DOI 10.1007/s10778-015-0687-5
International Applied Mechanics, Vol. 51, No. 2, March, 2015

BIFURCATIONS AND MULTISTABILITY OF THE OSCILLATIONS OF A THREE-DIMENSIONAL SYSTEM

A. A. Martynyuk and N. V. Nikitina

A generator with inertial nonlinearity is considered. The bifurcations are illustrated by simple examples using the comparison method and Lyapunov functions

Keywords: bifurcation, nonlinear monotonic systems, chaos, multistable system

Introduction. Bifurcation theory and qualitative analysis originated in the works of Puankare, Lyapunov, and Andronov. Elements of bifurcation theory can be found in a number of modern studies [1, 4–7, 9–13]. In [4], it was proposed to use variational equations with coefficients dependent on a partial solution in qualitative analysis. In the present paper, we will also try to classify physical objects generating multidimensional attractors. The problem to be considered involves multistability. In radio physics, multistability indicates that several attractors caused by certain initial conditions coexist in the phase space [2].

1. Problem Formulation. The following generalized equation of a generator consisting of a selective element (for example, an oscillatory circuit) and an amplifier is given in [1]:

$$\ddot{x} + F_1(x, z, \mu) + F_2(x, z, \mu) = 0, \qquad \dot{z} = F_3(x, z, \mu),$$

where x is the oscillating variable; μ is a set of parameters; F_i are, generally, nonlinear functions. The variable $z(t)$ is related to the variable $x(t)$ by a differential operator of the first order, i.e., the response $z(t)$ to the variable $x(t)$ is inertial.

Consider a generator with quadratic inertial nonlinearity described by a dimensionless system presented in [1]:

$$\frac{dx}{dt} = mx - xz + y, \qquad \frac{dy}{dt} = -x, \qquad \frac{dz}{dt} = -b(z - x^2), \tag{1}$$

where m and b are positive constant coefficients. The oscillatory circuit is described by the equation

$$\ddot{x} - (m - z)\dot{x} + (\dot{z} + 1)x = 0. \tag{2}$$

The inertial transformer is represented by the operator $\dot{z} = -b(z - x^2)$.

A generator with exponential inertial nonlinearity described by a dimensionless system is presented in [2]:

$$\frac{dx}{dt} = mx - xz + y, \qquad \frac{dy}{dt} = -x, \qquad \frac{dz}{dt} = -b(z - e^x + 1). \tag{3}$$

Here the oscillatory circuit (2) takes place. The inertial transformer is represented by the operator $\dot{z} = -b(z - e^x + 1)$.

The attractors associated with the mechanism of orbital instability in a system with two singular points have been studied well. In the well-know Lorenz problem, the passage of unstable trajectories from one saddle-focus to another gives rise to the obvious mechanism of orbital instability that generates a strange attractor. The Lorenz problem was qualitatively described in

S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterova St., Kyiv, Ukraine 03057, e-mail: center@inmech.kiev.ua. Translated from Prikladnaya Mekhanika, Vol. 51, No. 2, pp. 122–132, March–April 2015. Original article submitted November 21, 2012.

the monograph [11]. A similar mechanism including two singular points occurs in a conservative system of a bistable oscillator and a periodic force [4]. The Rikitake system also includes two center-nodes [10].

Let $x = \bar{x}(t)$ be a periodic solution of an n-dimensional autonomous system:

$$\frac{dx}{dt} = F(x), \tag{4}$$

where $x(t) \in R^n$ is the state vector of the system at an instant $t \in R$, $F: U \to R^n$ is a smooth function defined on some subset $U \subseteq R^n$. We introduce a small deviation $\delta x_i = x_i(t) - \bar{x}_i(t)\,(i = 1, 2, \dots, n)$ in the neighborhood of the solutions $\bar{x}_i\,(i = 1, 2, \dots, n)$. Let δx_i be new coordinates. In the coordinates δx_i (linear), the system takes the form $d\delta x / dt = A(\bar{x})\delta x$, $\delta x \in R^n$, where $A(\bar{x}) = \partial F / \partial x|_{x=\bar{x}}$ is a variational system [4]. By analyzing the roots of the characteristic equation of the matrix $A(\bar{x})$, it is possible to plot a curve (separatrix) that separates qualitatively different domains of the phase section and to estimate the parameters of an orbitally stable system.

In analytically defined systems of ordinary differential equations $dx / dt = A(t)x$, we introduce characteristic numbers of nontrivial solutions: $\Lambda_j = \overline{\lim}_{t\to\infty} [t^{-1} \ln||x_j(t)||]$ $(j = 1, 2, \dots, n)$, where $x_j(t)$ is the jth fundamental solution of the system, $||0||$ is the norm. The numbers Λ_j are called generalized characteristic numbers of an arbitrary system. For the vatiational system describing the evolution of the perturbations δx near a partial solution $\bar{x}(t)$ of a nonlinear system, the set Λ_j is called the Lyapunov characteristic exponents (LCEs) of the solution $\bar{x}(t)$ (or a phase path).

To describe the mechanism of occurrence of compound motions in a system with one singular point, we need to:

– plot separatrices that separate the field on the coordinate plane that is associated with the bifurcation process;

– analyze the trajectories of the generator for stability;

– prove the existence of the domain into which the trajectories of the generator are attracted [11];

– describe the phenomenon of multistability in terms of the qualitative results obtained.

The existence of the domain into which unstable trajectories are attracted [11] indicates that one sign in the signature of the LCE spectrum is a minus (–). Since we are considering the oscillatory process in a three-dimensional dissipative system, the signature of the LCE spectrum has two signs (0, –), i.e., the characteristic exponents are $\Lambda_2 = 0$ and $\Lambda_3 < 0$. An orbitally stable trajectory is characterized by the inequality $\Lambda_1 + \Lambda_2 + \Lambda_3 < 0$, and an orbitally unstable trajectory by the inequality $\Lambda_1 + \Lambda_2 + \Lambda_3 > 0$.

Proving the existence of the domain into which the trajectory is attracted is sufficient to state that an attractor exists, provided that there is one singular point. The Lyapunov instability of the trajectory in this case may be identical to orbital instability and is represented by a plus (+) in the signature of the LCE spectrum.

Let us outline another approach to proving the existence of an attractor in a three-dimensional system.

Assumption 1. System (4) is unstable in the neighborhood of zero. System (4) with $n = 3$ has one singular point. The saddle-focus has characteristic exponents $\mathrm{Re}\,\lambda_1 > 0, \mathrm{Re}\,\lambda_2 > 0, \lambda_3 < 0$ and saddle value $\sigma = \mathrm{Re}\,\lambda_1 + \mathrm{Re}\,\lambda_2 + \lambda_3 < 0$.

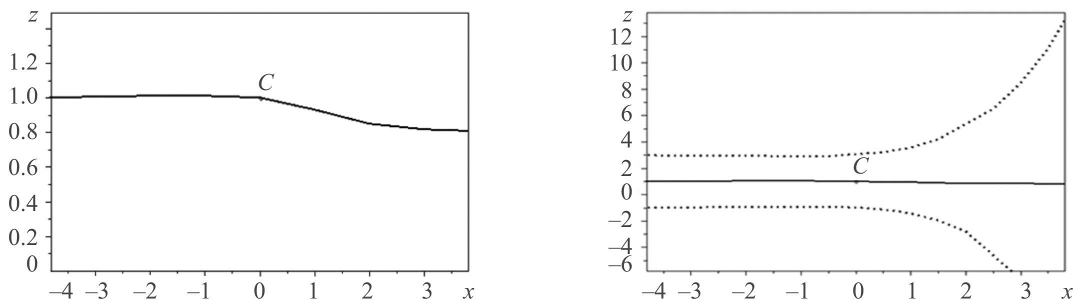
Assumption 2. On one coordinate plane and planes parallel to it, system (4) has a circular trajectory around a singular point, representing the damped oscillations of an oscillator. On the other two planes and planes parallel to them, the trajectory does not go to ∞.

Statement. Let the differential system (4) satisfy Assumptions 1 and 2. Then an attractor exists in the neighborhood of the singular point (saddle-focus).

Proof. According to Assumption 1, the circular trajectory of system (4) around the singular point, which is a saddle-focus, is unstable. According to Assumption 2, the existence of a damped circular trajectory on one of the coordinate planes is indicative of the dissipative nature of the motion. Assumption 2 suggests that the trajectory does not go to infinity, but rather remains in some neighborhood of the saddle-focus. Thus, in system (4), there is an attracting trajectory that does not go to ∞, but can be orbitally unstable.

2. Bifurcations on Phase Planes. Each of systems (1) and (3) has one singular point $O(0, 0, 0)$. Introducing small deviations $\delta x, \delta y, \delta z$ from the partial solutions $\bar{x}(t)$, $\bar{y}(t)$, $\bar{z}(t)$ in (1) and (3), we set up variational equations:

$$\frac{d\delta x}{dt} = (m - \bar{z})\delta x + \delta y - \bar{x}\delta z, \quad \frac{d\delta y}{dt} = -\delta x, \quad \frac{d\delta z}{dt} = -b(\delta z - 2\bar{x}\delta x), \tag{5}$$



a *b*

Fig. 1

$$\frac{d\delta x}{dt} = (m - \bar{z})\delta x + \delta y - \bar{x}\delta z, \qquad \frac{d\delta y}{dt} = -\delta x, \qquad \frac{d\delta z}{dt} = -b(\delta z - e^{\bar{x}}\delta x). \tag{6}$$

The characteristic equation of system (5) is

$$\lambda^3 + \lambda^2(b + \bar{z} - m) + \lambda(b(\bar{z} - m) + 2b\bar{x}^2 + 1) + b = 0. \tag{7}$$

The characteristic equation of system (6) is

$$\lambda^3 + \lambda^2(b + \bar{z} - m) + \lambda(b(\bar{z} - m) + be^{\bar{x}}\bar{x} + 1) + b = 0. \tag{8}$$

The roots of systems (7) and (8) corresponding to the singular point $O(0, 0, 0)$ can be found from the equation $\lambda^3 + \lambda^2(b - m) + \lambda(-bm + 1) + b = 0$ or $(\lambda + b)(\lambda^2 - m\lambda + 1) = 0$. They are $\lambda_{1,2} = m/2 \pm \sqrt{(m/2)^2 - 1}$ and $\lambda_3 = -b$. If

$$0 < m < 2, \quad b = 0.2, \tag{9}$$

then the roots λ_1 and λ_2 are complex. Let us show that there are points on the z-axis at which Eqs. (7) and (8) divide into two. Equations (7) and (8) have the form $(\lambda + b)(\lambda^2 + b\lambda + 1) = 0$ at the point $A(x = 0, \ y = 0, \ z = m + b)$. At this point A, the characteristic numbers $\lambda_{1,2} = -b/2 \pm \sqrt{(b/2)^2 - 1}$ and $\lambda_3 = -b$. At the point $C(x = 0, \ y = 0, \ z = m)$, Eqs. (7) and (8) have the form $(\lambda + b)(\lambda^2 + 1) = 0$ and the characteristic numbers $\lambda_{1,2} = \pm i$ and $\lambda_3 = -b$. The characteristic equations (7) and (8) do not have the partial solution $\bar{y}(t)$, which indicates that the bifurcation process is not related to the partial solution $\bar{y}(t)$.

Thus, the real part of the two complex-conjugate roots reverses sign at the point C. This facilitates the plotting of the separatrix on the plane xz. We use the characteristic equation (8) to plot curves that divide the plane xz into domains with different behavior of points.

Figure 1*a* shows the separatrix that divides the plane xz of an exponentially nonlinear generator. The points lying below the separatrix correspond to the positive real part of the complex roots. The roots on the separatrix are $\mathrm{Re}\,\lambda_{1,2} = 0$ and $\lambda_3 < 0$. The dotted curves in Fig. 1*b* are those separatrices on which the complex roots become real. On the upper branch, we have $\lambda_{1,2} < 0$ and $\lambda_3 < 0$ (hereafter, $(m, b) = (1, 0.2)$). The absolute values of the real roots increase on the curves from left to right; for example, the characteristic numbers have the following values at some points: $\lambda_{1,2} = -0.9761$ and $\lambda_3 = -0.2099$ at $x = -3$, $z = 2.9621$; $\lambda_{1,2} = -1$ and $\lambda_3 = -0.2$ at $x = 0$, $z = 3$; $\lambda_{1,2} = -3.7984$ and $\lambda_3 = -0.0139$ at $x = 3$, $z = 8.4107$. On the lower branch, we have $\lambda_{1,2} > 0$ and $\lambda_3 < 0$. The absolute values of the real roots increase on the curves from left to right; for example, the characteristic numbers have the following values at some points: $\lambda_{1,2} = 0.9895$ and $\lambda_3 = -0.2042$ at $x = -3$, $z = -0.9747$; $\lambda_{1,2} = 1$ and $\lambda_3 = -0.2$ at $x = 0$, $z = -1$; $\lambda_{1,2} = 3.4293$ and $\lambda_3 = -0.0170$ at $x = 3$, $z = -6.04115$. The field on the plane xz is asymmetric. This gives rise to an asymmetric domain into which trajectories can be attracted after the formation of an attractor.

3. Proving the Existence of the Domain into Which the Trajectories of a Quadratically Nonlinear Generator are Attracted. Introduce a variable $\zeta = z - Km$, where K is a positive number. Let us represent Eqs. (1) as

$$\frac{dx}{dt} = mx - x(\zeta + Km) + y, \quad \frac{dy}{dt} = -x, \quad \frac{d\zeta}{dt} = -b(\zeta + Km) + bx^2. \tag{10}$$

Introduce a function

$$2V = bx^2 + by^2 + \zeta^2 \tag{11}$$

and define its total derivative along the solutions of system (10) as

$$\frac{dV}{dt} = -bm\,(K-1)x^2 - b\,(\zeta^2 + \zeta Km). \tag{12}$$

The variable y does not appear in (12). Let us show that the trajectory of system (10) is attracted into the ellipse defined by the equation $dV\,/\,dt = 0$.

Using Eqs. (12), we get

$$-bm(K-1)x^2 - b\left(\zeta^2 + \zeta Km + \left(\frac{Km}{2}\right)^2\right) + b\left(\frac{Km}{2}\right)^2 = 0$$

(where a constant has been added and subtracted), and, next,

$$m(K-1)x^2 + \left(\zeta + \frac{Km}{2}\right)^2 = \left(\frac{Km}{2}\right)^2. \tag{13}$$

The ellipse is described by the equation

$$\frac{x^2}{A^2} + \left(\zeta + \frac{Km}{2}\right)^2 \Big/ B^2 = 1 \quad \left(A = \frac{Km}{2\sqrt{m(K-1)}}, B = \frac{Km}{2}\right).$$

Its center is at the point $x = 0, \zeta = -Km\,/\,2$. The domain inside ellipse (13) attracts the trajectory of system (10). This domain generates an attractor. The existence of a regular attractor can be proved using the symmetry principle.

4. Symmetry Theorem. The principle of symmetry in three-dimensional systems is to find a coordinate plane onto which a spatial integral curve is projected as a closed symmetric curve. On two other coordinate planes, the process must be stable and can be accompanied by symmetry.

Let us transfer the origin of coordinates to the point C and introduce a new coordinate system with Cx-, Cy-, and Cz-axes parallel to the Ox-, Oy-, and Oz-axes. The system of equations (1) becomes

$$\frac{dx}{dt} = -xz + y, \quad \frac{dy}{dt} = -x, \quad \frac{dz}{dt} = -b(z + m - x^2).$$

The system of three differential nonlinear equations is represented as

$$dx\,/\,dt = -xz + y, \quad dy\,/\,dt = -x, \quad dz\,/\,dt = -b(z + m + f(x)), \tag{14}$$

where m and b are positive parameters.

Assume that the function $f(x)$ is defined, continuous, and satisfies the uniqueness condition for solutions at each point x. Let a closed trajectory lie on a surface and possess some type of symmetry. It is possible to identify a plane onto which the trajectory is projected as a closed curve. The closed trajectory has certain projections onto the other two planes. The last two projections onto the coordinate planes can be symmetric and nonclosed. Another assumption is related to the instability of system (14) in the neighborhood of zero. Since two-dimensional projections allow predicting geometrical symmetry and stability, what has been said above can be formulated as a theorem.

Theorem. Consider system (14) under the following assumptions:

(i) system (14) has an unstable solution in the neighborhood of zero;

(ii) the motion of system (14) on the plane xz is described by the system

$$dx/dt = -xz, \quad dz/dt = -b\,(z + m + f(x)), \tag{15}$$

which contains two stable symmetric equilibrium positions. Then a closed integral curve exists in the three-dimensional system (15).

Proof. The motion of system (14) on the plane *xy* is described by the system

$$dx/dt = y, \quad dy/dt = -x, \tag{16}$$

which has a symmetric closed trajectory: the singular point is a center; the trajectory on the plane is symmetric about the *x*- and *y*-axes. The motion of system (14) on the plane *yz* is described by the system

$$dy/dt = 0, \quad dz/dt = -bz - m. \tag{17}$$

The singular point of system (17) has characteristic numbers $\lambda_1 = 0$ and $\lambda_2 = -b$. Let the initial conditions of system (14) generate a closed symmetric trajectory of system (16). Then system (15) is perturbed, and trajectories tending to the singular points $(\pm x_0, 0)$ form. The oscillations of system (15) become a stable process characterized by an integral curve symmetric in the space *xyz*. The projection of system (17) onto the plane *yz* does not counteract this process. The theorem is proved.

Consider the motion of system (14) on the plane *xz* described by the system of equations

$$dx/dt = -xz, \quad dz/dt = -b(z + m - x^2). \tag{18}$$

Let us find the singular points of system (18). Let $z = 0$. System (18) has two singular points with coordinates $x = \pm\sqrt{m}$ and $z = 0$. The singular points $C_1(\sqrt{m}, 0)$ and $C_2(-\sqrt{m}, 0)$ have characteristic numbers $\lambda_{1,2} = -b/2(1 \pm \sqrt{1 - 8m/b})$. The occurrence of a closed trajectory in system (14) is associated with the initial perturbation of the variable *x*, which perturbs the variable *y*. Since all the conditions of the theorem are satisfied, it may be stated that a closed curve exists in system (14).

Let us apply the comparison method to prove the instability of system (1). Let us introduce functions $V_1 = x^2/2, V_2 = y^2/2, V_3 = x^2/2 + z^2/2$. The derivatives of the functions $V_j\,(j = 1, 2, 3)$ along the solutions of system (1) can be estimated as

$$\frac{dV_1}{dt} = mx^2 - x^2 z + xy \le mx^2 + y^2/2 + x^2/2 + z^2/2 + x^4/2,$$

$$\frac{dV_2}{dt} = -xy \le x^2/2 + y^2/2,$$

$$\frac{dV_3}{dt} = bx^2 z + mx^2 - x^2 z + xy - bz^2 \le (m + b/2)x^2/2 + y^2/2 + (1-b)(x^2 + z^2)/2 + (1+b)x^4/2.$$

Here derivatives of two variables are replaced by sums:

$$\pm bx^2 z \le bx^4/2 + bz^2/2, \quad \pm xy \le x^2/2 + y^2/2, \quad \pm x^2 z \le x^4/2 + z^2/2.$$

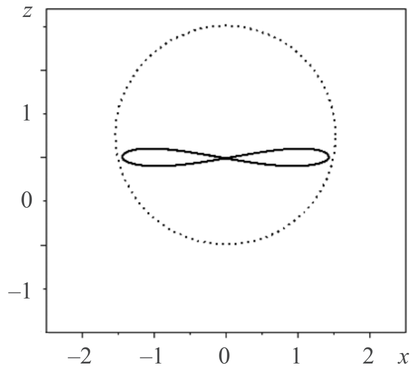
Consider the system of equations

$$\frac{d\vartheta_1}{dt} = 2m\vartheta_1 + \vartheta_2 + \vartheta_3 + 2\vartheta_1^2,$$

$$\frac{d\vartheta_2}{dt} = \vartheta_1 + \vartheta_2,$$

$$\frac{d\vartheta_3}{dt} = (2m + b)\vartheta_1 + \vartheta_2 + (1-b)\vartheta_3 + 2(1+b)\vartheta_1^2. \tag{19}$$

If $b < 1$, then the comparison system (19) shows that the trajectory goes away from zero. Let $(m, b) = (1, 0.2)$.



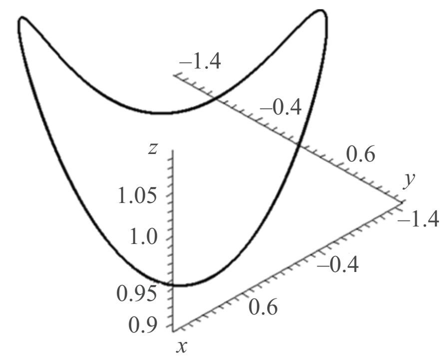


Fig. 2

The symmetry theorem is of geometrical nature. Consider the mechanism of formation of a periodic three-dimensional curve with symmetry. In analyzing motions on the coordinate planes, the basic goal is to find a plane on which a closed curve symmetric about two axes forms (plane *xy*). The other two coordinate planes stabilize the qualitative pattern and symmetry when the singular points of systems (15) and (17) are stable.

Figure 2*a* shows a section of the limit cycle of the generator and the boundary of the domain into which the cycle (K = 2.5, $A = 1.53$, $B = 1.25$) is attracted when $(m, b) = (1, 0.2)$. The initial perturbation $x_0 = -2.5$. Figure 2*b* illustrates the limit cycle.

5. Proving the Instability of the Trajectory of an Exponentially Nonlinear Generator. Let us analyze the trajectory of system (3) for stability using the comparison method. Recall some principles and a theorem used for this purpose. The comparison method involves setting up comparison equations (Wazewski-type equations) that are quasimonotonic. Major references and results on the stability of monotonic systems are presented in [3, 8].

Consider the system

$$\frac{d\vartheta_j}{dt} = q_j(\vartheta_1, \ldots, \vartheta_k), \quad j = 1, \ldots, k, \tag{20}$$

under the following assumptions:

1. System (20) is a Wazewski system, i.e., the components of the vector function $q(\vartheta)$ are quasimonotonically increasing functions. For the function $q(\vartheta)$ to quasimonotonically increase, it is necessary and sufficient that $\partial q_j / \partial \vartheta_i \geq 0$ for $j \neq i$.

2. The right-hand side of system (20) is continuous and the solution of the Cauchy problem for any $\vartheta_0 \in R^k$ is locally unique.

Theorem on Instability of Wazewski System in a Cone K (see [3, 8]). If the Wazewski system satisfies Assumptions 1 and 2 and there exists a sequence of points $\vartheta_m \in K, \vartheta_m \to 0$ as $m \to \infty$, such that the following inequalities hold for each m:

$$q_j(\vartheta_m) \geq 0, j = 1, \ldots, k, \tag{21}$$

and at least for one j the inequality is strict and there exists a neighborhood V of zero such that the vector field is not equal to zero on the set $K_{\vartheta_m} \cap V$, then the zero solution of system (20) is unstable in the cone.

Let us introduce functions $x^2/2, y^2/2, z^2/2$. The derivatives of these functions along the solutions of system (3) can be estimated as

$$x\frac{dx}{dt} = mx^2 - x^2 z + xy < (m + 1/2)x^2 + y^2/2 + z^2/2 + x^4/2,$$

$$y\frac{dy}{dt} = -xy \leq x^2/2 + y^2/2,$$

$$z\frac{dz}{dt} = -bz^2 + be^x z - bz < (-b+1)z^2/2 + b/2(e^x)^2 + b^2/2, \tag{22}$$

where the products of two variables $x^2 z$, xy, $e^x z$ are replaced by a sum of squares, such as $\pm x^2 z \le x^4/2 + z^2/2$. Also $\pm e^x z < (e^x)^2/2 + z^2/2$ and $\pm bz < b^2/2 + z^2/2$.

Consider the system

$$\frac{d\vartheta_1}{dt} = 2(m+1/2)\vartheta_1 + \vartheta_2 + \vartheta_3 + 2\vartheta_1^2,$$

$$\frac{d\vartheta_2}{dt} = \vartheta_1 + \vartheta_2,$$

$$\frac{d\vartheta_3}{dt} = (-b+1)\vartheta_3 + b/2(e^{\sqrt{2\vartheta_1}})^2 + b^2/2. \tag{23}$$

If the functions on the right-hand side of (23) satisfy the conditions of the instability theorem, then system (3) is unstable.

6. Proving the Existence of the Domain into Which the Trajectories of an Exponentially Nonlinear Generator are Attracted. Introduce a variable $\zeta = z - (Km + b)$, where K is a positive number. If the nonlinearity is represented by a power series $e^x = 1 + x + x^2/2$, then Eqs. (3) become

$$\frac{dx}{dt} = mx - x(\zeta + Km + b) + y,$$

$$\frac{dy}{dt} = -x,$$

$$\frac{d\zeta}{dt} = -b(\zeta + Km + b) + bx + bx^2/2. \tag{24}$$

We introduce the function

$$V = \frac{bx^2}{2} + \frac{by^2}{2} + \zeta^2.$$

The total derivative of the function V along the solutions of system (24) is defined by

$$\frac{dV}{dt} = -b(m(K-1)+b)x^2 - 2b\zeta^2 - 2b\zeta(Km+b) + 2bx\zeta. \tag{25}$$

The variable y does not appear in (25).

The derivatives of two variables $2x\zeta$ in (25) are replaced by a sum of squares because $\pm 2x\zeta \le x^2 + \zeta^2$. Then the derivative of the function can be estimated as

$$\frac{dV}{dt} \le -b(m(K-1)-1+b)x^2 - b\zeta^2 - 2b\zeta(Km+b). \tag{26}$$

The right-hand side of inequality (26) is continuous for $x \ge 0$ and $\zeta \ge 0$ such that the comparison equation

$$\frac{d\vartheta}{dt} = -b(m(K-1)-1+b)x^2 - b\zeta^2 - 2b\zeta(Km+b) \tag{27}$$

has a unique solution satisfying the initial condition $\vartheta = \vartheta(x_0, \zeta_0)$ at each point in the domain of definition.

It is easy to show that the trajectories lie inside the ellipse defined by the equation $d\vartheta/dt = 0$. Using Eq. (27), we get

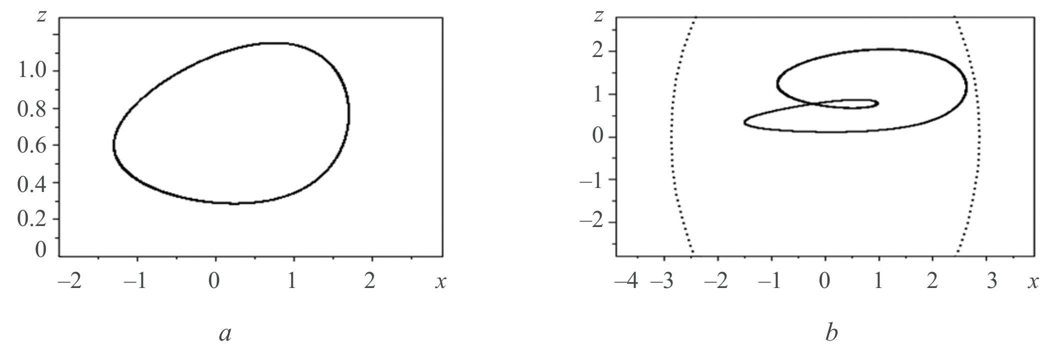


Fig. 3

TABLE 1

x_0	Stable motion
0.1	Strange attractor
–0.1	Limit cycle
0.5	Limit cycle
–0.5	Strange attractor
±1	Limit cycle
±1.5	Limit cycle
2.5	Strange attractor
–2.5	Limit cycle

$$(m(K-1)-1+b)x^2+\zeta^2+2\zeta(Km+b)=0,$$

$$(m(K-1)-1+b)x^2+(\zeta+(Km+b))^2=(Km+b)^2.$$

The ellipse is described by the equation

$$\frac{x^2}{A^2}+\frac{(\zeta+(Km+b))^2}{B^2}=1$$

$$\left(A=\frac{Km+b}{\sqrt{(m(K-1)-1+b)}},\, B=Km+b\right). \qquad (28)$$

Its center is at the point $x=0$, $\zeta=-(Km+b)$. The domain inside ellipse (28) attracts the trajectories of system (3). For this domain, $d\vartheta/dt<0$. This domain is approximate because the nonlinear term is represented by a power series. System (3) describes an oscillatory process in the neighborhood of zero, and the function $\exp^x$ is bounded $(-\infty<x<+\infty)$. Therefore, the Lyapunov unstable trajectory does not go to ∞ if there is an initial perturbation in x. Thus, it has been proved that the domain into which the trajectory of system (3) is attracted exists. This domain gives rise to an attractor.

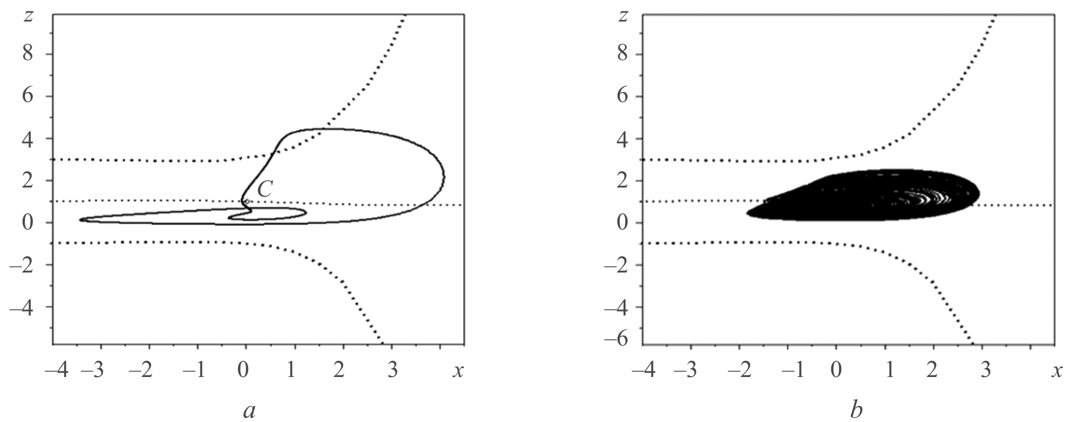


Fig. 4

The symmetry theorem does not state whether a regular attractor exists in system (3) or not. A regular attractor may form if the bifurcation process makes the separatrices symmetric (Fig. 1*a*, *b*). It may be assumed that weak asymmetry in the neighborhood of zero still allows the trajectory to close. The parameter m in (2) defines the level of negative dissipation. Decreasing the value of m in (3) may cause the trajectory to close. As the parameter of negative dissipation is increased, the system becomes orbitally unstable. Let at $m \le m_k$ the process be stable and, as an experiment shows, a closed trajectory exist in system (3). Figure 3*a* shows the trajectory for $(m, b) = (0.7, 0.2)$. As the parameter m is increased, the representative point slows down and the trajectory displays period multiplication. The slowing down is associated with the bifurcations on the plane *xz* (Fig. 1). Figure 3*b* shows the boundary of the approximate domain for $(m, b) = (0.92, 0.2)$ and the projection of the attractor onto the plane *xz*. Thus, the way to the strange attractor lies through period multiplication.

On Multistability. The physical phenomenon of multistability is associated with the asymmetry of the field about the *z*-axis and is that the asymmetric function appearing in the inertial transformer generates an asymmetric field (Fig. 1*a*, *b*) that forms an asymmetric attracting domain. Limit cycles are born in one case (if negative dissipation is relatively weak) and a limit cycle and a strange attractor are born in the other case (if negative dissipation is strong). Table 1 demonstrates that the sign and value of the initial perturbation (x_0) affect the behavior of the attractor ($(m, b) = (1, 0.2)$).

Figures 4*a* and 4*b* show the limit cycle and the strange attractor represented in Table 1. The regular and strange attractors are shown against the separatrices that separate the field of the plane *xz* (Fig. 1).

Thus, when $m > m_k$, a limit cycle (Fig. 4*a*) and a strange attractor (Fig. 4*b*) generated by orbital instability can exist.

In summary, it should be pointed out that the qualitative analysis of bifurcations for three-dimensional systems can also describe the mechanism of occurrence of compound motions.

REFERENCES

1. V. S. Anishchenko, *Complex Oscillations in Simple Systems* [in Russian], Nauka, Moscow (1990).
2. S. A. Koblyanskiy, A. V. Shabunin, and V. V. Astakhov, “Forced synchronization of periodic oscillations in a system with phase multistability,” *Rus. J. Nonlin. Dyn.*, **6**, No. 2, 277–289 (2010).
3. A. A. Martynyuk and A. Yu. Obolenskii, “Stability of solutions of Wazewski’s autonomous systems,” *Diff. Uravn.*, **16**, No. 8, 1392–1407 (1980).
4. N. V. Nikitina, *Nonlinear Systems with Complex and Chaotic Behavior of Trajectories* [in Russian], Feniks, Kyiv (2012).
5. V. S. Anishchenko, S. V. Astakhov, and T. E. Vadivasova, “Diagnostics of the degree of noise influence on a nonlinear system using relative metric entropy,” *Regul. Chaot. Dynam.*, **15**, No. 2–3, 263–276 (2010).
6. V. A. Krys’ko, T. V. Yakovleva, V. V. Dobriyan, and I. V. Papkova, “Wavelet-analysis-based chaotic synchronization of vibrations of multilayer mechanical structures,” *Int. Appl. Mech.*, **50**, No. 6, 706–720 (2014).
7. G. A. Leonov, *Strange Attractors and Classical Stability Theory*, University Press, St. Peterburg (2008).

8. A. A. Martynyuk, “Asymptotic stability criterion for nonlinear monotonic systems and its applications (review),” *Int. Appl. Mech.*, **47**, No. 5, 475–534 (2011).
9. A. A. Martynyuk and N. V. Nikitina, “Bifurcation processes in periodically perturbed systems,” *Int. Appl. Mech.*, **49**, No. 1, 114–121 (2013).
10. A. A. Martynyuk and N. V. Nikitina, “Stability and bifurcation in a model of the magnetic field of the Earth,” *Int. Appl. Mech.*, **50**, No. 6, 721–729 (2014).
11. Yu. I. Neimark and P. S. Landa, *Stochastic and Chaotic Oscillations*, Kluwer, Dordrecht (1992).
12. L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, and L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, Part I, World Scientific (1998).
13. L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, and L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II, World Scientific (2001).