coordinate system to track this section (shown by a solid line in Fig. 2) of the reference trajectory.

In this coordinate system, the condition $\dot{x} > 0$ is satisfied. In the same figure, the dash-and-dot line shows the trajectory of the aircraft (the advance Δ is defined by (4.6)). Figure 3 shows, in the same coordinate system, the heading angle θ defined by the first formula in (2.5). Figure 4 shows the bank angle φ determined from (1.2) for velocity $V = 10$ m/sec (ψ is defined by the second formula in (2.5)).

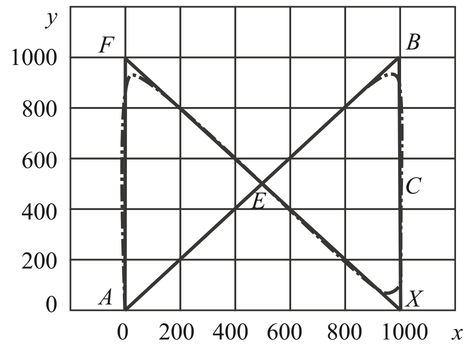

Fig. 6

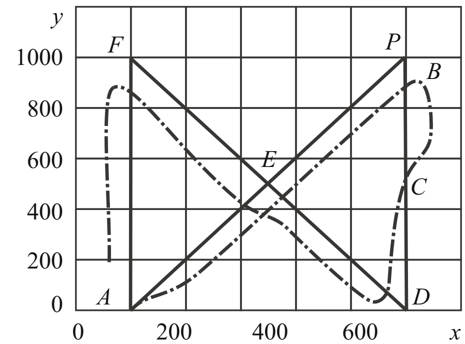

Fig. 7

Figure 5 shows, in the original coordinate system, the section *ABC* of the reference trajectory (solid line) and the trajectory of the aircraft (dash-and-dot line). Using similar procedures, we obtain the trajectory of the aircraft on the sections *CDE* and *EFA* (on these sections, the angle of rotation of the coordinate system that ensures that $\dot{x} > 0$ is equal to $\pi / 8 - \pi$). The whole trajectory of the aircraft is shown by a dash-and-dot line in Fig. 6 (the solid line represents the reference trajectory). Comparing Fig. 6 and [3, Fig. 11.6] indicates that the algorithm described above and the algorithm from [3, algorithm 6] provide equal accuracy of tracking the reference trajectory.

To demonstrate the improved accuracy of tracking owing to feeding, according to (4.4), the reference signal with an advance Δ defined by (4.6), Fig. 7 shows the solution for $\Delta = 0$, i.e., when the servo system is described by (4.1).

Comparing these figures suggests that feeding the reference signal with an advance allows considerable improvement of the tracking accuracy.

Conclusions. The aircraft's capability of tracking the reference trajectory has been improved by feeding the trajectory signal with a certain advance to the servo system. The algorithm is tested by way of example of an aircraft tracking a reference trajectory similar to that in [3, Fig. 11.6]. It is significant that this approach imposes no constraints on the differentiability of the reference signal.

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