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THIRD APPROXIMATION IN THE ANALYSIS OF A QUADRATIC NONLINEAR HYPERELASTIC CYLINDRICAL WAVE

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The perturbation method is applied to solve the problem of the propagation of a plane longitudinal harmonic wave in a hyperelastic material described by the classical Murnaghan model. The exact expression of the second-order solution in terms of Hankel functions of zero and first order and their products is derived. A simplification of the expression is considered

Keywords: cylindrical wave, quadratically nonlinear wave equation, Murnaghan model, perturbation method, first three approximations

Introduction. The subject of the present study is a cylindrical wave [2, 3, 15] in a quadratic nonlinear hyperelastic material described by the Murnaghan model. We will consider different problem formulations (plane strain, axisymmetric, and other states) [18–20].

Let the state be axisymmetric. A second-order solution of such a problem was analyzed in [14, 23]. Since the zero (linear) approximation is expressed in terms of the Hankel function that can be represented as power series, such series expansions were used in [20, 23]. A general approach to the analysis of many approximations was presented in [14] where the exact analytical expression for the second-order solution in terms of Hankel functions was also derived.

It should be noted that recent trends have been toward more intensive study of nonlinear elastic waves. Examples are publications on the propagation of elastic waves in classical materials [4, 8], in microstructural materials [6, 24], in materials of new type [5], in geological media [9, 26], in structural members [12, 25], on simple and complex types of waves [4, 10, 12, 15, 16, 17, 21]. Plane and cylindrical waves in Murnaghan materials were earlier analyzed using methods well known in the nonlinear theory of waves: the method of successive approximations (pertubation method) and the method of slowly varying amplitudes (Van der Pol method) [7]. In the former case, the solution was restricted to the first two (zero (linear) and first) approximations because of the following two reasons [7]: (i) the first-order approximation coincides with the solution of the evolutionary equation found with the method of slowly varying amplitudes; (ii) the basic nonlinear wave phenomena described by the first-order solution and the solution of the evolutionary equation are in agreement with experimental observations. The analysis of the effect of the subsequent approximations is still an open issue. The publication [22] may be regarded as just the beginning of such an analysis.

1. Finding the First Three Approximations for a Cylindrical Wave. We start with the nonlinear wave equation for an axisymmetric state (depending only on the coordinate *r* and having the axis of symmetry *Oz*) of the continuum (this state is characteristic of a classical cylindrical wave) with physical and geometrical nonlinearities [19, 20]:

$$u_{r,tt} - \frac{\lambda + 2\mu}{\rho}\left(u_{r,r} + \frac{u_r}{r}\right)_{,r} = S\left(u_r, u_{r,r}, u_{r,rr}\right),$$

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$$S\left(u_r, u_{r,r}, u_{r,rr}\right) = -\tilde{N}_1 u_{r,rr} u_{r,r} - \tilde{N}_2 \frac{1}{r} u_{r,rr} u_r - \tilde{N}_3 \frac{1}{r^2} u_{r,r} u_r - \tilde{N}_4 \frac{1}{r}\left(u_{r,r}\right)^2 - \tilde{N}_5 \frac{1}{r^3}\left(u_r\right)^2,$$

$$\tilde{N}_1 = \left[3 + \frac{2(A+3B+C)}{\lambda+2\mu}\right], \quad \tilde{N}_2 = \frac{\lambda+2B+2C}{\lambda+2\mu}, \quad \tilde{N}_3 = \frac{\lambda}{\lambda+2\mu},$$

$$\tilde{N}_4 = \frac{2\lambda+3\mu+A+2B+2C}{\lambda+2\mu}, \quad \tilde{N}_5 = \frac{2\lambda+3\mu+A+2B+C}{\lambda+2\mu}, \tag{1}$$

where $u_r(r,t)$ is the radial displacement; r is the distance traveled by the wave; t is the time of propagation; ρ is constant density; λ, μ are the elastic constants of the second order (Lame constants); A, B, C are the elastic constants of the third order (Murnaghan constants); the overdot denotes differentiation with respect to time t; the comma after the index denotes differentiation with respect to the coordinate r.

Consider harmonic cylindrical waves in a hyperelastic medium described in [19, 20] and having a cylindrical cavity of radius r_o to which a harmonic load $\sigma^{rr}(r_o, t) = p_o e^{i\omega t}$ is applied or a harmonic radial displacement $u_r(r_o, t) = u_{ro} e^{i\omega t}$ is imparted. In the linear case, such waves are analytically described by the solution to the linear wave equation

$$u_{r,tt} - (v_{ph})^2 \left(u_{r,r} + \frac{u_r}{r}\right)_{,r} = 0 \tag{2}$$

in terms of first-order Hankel functions of the first kind [7, 11, 13, 15]:

$$u_r^{(0)}(r,t) = u_{ro} H_1^{(1)}(k_L r) e^{i\omega t}, \tag{3}$$

where u_{ro} is an arbitrary amplitude determined from the boundary condition on the surface of the cavity

$$u_{ro} = -\frac{p_o k_L}{k_L(\lambda+2\mu) H_0^{(1)}(k_L r_o) - \frac{2\mu}{r_o} H_1^{(1)}(k_L r_o)};$$

$k_L = (\omega / v_L)$ is the wave number of a longitudinal plane wave.

A specific feature of the cylindrical wave (3) is that it is not harmonic (the properties of the Hankel function suggest that it is asymptotically harmonic) and its intensity decreases with time.

When analyzed with the method of successive approximations, a specific feature of nonlinear plane and cylindrical waves is an increase in the amplitude (intensity) of the wave with the distance traveled.

Thus, the analysis of a cylindrical wave undertaken here is expected to reveal the competition between two processes: increase in the wave amplitude due to nonlinear deformation and decrease in the wave amplitude due to the nature of the cylindrical wave.

The first approximation can be found as the solution of the inhomogeneous linear wave equation:

$$u_{r,tt}^{(1)} - (v_{ph})^2 \left(u_{r,r}^{(1)} + \frac{u_r^{(1)}}{r}\right)_{,r} = S\left(u_r^{(0)}, u_{r,r}^{(0)}, u_{r,rr}^{(0)}\right), \tag{4}$$

$$S\left(u_r, u_{r,r}, u_{r,rr}\right) = -\tilde{N}_1 u_{r,rr} u_{r,r} - \tilde{N}_2 \frac{1}{r} u_{r,rr} u_r - \tilde{N}_3 \frac{1}{r^2} u_{r,r} u_r - \tilde{N}_4 \frac{1}{r}(u_{r,r})^2 - \tilde{N}_5 \frac{1}{r^3}(u_r)^2.$$

Then the propagation of a cylindrical wave is described by the following first-order solution:

$$u_r(r,t) = u_r^{(0)}(r,t) + u_r^{(1)}(r,t) = u_{ro} H_1^{(1)}(k_L r) e^{i\omega t}$$

$$+ \left\{B_{00}[H_0^{(1)}(k_L r)]^2 + B_{11}[H_1^{(1)}(k_L r)]^2 + B_{01} H_0^{(1)}(k_L r) H_1^{(1)}(k_L r)\right\} e^{2i\omega t}, \tag{5}$$

where

$$B_{11} = k_{\mathrm{L}} \left(-2 + \frac{3}{(k_{\mathrm{L}} r)^2} - 2\left(\frac{1}{1-2(k_{\mathrm{L}} r)^2} \right) + 8\left(\frac{1}{1-4(k_{\mathrm{L}} r)^2} \right) \right)^{-1}$$

$$\times \left\{ \left(-\tilde{N}_1 + \tilde{N}_2 \right) \frac{1}{k_{\mathrm{L}} r} + \left(2\tilde{N}_1 - 2\tilde{N}_2 + \tilde{N}_3 - \tilde{N}_4 - \tilde{N}_5 \right) \frac{1}{(k_{\mathrm{L}} r)^3} - 2\left(\tilde{N}_1 + \tilde{N}_4 \right) \left(\frac{(k_{\mathrm{L}} r)^2}{1-2(k_{\mathrm{L}} r)^2} \right) \frac{1}{(k_{\mathrm{L}} r)} \right.$$

$$\left. -\tilde{N}_1 \frac{2}{k_{\mathrm{L}} r} \left(\frac{(k_{\mathrm{L}} r)^2}{1-4(k_{\mathrm{L}} r)^2} \right) - \left(-\tilde{N}_1 + \tilde{N}_2 - \tilde{N}_3 + 2\tilde{N}_4 \right) \frac{2}{k_{\mathrm{L}} r} \frac{1}{(k_{\mathrm{L}} r)^2} \left(\frac{(k_{\mathrm{L}} r)^2}{1-4(k_{\mathrm{L}} r)^2} \right) \right\},$$

$$B_{00} = \left(\frac{1}{(k_{\mathrm{L}} r)^2} - 2 \right)^{-1} \left(k_{\mathrm{L}} \left(\tilde{N}_1 + \tilde{N}_4 \right) \frac{1}{(k_{\mathrm{L}} r)} - 2B_{11} \right),$$

$$B_{01} = \left(\frac{1}{(k_L r)^2} - 4 \right)^{-1} \left(k_L \left[\tilde{N}_1 + \left(-\tilde{N}_1 + \tilde{N}_2 - \tilde{N}_3 + 2\tilde{N}_4 \right) \frac{1}{(k_L r)^2} \right] + \frac{4}{k_L r} B_{11} \right).$$

Solution (5) can be simplified provided that the distance $(r - r_{\mathrm{o}})$ traveled by the wave is related to the wavelength λ_{L} so that

$$k_{\mathrm{L}} = 2\pi / \lambda_{\mathrm{L}} \to k_{\mathrm{L}} r = 2\pi r / \lambda_{\mathrm{L}} \to r > 3\lambda_{\mathrm{L}} \to k_{\mathrm{L}} r > 20. \tag{6}$$

In this case, two terms with coefficients B_{00}, B_{11} and some factors of B_{01} can be neglected in (5). As a result, we obtain the equation

$$u_r(r,t) = u_{r\mathrm{o}} H_1^{(1)}(k_{\mathrm{L}} r) e^{i\omega t} - \frac{1}{4} (u_{\mathrm{o}})^2 k_{\mathrm{L}} \frac{N_1}{\lambda + 2\mu} H_0^{(1)}(k_{\mathrm{L}} r) H_1^{(1)}(k_{\mathrm{L}} r) e^{2i\omega t}. \tag{7}$$

Consider the equation for the third approximation

$$u_{r,tt}^{(2)} - (v_{\mathrm{ph}})^2 \left(u_{r,r}^{(2)} + \frac{u_r^{(2)}}{r} \right)_{,r} = S\left(u_r^{(1)}, u_{r,r}^{(1)}, u_{r,rr}^{(1)} \right). \tag{8}$$

The right-hand side of Eq. (8) is represented in terms of Hankel functions:

$$S(u_r^{(1)}, u_{r,r}^{(1)}, u_{r,rr}^{(1)}) = (k_{\mathrm{L}})^3 e^{4i\omega t} \left\{ \left(\tilde{N}_1 - \tilde{N}_4 \right) (1/k_{\mathrm{L}} r) \left(H_0^{(1)}(k_{\mathrm{L}} r) \right)^4 \right.$$

$$+ \left(3\tilde{N}_1 + \tilde{N}_4 \right) (1/k_{\mathrm{L}} r) \left(H_1^{(1)}(k_{\mathrm{L}} r) \right)^4$$

$$+ \left(-\left(\frac{3}{k_{\mathrm{L}}^2 r^2} - 4 \right) \tilde{N}_1 + \frac{1}{k_{\mathrm{L}}^2 r^2} \tilde{N}_2 - \frac{1}{k_{\mathrm{L}}^2 r^2} \tilde{N}_3 + \frac{1}{k_{\mathrm{L}}^2 r^2} \tilde{N}_4 \right) (H_0^{(1)})^3 H_1^{(1)}$$

$$+ \left(-\left(4 - \frac{5}{k_{\mathrm{L}}^2 r^2} \right) \tilde{N}_1 - 3\frac{1}{k_{\mathrm{L}}^2 r^2} \tilde{N}_2 + \frac{1}{k_{\mathrm{L}}^2 r^2} \tilde{N}_3 - \frac{1}{k_{\mathrm{L}}^2 r^2} \tilde{N}_4 \right) H_0^{(1)} (H_1^{(1)})^3$$

$$+\left(-\frac{1}{k_{\rm L} r}\left(8-\frac{2}{k_{\rm L}^2 r^2}\right)\tilde{N}_1-\frac{1}{k_{\rm L} r}\left(\frac{2}{k_{\rm L}^2 r^2}-4\right)\tilde{N}_2\right.$$

$$\left.+\tilde{N}_3\frac{1}{k_{\rm L}^3 r^3}+\frac{1}{k_{\rm L} r}\left(1+\frac{1}{k_{\rm L}^2 r^2}\right)\tilde{N}_4-\tilde{N}_5\frac{1}{k_{\rm L}^3 r^3}\right)(H_0^{(1)})^2(H_1^{(1)})^2. \qquad (9)$$

Thus, the right-hand side of the linear wave equation (4) includes three terms with Hankel functions of the zero and first orders to the fourth power and their products to the first, second, and third powers.

The coefficients in these terms depend on time through the exponential $e^{4i\omega t}$ (the time factor of the second harmonic of a longitudinal plane wave), the traveled distance r, and constant parameters (elastic constants and wave number of a plane longitudinal wave).

Since the powers of Hankel functions of the zero and first orders and their products $H_0^{(1)}(k_{\rm L} r)H_1^{(1)}(k_{\rm L} r)$, $(H_0^{(1)})^4$, $(H_1^{(1)})^4$, $(H_0^{(1)})^3 H_1^{(1)}$, $H_0^{(1)}(H_1^{(1)})^3$, $(H_0^{(1)})^2(H_1^{(1)})^2$ are not solutions of the homogeneous wave equation (2), the procedure of solving the inhomogeneous equation (4) differs from the procedure of solving the related problem for a longitudinal plane wave.

A partial solution of the inhomogeneous equation for a cylindrical wave can be selected in a form corresponding to the right-hand side (9):

$$u=\left\{B_{04}(H_0^{(1)})^4+B_{14}(H_1^{(1)})^4+B_{22}(H_0^{(1)})^2(H_1^{(1)})^2\right.$$

$$\left.+B_{13}H_0^{(1)}(H_1^{(1)})^3+B_{31}(H_0^{(1)})^3H_1^{(1)}\right\}e^{4i\omega t}. \; (10)$$

Then the propagation of a cylindrical wave is described by the following second-order solution:

$$u_r(r,t)=u_r^{(0)}(r,t)+u_r^{(1)}(r,t)+u_r^{(2)}(r,t)=u_{ro}H_1^{(1)}(k_{\rm L} r)e^{i\omega t}$$

$$+\left\{B_{00}[H_0^{(1)}(k_{\rm L} r)]^2+B_{11}[H_1^{(1)}(k_{\rm L} r)]^2+B_{01}H_0^{(1)}(k_{\rm L} r)H_1^{(1)}(k_{\rm L} r)\right\}e^{2i\omega t}$$

$$+\left\{B_{04}(H_0^{(1)})^4+B_{14}(H_1^{(1)})^4+B_{22}(H_0^{(1)})^2(H_1^{(1)})^2+B_{13}H_0^{(1)}(H_1^{(1)})^3+B_{31}(H_0^{(1)})^3H_1^{(1)}\right\}e^{4i\omega t}. \qquad (11)$$

The unknown coefficients $B_{04}, B_{14}, B_{22}, B_{13}, B_{31}$ are determined by substituting (11) into the left-hand side of Eq. (8) and comparing with the right-hand side (9).

This comparison leads to the equality

$$\left(\frac{3}{(k_{\rm L} r)^2}+12\right)B_{04}+2B_{22}k_{\rm L}^2=-k_{\rm L}^2\frac{1}{\rho\omega^2 r}\left(\tilde{N}_1-\tilde{N}_4\right),$$

$$\left(\frac{3}{(k_{\rm L} r)^2}+\frac{12}{r^2}+12\right)B_{14}+2B_{22}k_{\rm L}^2+6B_{13}k_{\rm L}\frac{1}{r}=-k_{\rm L}^2\frac{1}{\rho\omega^2 r}\left(3\tilde{N}_1+\tilde{N}_4\right),$$

$$\left(\frac{3}{(k_{\rm L} r)^2}+\frac{2}{r^2}-8k_{\rm L}^2+12\right)B_{22}+12k_{\rm L}^2B_{04}+12k_{\rm L}^2B_{14}+6B_{31}k_{\rm L}\frac{1}{r}-12B_{13}k_{\rm L}\frac{1}{r}$$

$$=k_{\rm L}^3\frac{1}{\rho\omega^2}\left(-\frac{1}{k_{\rm L} r}\left(8-\frac{2}{k_{\rm L}^2 r^2}\right)\tilde{N}_1-\frac{1}{k_{\rm L} r}\left(\frac{2}{k_{\rm L}^2 r^2}-4\right)\tilde{N}_2\right.$$

TABLE 1

Material	$\rho \cdot 10^{-4}$	$\lambda \cdot 10^{-10}$	$\mu \cdot 10^{-10}$	$A \cdot 10^{-11}$	$B \cdot 10^{-11}$	$C \cdot 10^{-11}$
Tungsten	1.89	7.5	7.3	–1.08	–1.43	–9.08
Molybdenum	1.02	15.7	1.1	–0.26	–2.83	3.72
Copper	0.893	10.7	4.8	–2.8	–1.72	–2.4
Steel	0.78	9.4	7.9	–3.25	–3.1	–8.0
Aluminum	0.27	5.2	2.7	–0.65	–2.05	–3.7
Polystyrene	0.105	0.369	0.114	–0.108	–0.0785	–0.0981

$$+\tilde{N}_3 \frac{1}{k_L^3 r^3} + \frac{1}{k_L r}\left(1+\frac{1}{k_L^2 r^2}\right)\tilde{N}_4 - \tilde{N}_5 \frac{1}{k_L^3 r^3}\Bigg),$$

$$\left(\frac{3}{(k_L r)^2} + \frac{6}{r^2} - 6k_L^2 + 12\right)B_{13} + 6k_L^2 B_{31} - 24B_{14}k_L \frac{1}{r} + 8B_{22}k_L \frac{1}{r}$$

$$= k_L^3 \frac{1}{\rho\omega^2}\left(-\left(4-\frac{5}{k_L^2 r^2}\right)\tilde{N}_1 - 3\frac{1}{k_L^2 r^2}\tilde{N}_2 + \frac{1}{k_L^2 r^2}\tilde{N}_3 - \frac{1}{k_L^2 r^2}\tilde{N}_4\right),$$

$$\left(\frac{3}{(k_L r)^2} - 6k_L^2 + 12\right)B_{31} + 6B_{13}k_L^2 - 4B_{22}k_L \frac{1}{r}$$

$$= k_L^3 \frac{1}{\rho\omega^2}\left(-\left(\frac{3}{k_L^2 r^2} - 4\right)\tilde{N}_1 + \frac{1}{k_L^2 r^2}\tilde{N}_2 - \frac{1}{k_L^2 r^2}\tilde{N}_3 + \frac{1}{k_L^2 r^2}\tilde{N}_4\right). \quad (12)$$

Substituting the equality $\rho\omega^2 = (\lambda + 2\mu)(k_L)^2$ into the expression $(u_o)^2(\rho\omega^2)^{-1}(k_L)^3$, comparing and simplifying as indicated above, we obtain the following expressions for the coefficients:

$$B_{04}, B_{14}, B_{22} = 0, \quad B_{13} = \frac{1}{3}k\tilde{N}_1, \quad B_{31} = -\frac{1}{3}k\tilde{N}_1.$$

Then the propagation of a cylindrical wave is described by the second-order formula

$$u_r(r,t) = u_r^{(0)}(r,t) + u_r^{(1)}(r,t) + u_r^{(2)}(r,t) = u_o H_1^{(1)}(k_L r)e^{i\omega t}$$

$$-\frac{1}{4}u_o^2 k_L \frac{N_1}{\lambda + 2\mu} H_0^{(1)}(k_L r) H_1^{(1)}(k_L r) e^{2i\omega t}$$

$$+\frac{1}{48}u_o^4 k^3 \left(\frac{N_1}{\lambda + 2\mu}\right)^3 H_0^{(1)}(k_L r) H_1^{(1)}(k_L r)\left(\left(H_1^{(1)}(k_L r)\right)^2 - \left(H_0^{(1)}(k_L r)\right)^2\right)e^{4i\omega t}. \quad (13)$$

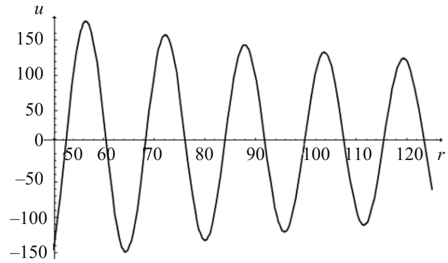

Fig. 1

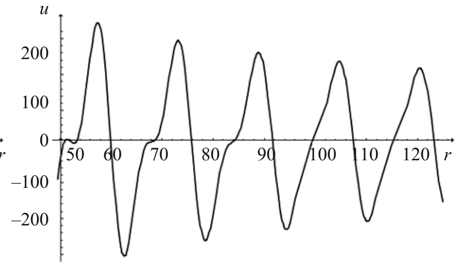

Fig. 2

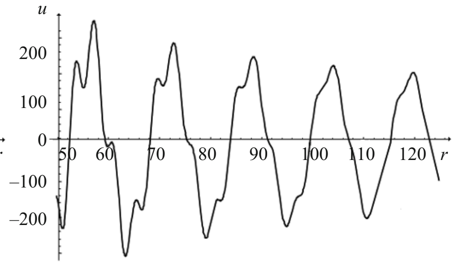

Fig. 3

Thus, solution (13) is expressed in terms of Hankel functions of the zero and first order and contains the first, second, and fourth harmonics.

Let us now discuss results from the primary numerical analysis of the evolution of a cylindrical wave profile in different materials with characteristics (density, two Lamé constants, three Murnaghan constants) summarized in Table 1.

Input data (in SI units): initial amplitude $u_o = 1 \cdot 10^{-3}$ m, wave frequency $\omega = 1.5$ MHz, wave number $k_L = 4000\,\text{m}^{-1}$, $r_o = 5 \cdot 10^{-3}$ m. Figures 1–6 demonstrate the distortion of the profile of a cylindrical wave in aluminum (fifth material in Table 1).

In all the figures, the abscissa axis indicates r and the ordinate axis indicates u_r.

Figure 1 shows the initial profile $u_r(r,t) = u_r^{(0)}(r,t) = u_o H_1^{(1)}(k_L r) e^{i\omega t}$. Figures 2 and 3 show the wave profile affected by the second and third approximations, respectively, i.e.,

$$u_r(r,t) = u_r^{(0)}(r,t) + u_r^{(1)}(r,t) = u_o H_1^{(1)}(k_L r) e^{i\omega t} - \frac{1}{4} u_o^2 k_L \frac{N_1}{\lambda + 2\mu} H_0^{(1)}(k_L r) H_1^{(1)}(k_L r) e^{2i\omega t},$$

and

$$u_r(r,t) = u_r^{(0)}(r,t) + u_r^{(2)}(r,t) = u_o H_1^{(1)}(k_L r) e^{i\omega t}$$

$$+ \frac{1}{48} u_o^4 k^3 \left(\frac{N_1}{\lambda + 2\mu} \right)^3 H_0^{(1)}(k_L r) H_1^{(1)}(k_L r) \left(\left(H_1^{(1)}(k_L r) \right)^2 - \left(H_0^{(1)}(k_L r) \right)^2 \right) e^{4i\omega t}.$$

Figure 4 demonstrates the joint effect of the second and third approximations,

$$u_r(r,t) = u_r^{(0)}(r,t) + u_r^{(1)}(r,t) + u_r^{(2)}(r,t)$$

$$= u_o H_1^{(1)}(k_L r) e^{i\omega t} - \frac{1}{4} u_o^2 k_L \frac{N_1}{\lambda + 2\mu} H_0^{(1)}(k_L r) H_1^{(1)}(k_L r) e^{2i\omega t}$$

$$+ \frac{1}{48} u_o^4 k^3 \left(\frac{N_1}{\lambda + 2\mu} \right)^3 H_0^{(1)}(k_L r) H_1^{(1)}(k_L r) \left(\left(H_1^{(1)}(k_L r) \right)^2 - \left(H_0^{(1)}(k_L r) \right)^2 \right) e^{4i\omega t}.$$

Figure 5 shows both initial and distorted wave profiles.

It can be seen from Figs. 1 and 4 that the frequency almost doubles very quickly and the amplitude decreases with time, as demonstrated by Fig. 6, which shows the initial and distorted profiles at a distance from $2.5r_o$ to $5r_o$.

Unlike a plane longitudinal wave, which is initially a linear harmonic one, and then, with distance/time of travel, the first, second, and fourth harmonics are superimposed to produce a weakly modulated wave, the basic wave effects for a cylindrical wave are that initially, the effect of the third harmonic stronger and the effect of the second harmonic is weaker, and then, as time passes, their effect weakens, and the wave profile tends to the first approximation, which can be observed between $2.5r_o$ to $5r_o$.

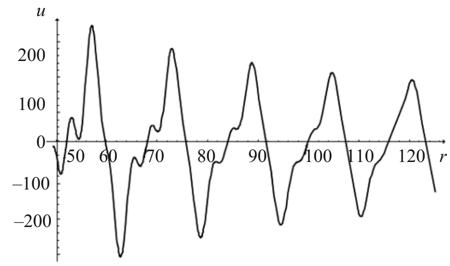

Fig. 4

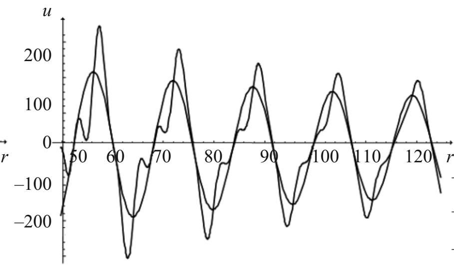

Fig. 5

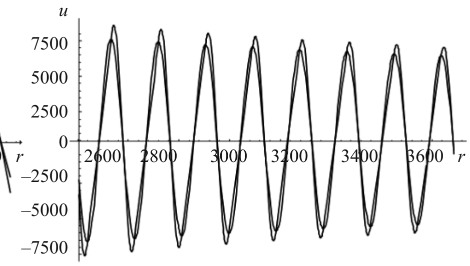

Fig. 6

Conclusions. Thus, we have used the perturbation method (small-parameter method) to analyze the propagation of a harmonic longitudinal plane wave in a nonlinear quadratic hyperelastic material described by the classical Murnaghan model. We have derived the exact analytic expression of the second-order solution in terms of Hankel functions of the zero and first orders and their products. A simplification of the expression has been proposed.

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