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INFLUENCE OF AN ELASTIC FOUNDATION ON THE DISPERSION OF HARMONIC WAVES IN LONGITUDINALLY REINFORCED CYLINDRICAL SHELLS

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The effect of discrete ribs and Winkler and Pasternak foundations on the number and shape of the dispersion curves of harmonic waves propagating along a stringer-reinforced cylindrical shell is studied. The following cases of deformation are considered: (i) the stringers bend and twist, (ii) the stringers only bend, and (iii) the stringers only twist. The study of the effect of elastic foundation on the wave numbers shows that as the coefficients of subgrade reaction increase, the cutoff frequencies increase within a given range of excitation frequencies and the shape of the dispersion curves changes

Keywords: cylindrical shell, longitudinal ribs, Winkler and Pasternak coefficients of subgrade reaction, harmonic waves, dispersion curves, cutoff frequencies

Introduction. To analyze the effect of harmonic loads on shell structures, the dynamic characteristics of propagating waves [2, 5–7, 11] and, in particular, their wave numbers must be known. After determining them, the dispersion curves can be plotted. The dispersion curves for a rib-reinforced closed circular cylindrical shell were addressed in [19, 20], where the effect of discrete ribs on the number and shape of dispersion curves was also analyzed. The experimental studies [4, 8] suggest that the ambient medium has a strong effect on both the dynamic characteristics of reinforced shells and the wave processes in them. The effect of an elastic foundation on the natural frequencies and vibration modes, stability, and stress–strain state of ribbed cylindrical shells was analyzed in [3, 9, 10, 12, 14–17]. The free vibrations of longitudinally reinforced cylindrical shells were studied in [13, 18].

Here we will study the effect of Winkler and Pasternak foundation on the cutoff frequencies and the number and shape of the dispersion curves for harmonic waves propagating along a stringer-reinforced closed cylindrical shell.

1. Problem Formulation. Basic Equations. Consider a closed circular cylindrical shell regularly reinforced with identical stringers. The shell is hinged at the ends and resting on an elastic foundation characterized by Winkler or Pasternak coefficients of subgrade reaction (C_1 and C_2).

The equations of motion can be derived from applied theories of shells and rods based on the Kirchhoff and Kirchhoff–Clebsch hypotheses, respectively [1]:

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial\theta^2}\right)u+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial\xi\partial\theta}-\nu\frac{\partial w}{\partial\xi}-\frac{\partial^2 u}{\partial t_1^2}+\sum_{j=1}^{k_1}\delta(\theta-\theta_j)$$

$$\times\left(\gamma_s\frac{\partial^2 u}{\partial\xi^2}-\delta_s\frac{\partial^2 w}{\partial\xi^3}-\bar{\rho}_s\bar{\gamma}_s\frac{\partial^2 u}{\partial t_1^2}+\bar{\rho}_s\bar{\delta}_s\frac{\partial^3 w}{\partial t_1^2\partial\xi}\right)\Bigg|_{\theta=\theta_j}=0,$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{\partial\xi\partial\theta}+\left[\frac{1-\nu}{2}(1+4a^2)\frac{\partial^2}{\partial\xi^2}+(1+a^2)\frac{\partial^2}{\partial\theta^2}\right]v$$

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$$+\left\{-\frac{\partial}{\partial\theta}+a^2\left[(2-\nu)\frac{\partial^3}{\partial\xi^2\partial\theta}+\frac{\partial^3}{\partial\theta^3}\right]\right\}w-\frac{\partial^2 u}{\partial t_1^2}+\sum_{j=1}^{k_1}\delta(\theta-\theta_j)$$

$$\times\left[-\left(1-\frac{h_\mathrm{s}}{r}\right)^2\lambda_{1\mathrm{s}}\frac{\partial^4 v}{\partial\xi^4}+\left(1-\frac{h_\mathrm{s}}{r}\right)\lambda_{2\mathrm{s}}\frac{\partial^5 w}{\partial\xi^4\partial\theta}+\mu_\mathrm{s}\left(\frac{\partial^3 w}{\partial\xi^2\partial\theta}+\frac{\partial^2 v}{\partial\xi^2}\right)\right.$$

$$\left.-\bar{\rho}_\mathrm{s}\bar{\gamma}_\mathrm{s}\left(1-\frac{h_\mathrm{s}}{r}\right)\frac{\partial^2 v}{\partial t_1^2}+\bar{\rho}_\mathrm{s}\bar{\delta}_\mathrm{s}\left(1-\frac{h_\mathrm{s}}{r}\right)\frac{\partial^2 w}{\partial t_1^2\partial\theta}-\bar{\rho}_\mathrm{s}\bar{\mu}_\mathrm{s}\left(\frac{\partial^3 w}{\partial t_1^2\partial\theta}+\frac{\partial^2 v}{\partial t_1^2}\right)\right]\Bigg|_{\theta=\theta_j}=0,$$

$$-\nu\frac{\partial u}{\partial\xi}+\left\{-\frac{\partial}{\partial\theta}+a^2\left[(2-\nu)\frac{\partial^3}{\partial\xi^2\partial\theta}+\frac{\partial^3}{\partial\theta^3}\right]\right\}v+w+a^2\Delta\Delta w+\bar{C}_1 w$$

$$-\bar{C}_2\left(\frac{\partial^2 w}{\partial\xi^2}+\frac{\partial^2 w}{\partial\theta^2}\right)+\frac{\partial^2 w}{\partial t_1^2}+\sum_{j=1}^{k_1}\left\{\delta(\theta-\theta_j)\left[\eta_\mathrm{s}\frac{\partial^4 w}{\partial\xi^4}-\delta_\mathrm{s}\frac{\partial^3 u}{\partial\xi^3}+\bar{\rho}_\mathrm{s}\bar{\gamma}_\mathrm{s}\frac{\partial^2 w}{\partial t_1^2}\right.\right.$$

$$\left.+\bar{\rho}_\mathrm{s}\bar{\delta}_\mathrm{s}\frac{\partial^3 u}{\partial t_1^2\partial\xi}-\bar{\rho}_\mathrm{s}\bar{\eta}_\mathrm{s}\frac{\partial^4 w}{\partial\xi^2\partial t_1^2}\right]\Bigg|_{\theta=\theta_j}+\frac{d\delta(\theta-\theta_j)}{d\theta}\left[\mu_\mathrm{s}\left(\frac{\partial^3 w}{\partial\xi^2\partial\theta}+\frac{\partial^2 v}{\partial\xi^2}\right)\right.$$

$$+\left(1-\frac{h_\mathrm{s}}{r}\right)\lambda_{2\mathrm{s}}\frac{\partial^4 v}{\partial\xi^4}-\lambda_{3\mathrm{s}}\frac{\partial^5 w}{\partial\xi^4\partial\theta}-\bar{\rho}_\mathrm{s}\bar{\mu}_\mathrm{s}\left(\frac{\partial^3 w}{\partial t_1^2\partial\theta}+\frac{\partial^2 v}{\partial t_1^2}\right)$$

$$\left.+\bar{\rho}_\mathrm{s}\bar{\delta}_\mathrm{s}\left(1-\frac{h_\mathrm{s}}{r}\right)\frac{\partial^2 v}{\partial t_1^2}-\bar{\rho}_\mathrm{s}\bar{\eta}_\mathrm{s}\frac{\partial^3 w}{\partial t_1^2\partial\theta}\right]\Bigg|_{\theta=\theta_j}=0, \qquad (1)$$

where u, v, and w are the displacements of a particle of the shell's mid-surface; $\xi = x/r$, $\theta = y/r$, x and y are the Cartesian coordinates of this particle; $t_1 = t\omega_0$, t is time,

$$a^2=\frac{h^2}{12r^2},\quad \bar{\rho}_\mathrm{s}=\frac{\rho_\mathrm{s}}{\rho_0},\quad \bar{\gamma}_\mathrm{s}=\frac{F_\mathrm{s}k_1}{2\pi rh},\quad \bar{\delta}_\mathrm{s}=\frac{h_\mathrm{s}}{r}\bar{\gamma}_\mathrm{s},\quad \bar{\eta}_\mathrm{s}=\left(\frac{h_\mathrm{s}}{r}\right)^2\bar{\gamma}_\mathrm{s},$$

$$\bar{\mu}_\mathrm{s}=\frac{I_\mathrm{tws}k_1}{2\pi r^3 h},\quad \gamma_\mathrm{s}=\frac{E_\mathrm{s}}{E}(1-\nu^2)\bar{\gamma}_\mathrm{s},\quad \delta_\mathrm{s}=\frac{h_\mathrm{s}}{r}\gamma_\mathrm{s},\quad \eta_\mathrm{s}=\frac{E_\mathrm{s}(I_{y\mathrm{s}}+h_\mathrm{s}^2F_\mathrm{s})k_1}{2\pi r^3 hE}(1-\nu^2),$$

$$\lambda_{1\mathrm{s}}=\frac{E_\mathrm{s}I_{z\mathrm{s}}k_1(1-\nu^2)}{2\pi r^3 hE},\quad \lambda_{2\mathrm{s}}=\frac{h_\mathrm{s}}{r}\lambda_{1\mathrm{s}},\quad \lambda_{3\mathrm{s}}=\left(\frac{h_\mathrm{s}}{r}\right)^2\lambda_{1\mathrm{s}},\quad \mu_\mathrm{s}=\frac{G_\mathrm{s}}{E_\mathrm{s}}(1-\nu^2)\bar{\mu}_\mathrm{s},$$

$$\bar{C}_1=\frac{(1-\nu^2)r^2}{Eh}C_1,\quad \bar{C}_2=\frac{(1-\nu^2)}{Eh}C_2,$$

h and r are the thickness and mid-radius of the shell; E, ν, and ρ_0 are the elastic modulus, Poisson's ratio, and density of the material of the shell; F_s, I_{ys}, I_{zs}, and I_tws are the cross-sectional area of a stringer, its moments of inertia in bending in the radial plane and in the plane equidistant to the tangent to the shell's mid-surface, and the twisting moment of inertia; h_s is the eccentricity of a stringer (the distance from the shell mid-surface to the stringer axis; $h_\mathrm{s} > 0$ if the stringers are attached to the inside surface of the shell); k_1 is the number of stringers; E_s, G_s, ρ_s are the elastic and shear moduli and density of the material of the stringers; $\delta(\theta-\theta_j)$ is the Dirac delta function, $\theta_j=\frac{2\pi}{k_1}j$.

2. Problem-Solving Method. Let the candidate solution of system (1) be represented by series:

$$u = e^{ik\xi} \sum_{n=0}^{\infty} (u_{n1} \cos n\theta + u_{n2} \sin n\theta) \cos \omega_1 t_1,$$

$$v = e^{ik\xi} \sum_{n=0}^{\infty} (v_{n1} \sin n\theta + v_{n2} \cos n\theta) \cos \omega_1 t_1,$$

$$w = e^{ik\xi} \sum_{n=0}^{\infty} (w_{n1} \cos n\theta + w_{n2} \sin n\theta) \cos \omega_1 t_1, \tag{2}$$

where u_{ns}, v_{ns}, w_{ns} (s = 1, 2) are unknown constants; k is the dimensionless wave number (the corresponding wavelength $\lambda = 2\pi r/k$), $\omega_1 = \omega/\omega_0$, $\omega_0 = \sqrt{E/[(1-\nu^2)\rho_0 r^2]}$.

Substituting (2) into (1), we reduce the problem to infinite systems of homogeneous linear algebraic equations for u_{ns}, v_{ns}, and w_{ns}. In [1] it was shown that these systems have an exact solution. It was used in [19] to derive dispersion equations (from which the characteristic (wave) numbers are calculated) for the following three cases of deformation:

(a) the shell undergoes arbitrary circumferential deformation (general case of deformation, according to [1]); the wave numbers depend on all the geometrical and mechanical characteristics of the stringers;

(b) the shell undergoes such circumferential deformation that the stringers are at the deflection antinodes (first special case of deformation); the wave numbers depend only on the stiffnesses of the stringers in tension/compression and radial in-plane bending and on the geometrical and mechanical parameters on which the inertial characteristics of the stringers depend;

(b) the shell undergoes such circumferential deformation that the stringers are at the deflection nodes (second special case of deformation); the wave numbers depend only on the stiffnesses of the stringers in twisting and in bending in the plane equidistant to the tangent to the shell's mid-surface and on the geometrical and mechanical parameters on which the inertial characteristics of the stringers depend.

We will use a simplified version of Eqs. (1). We will assume that the characteristic numbers have a weak effect on the stiffness of the stringers in twisting and in bending in the plane equidistant to the tangent to the mid-surface and on the inertial characteristics. Then the characteristic equations for the general and first special cases of deformation have the same form, but correspond to different circumferential wave numbers.

Thus, the wave numbers can be determined by finding the roots of the following equations:

$$1 + L_n^{11} + L_n^{33} + L_n^{11} L_n^{33} + L_n^{13} L_n^{31} = 0, \tag{3}$$

$n = 1, \ldots, n_2$ (general case of deformation); $n = 0, (k_1/2)\delta_{k_1 2s_1}$ (first special case of deformation). In the second special case of deformation, these equations have the form

$$1 + L_n^{22} + L_n^{44} + L_n^{22} L_n^{44} - L_n^{24} L_n^{42} = 0 \quad (n = 0,\ \delta_{k_1 2s_1} k_1/2), \tag{4}$$

where $n_2 = k_1/2$ (if k_1 is even); $n_2 = (k_1 - 1)/2$ (if k_1 is odd); $2n$ is the number of nodal lines in the circumferential wave; $\delta_{k_1 2s_1}$ is the Kronecker delta; s_1 = 1, 2, … .

As shown in [1], the number of different characteristic (dispersion in our case) equations is restricted and determined by the periodicity conditions for the sums $\Phi_1^n(X_{n_1})$ below:

$$L_n^{11} = C_{11} \Phi_1^n \left(\frac{L_{n_1}}{1+\delta_{0n_1}} \right) - C_{22} \Phi_1^n \left(\frac{E_{n_1}}{1+\delta_{0n_1}} \right), \quad L_n^{13} = C_5 \Phi_1^n \left(\frac{E_{n_1}}{1+\delta_{0n_1}} \right) + C_{22} \Phi_1^n \left(\frac{L_{n_1}}{1+\delta_{0n_1}} \right),$$

$$L_n^{31} = C_{11} \Phi_1^n \left(\frac{E_{n_1}}{1+\delta_{0n_1}} \right) + C_{22} \Phi_1^n \left(\frac{F_{n_1}}{1+\delta_{0n_1}} \right), \quad L_n^{33} = C_5 \Phi_1^n \left(\frac{F_{n_1}}{1+\delta_{0n_1}} \right) - C_{22} \Phi_1^n \left(\frac{E_{n_1}}{1+\delta_{0n_1}} \right),$$

TABLE 1

$\overline{C}_1$	$\omega_1 \cdot 10$			
	$n = 0$	$n = 1$		$n = 2$
0	0.099	0.053	0.841	0.018
	0.427	0.165	1.180	0.236
	0.978	0.334	1.576	0.676
	1.753	0.559	—	1.339
0.001	0.305	0.220	0.638	0.268
	0.521	0.295	0.896	0.380
	1.024	0.345	1.220	0.740
	1.780	0.453	1.606	1.374
0.005	0.644	0.490	0.888	0.590
	0.798	0.641	1.089	0.708
	1.190	0.699	1.368	0.957
	1.881	0.764	1.722	1.503
0.010	0.892	0.691	1.126	0.825
	1.051	0.892	1.291	0.973
	1.372	0.975	1.534	1.176
	—	1.031	1.857	1.657

$$L_n^{22} = C_3 \Phi_1^n \left(\frac{N_{n_1}}{1 + \delta_{0n_1}} \right) - C_4 \Phi_1^n (n_1 M_{n_1}), \quad L_n^{24} = C_6 \Phi_1^n (n_1 M_{n_1}) + C_4 \Phi_1^n \left(\frac{N_{n_1}}{1 + \delta_{0n_1}} \right),$$

$$L_n^{42} = -C_3 \Phi_1^n (n_1 M_{n_1}) - C_4 \Phi_1^n (n_1^2 F_{n_1}), \quad L_n^{44} = C_6 \Phi_1^n (n_1^2 F_{n_1}) - C_4 \Phi_1^n (n_1 M_{n_1}), \tag{5}$$

$$\Phi_1^n (X_{n_1}) = X_n + \sum_{l=1}^{\infty} (X_{lk_1 + n} + X_{lk_1 - n}), \tag{6}$$

$$L_n = \frac{1}{D_n} \left[a_n^{22} a_n^{33} - (a_n^{23})^2 \right], \quad E_n = \frac{1}{D_n} (a_n^{12} a_n^{23} - a_n^{13} a_n^{22}), \quad F_n = -\frac{1}{D_n} \left[a_n^{11} a_n^{22} - (a_n^{12})^2 \right],$$

$$N_n = -\frac{1}{D_n} \left[a_n^{11} a_n^{33} - (a_n^{13})^2 \right], \quad M_n = -\frac{1}{D_n} (-a_n^{11} a_n^{23} + a_n^{13} a_n^{12}), \tag{7}$$

TABLE 2

$\overline{C}_1$	$\overline{C}_2$	$\omega_1 \cdot 10$			
		$n = 0$	$n = 1$		$n = 2$
0	0	0.099	0.053	0.841	0.018
		0.427	0.165	1.180	0.236
		0.978	0.334	1.576	0.676
		1.753	0.559	—	1.339
0.001	0.0001	0.476	0.231	1.079	0.319
		0.913	0.405	1.394	0.675
		1.529	0.587	1.757	1.200
		—	0.811	—	1.913
	0.0005	0.872	0.269	1.118	0.466
		1.753	0.686	1.571	1.304
	0.0010	1.194	0.311	1.543	0.602
		—	0.923	—	1.805
0.010	0.0001	0.976	0.696	1.421	0.856
		1.281	0.952	1.674	1.115
		1.779	1.078	1.988	1.504
		—	1.225	—	—
	0.0005	1.224	0.711	1.436	0.928
		—	1.108	1.817	1.575
	0.0010	1.472	0.728	1.787	1.005
		—	1.270	—	—

$$D_n = -a_n^{11} a_n^{22} a_n^{33} - 2a_n^{12} a_n^{13} a_n^{23} + a_n^{11} (a_n^{23})^2 + a_n^{22} (a_n^{13})^2 + a_n^{33} (a_n^{12})^2, \qquad (8)$$

$$a_n^{11} = \left(k^2 + \frac{1-\nu}{2} n^2\right) - \omega_1^2, \quad a_n^{12} = \frac{1+\nu}{2} kn, \quad a_n^{13} = \nu k,$$

$$a_n^{22} = (1+a^2) n^2 + \frac{1-\nu}{2}(1+4a^2) k^2 - \omega_1^2, \quad a_n^{23} = n\left[1+(2-\nu) a^2 k^2 + a^2 n^2\right],$$

$$a_n^{33} = 1 + a^2 (k^2 + n^2)^2 + \overline{C}_1 + \overline{C}_2 (k^2 + n^2) - \omega_1^2,$$

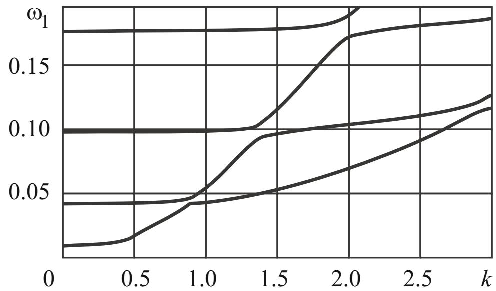

Fig. 1

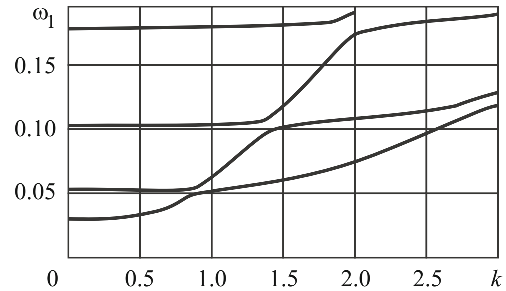

Fig. 2

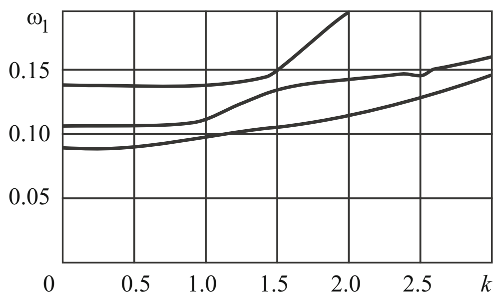

Fig. 3

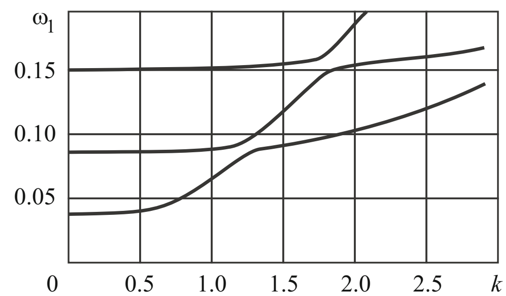

Fig. 4

$$C_{11} = -\gamma_s k^2 + \bar{\rho}_s \bar{\gamma}_s \omega_1^2, \quad C_{22} = -\delta_s k^3 + \bar{\rho}_s \bar{\delta}_s \omega_1^2 k, \quad C_5 = \eta_s k^4 - \bar{\rho}_s \bar{\gamma}_s \omega_1^2 + \bar{\rho}_s \bar{\eta}_s \omega_1^2 k^2,$$

$$C_3 = k^2 (\lambda_{1s} k^2 + \mu_s) - \bar{\rho}_s \bar{\gamma}_s \omega_1^2 \left(1 - \frac{h_s}{r}\right) - \bar{\rho}_s \bar{\mu}_s \omega_1^2,$$

$$C_4 = k^2 (\lambda_{2s} k^2 - \mu_s) - \bar{\rho}_s \bar{\delta}_s \omega_1^2 \left(1 - \frac{h_s}{r}\right) + \bar{\rho}_s \bar{\mu}_s \omega_1^2,$$

$$C_6 = k^2 (\lambda_{3s} k^2 + \mu_s) - \bar{\rho}_s (\bar{\mu}_s + \bar{\eta}_s)\, \omega_1^2 .$$

The equations for the cutoff frequencies can be derived from (3) and (4) where $k = 0$.

3. Analysis of the Numerical Results. The numerical results discussed here have been obtained for a shell reinforced with four stringers ($k_1 = 4$) attached to the inside surface of the shell having the following dimensionless geometrical and mechanical parameters: $h/r = 0.25 \cdot 10^{-2}$, $F_s / 2\pi rh = 0.16 \cdot 10^{-1}$, $h_s / r = 0.14 \cdot 10^{-1}$, $I_{tws} / 2\pi r^3 h = 0.53 \cdot 10^{-6}$, $I_{zs} / 2\pi r^3 h = 0.13 \cdot 10^{-6}$, $E_s = E$, $G_s = 0.3845E$, $\nu = 0.3$. The following range of excitation frequencies has been examined: $0 \le \omega_1 \le 0.2$.

The external loading was assumed to generate harmonic waves described by the dispersion equations (3). Since we are considering harmonic waves propagating along the shell, only the real roots of Eqs. (3) will be considered.

The following types of harmonic load on the ribbed shell were considered: (i) cyclically symmetric circumferential load with period $2\pi / k_1$ ($n = 0$); (ii) cyclically symmetric circumferential load with period $4\pi / k_1$ ($n = k_1 / 2$), (iii) antisymmetric circumferential load ($n = 1$). For numerical purposes, 101 term ($l_{max} = 100$) was retained in series (6).

Tables 1 and 2 summarize the cutoff frequencies for different values of the Winkler and Pasternak coefficients $\bar{C}_1$ and $\bar{C}_2$. It can be seen that with increase in the coefficients of subgrade reaction, the cutoff frequencies increase, compared with the frequencies for the shell without elastic foundation. The number of cutoff frequencies decreases at relatively great coefficients of subgrade reaction. The cutoff frequencies corresponding to the Pasternak foundation are higher than the cutoff frequencies corresponding to the Winkler foundation.

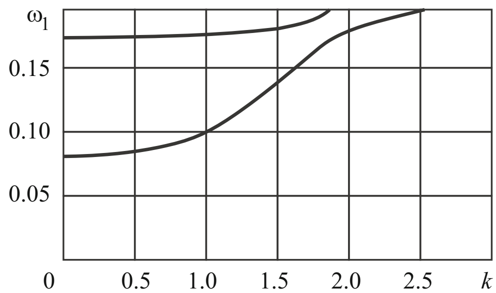

Fig. 5

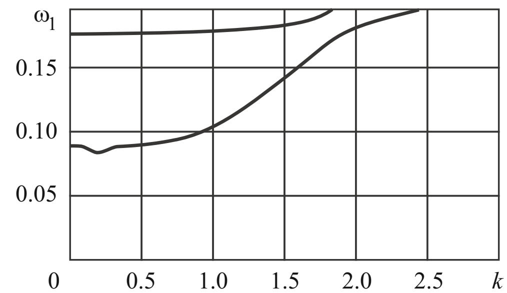

Fig. 6

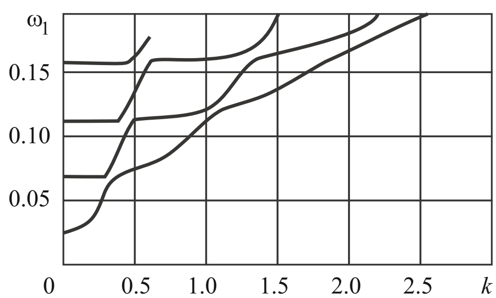

Fig. 7

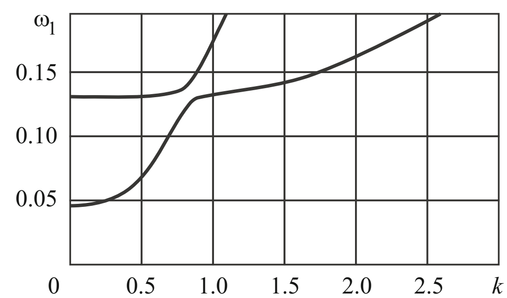

Fig. 8

Figures 1, 2, and 3 show the dispersion curves for $n = 0$ and for $\overline{C}_1 = 0$ (no Winkler foundation), $\overline{C}_1 = 0.001$, and $\overline{C}_1 = 0.01$, respectively. It can be seen that as the Winkler coefficient of subgrade reaction increases, the dispersion curves become more shallow.

Figures 4 and 5 show the dispersion curves for the same case of deformation, $\overline{C}_1 = 0$, and for $\overline{C}_2 = 0.0001$ and $\overline{C}_2 = 0.01$, respectively. With increase in the Pasternak coefficient of subgrade reaction $\overline{C}_2$, the number and shape of dispersion curves change too.

Figures 6, 7, and 8 show the dispersion curves of harmonic waves propagating along a stringer-reinforced cylindrical shell on a Pasternak foundation ($\overline{C}_1 = 0.001$ and $\overline{C}_2 = 0.0005$). Figure 7 represents harmonic circumferential waves of arbitrary profile (the wavelength is not multiple of the distance between stringers, which is the general case of deformation), and Figs. 6 and 8 represent waves having antinodes on the stringers (first special case of deformation). Comparing these figures reveals that the number of dispersion curves in the general case of deformation is greater than that in the first special case of deformation. In the first special case of deformation ($n = 0$) and ($n = 2$), the dispersion curves differ in shape, but not in number. For $n = 0$, the dispersion curves are more shallow than for $n = 2$.

Conclusions. Analyzing the results of the study, we may conclude that the Winkler and Pasternak foundations affect the cutoff frequencies for harmonic waves propagating along a stringer-reinforced shell. With increase in the coefficients of subgrade reaction, the values of the cutoff frequencies increase and their number in the chosen excitation frequency range decreases. The presence of discrete ribs made it possible to plot dispersion curves for different case of deformation of the shell. The greater the coefficients of subgrade reaction, the more shallow the dispersion curves in all the cases of deformation considered.

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