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EFFECT OF BOUNDARY CONDITIONS ON THE NATURAL FREQUENCIES AND VIBRATION MODES OF PIEZOELECTRIC PLATES WITH RADIALLY CUT ELECTRODES

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The general solution to the problem of the nonaxisymmetric electromechanical vibrations of a piezoceramic ring plate is found. The effect of boundary conditions (clamped edge–free edge, free edge–clamped edge, free edge–free edge) on the natural frequency spectra (for the first circumferential harmonics) of plates with radially cut electrode coating is numerically analyzed

Keywords: piezoceramic ring plate, radially cut electrode coating, nonaxisymmetric electromechanical vibrations, natural frequency spectra

Introduction. Thin piezoelectric planar transducers with thickness polarization are widely used in modern electromechanical converters of various functionality [2–4, 6–8, 10–12, etc.]. Disk- and ring-shaped vibrators with solid electrodes on the faces undergo axisymmetric vibrations [4, 9]. The vibrations will be nonaxisymmetric with respect to the circumferential coordinate if the electroelastic sectors of a ring plate with radially cut electrodes are excited in antiphase. The circumferential vibration modes and the associated frequencies are determined by the number of radial cuts in the electrodes [2, 3, 11, 12]. Here we will compare the frequency spectra of a plate with boundary conditions of three types.

1. Problem Formulation. Basic Equations. Consider a piezoceramic plate ($r_0 < r < r_1$) of thickness h. To describe the plate, we will use a cylindrical coordinate system r, θ, z with the plane $z = 0$ coinciding with the midsurface of the plate. This thickness-polarized thin plate with electroded faces $z = \pm h/2$ is in plane stress state ($u_r(r, \theta, t)$, $u_\theta(r, \theta, t)$, $\sigma_{zz} = \sigma_{z\theta} = \sigma_{zr} = 0$, $E_r = E_\theta = 0$, $E_z(r, \theta, t)$). As shown in [2, 5, 7, 10, 13], the displacements u_r and u_θ can be expressed in terms of potentials $\Phi(r, \theta, t)$ and $\Psi(r, \theta, t)$:

$$u_r = \frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \qquad u_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} - \frac{\partial \Psi}{\partial r}. \tag{1}$$

The functions $\Phi(r, \theta, t)$ and $\Psi(r, \theta, t)$ can be determined from wave equations:

$$\Delta\Phi - (1+\nu_E)\, d_{31} E_z = (1-\nu_E^2)\, s_{11}^E \rho \frac{\partial^2 \Phi}{\partial t^2},$$

$$\Delta\Psi = 2(1+\nu_E)\, s_{11}^E \rho \frac{\partial^2 \Psi}{\partial t^2}. \tag{2}$$

The mechanical stresses can be found from the formulas

$$\sigma_{rr} = \frac{1}{(1-\nu_E^2) s_{11}^E}\left[\frac{\partial u_r}{\partial r} + \nu_E\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) - (1+\nu_E) d_{31} E_z\right],$$

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$$\sigma_{\theta\theta} = \frac{1}{(1-\nu_E^2)s_{11}^E}\left[\left(\nu_E \frac{\partial u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) - (1+\nu_E)d_{31}E_z\right],$$

$$\sigma_{r\theta} = \frac{1}{2(1+\nu_E)s_{11}^E}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right). \tag{3}$$

The electric potential for a plate with solid electrodes on the faces $z = \pm h/2$ is given by $\varphi = h^{-1} z V_0(t)$, the edge effect being neglected. This potential corresponds, according to [2, 8], to an electric field with $E_r = E_\theta = 0$, $E_z = h^{-1} V_0(t)$; hence, as shown in [7–9, 12, 13], the term $(1+\nu_E) d_{31} E_z$ in Eq. (3) should be omitted.

The homogeneous boundary conditions for displacements and stresses (at $r = r_0$ and $r = r_1$) in a circular piezoceramic plate of radius r_1 with a hole of radius r_0 are taken one from each of the following two pairs ($j = 0, 1$):

$$u_r(r_j, \theta, t) = 0 \;\Lambda\; \sigma_{rr}(r_j, \theta, t) = 0,$$

$$u_\theta(r_j, \theta, t) = 0 \;\Lambda\; \sigma_{r\theta}(r_j, \theta, t) = 0. \tag{4}$$

The initial conditions for steady-state harmonic vibrations are not formulated.

2. Problem-Solving Method. Let the electrode coating on the faces $z = \pm h/2$ be cut into $2N$ sectors, and adjacent sectors be connected in antiphase so that $E_{za} = (-1)^{n-1} V_0 / h$, $n = 1, \ldots, 2N$. If vibrations are harmonic, $f(r, \theta, t) = \mathrm{Re}\, f^{a}(r, \theta) \exp i\omega t$, where ω is the angular frequency, then a candidate solution to Eqs. (2) (the term $(1+\nu) d_{31} E_z$ in the first equation should be equated to zero [3, 10]) in polar coordinates r, θ can be chosen in the form of series:

$$\Phi(r,\theta,t) = R^2 \,\mathrm{Re} \sum_m \{A_{m,1} J_m(k_1 r) + A_{m,2} Y_m(k_1 r)\} \sin m\theta \exp i\omega t,$$

$$\Psi(r,\theta,t) = R^2 \,\mathrm{Re} \sum_m \{A_{m,3} J_m(k_2 r) + A_{m,4} Y_m(k_2 r)\} \cos m\theta \exp i\omega t, \tag{5}$$

where $J_m(k_j r)$ and $Y_m(k_j r)$ are mth-order cylindrical functions of the first and second kinds [1]; $k_1^2 = (1-\nu_E^2) s_{11}^E \rho\omega^2$, $k_2^2 = 2(1+\nu_E) s_{11}^E \rho\omega^2$; $A_{m,i}$ are dimensionless constants.

From (1) and (3), we can find, according to [9, 12], the displacements

$$u_r = R\,\mathrm{Re} \sum_m [u_{m1}(k_1 r) A_{m,1} + u_{m2}(k_1 r) A_{m,2} - u_{m3}(k_2 r) A_{m,3} - u_{m4}(k_2 r) A_{m,4}] \sin m\theta \exp i\omega t,$$
$$u_\theta = R\,\mathrm{Re} \sum_m [l_{m3}(k_1 r) A_{m,1} + l_{m4}(k_1 r) A_{m,2} + l_{m1}(k_2 r) A_{m,3} + l_{m2}(k_2 r) A_{m,4}] \cos m\theta \exp i\omega t \tag{6}$$

and the stresses

$$\sigma_{rr}(r,\theta,t) = -\mathrm{Re} \frac{1}{(1-\nu_E^2)s_{11}^E}\left\{\sum_m \Big(a_{m1}(k_1 r) A_{m,1} + a_{m2}(k_1 r) A_{m,2}\right.$$

$$\left. + a_{m3}(k_2 r) A_{m,3} + a_{m4}(k_2 r) A_{m,4}\Big) \sin m\theta + \frac{4}{\pi} V_0 (1+\nu_E) d_{13} \sum_{n=1}^{\infty} \frac{\sin N(2n-1)\theta}{2n-1}\right\} \exp i\omega t,$$

$$\sigma_{\theta\theta}(r,\theta,t) = -\mathrm{Re} \frac{1}{(1-\nu_E^2)s_{11}^E}\left\{\sum_m \Big(b_{m1}(k_1 r) A_{m,1} + b_{m2}(k_1 r) A_{m,2}\right.$$

$$\left. + b_{m3}(k_2 r) A_{m,3} + b_{m4}(k_2 r) A_{m,4}\Big) \sin m\theta + \frac{4}{\pi} V_0 (1+\nu_E) d_{13} \sum_{n=1}^{\infty} \frac{\sin N(2n-1)\theta}{2n-1}\right\} \exp i\omega t,$$

$$\sigma_{r\theta}(r,\theta,t) = \mathrm{Re}\frac{1}{(1+\nu_E)s_{11}^E}\sum_m \Big(c_{m1}(k_1 r)A_{m,1} + c_{m2}(k_1 r)A_{m,2} + c_{m3}(k_2 r)A_{m,3}$$

$$+ c_{m4}(k_2 r)A_{m,4}\Big)\cos m\theta \exp i\omega t, \tag{7}$$

where

$$a_{m1}(k_1 r) = \Big[(1-\nu_E)k_1 r J_{m-1}(k_1 r) + \Big(k_1^2 r^2 - (1-\nu_E)m(m+1)\Big)J_m(k_1 r)\Big]R^2/r^2,$$

$$a_{m2}(k_1 r) = \Big[(1-\nu_E)k_1 r Y_{m-1}(k_1 r) + \Big(k_1^2 r^2 - (1-\nu_E)m(m+1)\Big)Y_m(k_1 r)\Big]R^2/r^2,$$

$$a_{m3}(k_2 r) = (1-\nu_E)m\big[k_2 r J_{m-1}(k_2 r) - (m+1)J_m(k_2 r)\big]R^2/r^2,$$

$$a_{m4}(k_2 r) = (1-\nu_E)m\big[k_2 r Y_m(k_2 r) - (m+1)Y_m(k_2 r)\big]R^2/r^2,$$

$$b_{m1}(k_1 r) = \Big[-(1-\nu_E)k_1 r J_{m-1}(k_1 r) + \Big(\nu_E k_1^2 r^2 + (1-\nu_E)m(m+1)\Big)J_m(k_1 r)\Big]R^2/r^2,$$

$$b_{m2}(k_1 r) = \Big[-(1-\nu_E)k_1 r Y_{m-1}(k_1 r) + \Big(\nu_E k_1^2 r^2 + (1-\nu_E)m(m+1)\Big)Y_m(k_1 r)\Big]R^2/r^2,$$

$$b_{m3}(k_2 r) = -(1-\nu_E)m\big[k_2 r J_{m-1}(k_2 r) - (m+1)J_m(k_2 r)\big]R^2/r^2,$$

$$b_{m4}(k_2 r) = -(1-\nu_E)m\big[k_2 r Y_{m-1}(k_2 r) - (m+1)Y_m(k_2 r)\big]R^2/r^2,$$

$$c_{m1}(k_1 r) = m\big[k_1 r J_{m-1}(k_1 r) - (m+1)J_m(k_1 r)\big]R^2/r^2,$$

$$c_{m2}(k_1 r) = m\big[k_1 r Y_{m-1}(k_1 r) - (m+1)Y_m(k_1 r)\big]R^2/r^2,$$

$$c_{m3}(k_2 r) = \left[k_2 r J_{m-1}(k_2 r) + \left(\frac{1}{2}k_2^2 r^2 - m(m+1)\right)J_m(k_2 r)\right]R^2/r^2,$$

$$c_{m4}(k_2 r) = \left[k_2 r Y_{m-1}(k_2 r) + \left(\frac{1}{2}k_2^2 r^2 - m(m+1)\right)Y_m(k_2 r)\right]R^2/r^2,$$

$$u_{m1}(k_1 r) = -m\frac{R}{r}J_m(k_1 r) + k_1 R J_{m-1}(k_1 r), \quad u_{m2}(k_1 r) = -m\frac{R}{r}Y_m(k_1 r) + k_1 R Y_{m-1}(k_1 r),$$

$$u_{m3}(k_2 r) = m\frac{R}{r}J_m(k_2 r), \quad u_{m4}(k_2 r) = m\frac{R}{r}Y_m(k_2 r),$$

$$l_{m1}(k_1 r) = u_{m3}(k_1 r), \; l_{m2}(k_1 r) = u_{m2}(k_1 r), \; l_{m3}(k_2 r) = -u_{m3}(k_2 r), \; l_{m4}(k_2 r) = -u_{m2}(k_2 r). \tag{8}$$

Since $E_z^{\mathrm{a}} = (-1)^{n-1}V_0 h^{-1}$ $(n = 1, 2, \ldots, 2N)$ and the electric-field strength $E_z = \mathrm{Re}\, E_z^{\mathrm{a}} \exp i\omega t$ can be expanded into a Fourier series in the angular coordinate θ:

$$E_z^{\mathrm{a}} = -\frac{2V_0}{\pi h}\sum_{n=1}^{\infty}\frac{\sin N(2n-1)\,\theta}{2n-1}, \tag{9}$$

we have $m = N(2n-1)$, $n = 1, 2, \ldots$, in formulas (6)–(8).

In the resonance case, the concept of complex moduli [2, 9] has to be used, i.e., the material constants should be considered complex ($\tilde{s}_{ij}^E = s_{ij}^E - is_{ij}^{E\,\mathrm{Im}}$, $\tilde{d}_{ij} = d_{ij} - id_{ij}^{\mathrm{Im}}$, $\tilde{\varepsilon}_{ij}^T = \varepsilon_{ij}^T - i\varepsilon_{ij}^{T\,\mathrm{Im}}$).

To determine the resonant frequencies, it is possible to neglect the loss tangents as small and to use the real values of the material constants.

When $N = 0$, the electroelastic vibrations are radial and azimuthal:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 u_r}{\partial t^2}, \tag{10}$$

$$\sigma_{rr} = \frac{1}{(1-\nu_E^2) s_{11}^E} \left(\frac{\partial u_r}{\partial r} + \nu_E \frac{u_r}{r} - (1+\nu_E)\ d_{31} E_z \right),$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = \rho \frac{\partial^2 u_\theta}{\partial t^2},$$

$$\sigma_{r\theta} = \frac{1}{2(1+\nu_E) s_{11}^E} \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right). \tag{11}$$

The natural frequencies of electroelastic radial vibrations (11) are analyzed in [12]. Azimuthal vibrations (11), which cannot be excited electrically, are addressed for a fuller analysis of the results.

For $N > 0$, the frequencies of radial (problem (10)) and azimuthal (problem (11)) vibrations will be called quasiradial and quasiazimuthal, respectively.

Consider a ring plate with the inner edge ($r = r_0$) clamped and the outer edge ($r = r_1$) free:

$$u_r(r_0, \theta, t) = 0, \quad u_\theta(r_0, \theta, t) = 0, \quad \sigma_{rr}(r_1, \theta, t) = 0, \quad \sigma_{r\theta}(r_1, \theta, t) = 0 \tag{12}$$

with the inner edge ($r = r_0$) free and the outer edge ($r = r_1$) clamped:

$$\sigma_{rr}(r_0, \theta, t) = 0, \quad \sigma_{r\theta}(r_0, \theta, t) = 0, \quad u_r(r_1, \theta, t) = 0, \quad u_\theta(r_1, \theta, t) = 0, \tag{13}$$

and with the inner ($r = r_0$) and outer ($r = r_1$) edges free:

$$\sigma_{rr}(r_0, \theta, t) = 0, \quad \sigma_{r\theta}(r_0, \theta, t) = 0, \quad \sigma_{rr}(r_1, \theta, t) = 0, \quad \sigma_{r\theta}(r_1, \theta, t) = 0. \tag{14}$$

Using expressions (6), (7) and boundary conditions (12), we obtain block systems of algebraic equations for the dimensionless constants $A_{N(2n-1),i}$ ($n = 1, 2, \ldots,\ i = 1, \ldots, 4$):

$$u_{N(2n-1),1}(k_1 r_0) A_{N(2n-1),1} + u_{N(2n-1),2}(k_1 r_0) A_{N(2n-1),2}$$
$$+ u_{N(2n-1),3}(k_2 r_0) A_{N(2n-1),3} + u_{N(2n-1),4}(k_2 r_0) A_{N(2n-1),4} = 0,$$
$$l_{N(2n-1),1}(k_1 r_0) A_{N(2n-1),1} + l_{N(2n-1),2}(k_1 r_0) A_{N(2n-1),2}$$
$$+ l_{N(2n-1),3}(k_2 r_0) A_{N(2n-1),3} + l_{N(2n-1),4}(k_2 r_0) A_{N(2n-1),4} = 0,$$
$$a_{N(2n-1),1}(k_1 r_1) A_{N(2n-1),1} + a_{N(2n-1),2}(k_1 r_1) A_{N(2n-1),2}$$
$$+ a_{N(2n-1),3}(k_2 r_1) A_{N(2n-1),3} + a_{N(2n-1),4}(k_2 r_1) A_{N(2n-1),4} = -\frac{4}{\pi} V_0 \frac{(1+\nu_E) d_{13}}{2n-1},$$
$$c_{N(2n-1),1}(k_1 r_1) A_{N(2n-1),1} + c_{N(2n-1),2}(k_1 r_1) A_{N(2n-1),2}$$
$$+ c_{N(2n-1),3}(k_2 r_1) A_{N(2n-1),3} + c_{N(2n-1),4}(k_2 r_1) A_{N(2n-1),4} = 0. \tag{15}$$

The resonant frequencies can be determined by equating the fourth-order determinants of the homogeneous (at $V_0 = 0$) systems of algebraic equations (15) to zero:

$$\begin{vmatrix} u_{m1}(k_1 r_0) & u_{m2}(k_1 r_0) & u_{m3}(k_2 r_0) & u_{m4}(k_2 r_0) \\ l_{m1}(k_1 r_0) & l_{m2}(k_1 r_0) & l_{m3}(k_2 r_0) & l_{m4}(k_2 r_0) \\ a_{m1}(k_1 r_1) & a_{m2}(k_1 r_1) & a_{m3}(k_2 r_1) & a_{m4}(k_2 r_1) \\ c_{m1}(k_1 r_1) & c_{m2}(k_1 r_1) & c_{m3}(k_2 r_1) & c_{m4}(k_2 r_1) \end{vmatrix} = 0. \tag{16}$$

Using the boundary conditions (13) and the expressions for the displacements and stresses, we obtain block systems of algebraic equations for the dimensionless constants:

$$a_{N(2n-1),1}(k_1 r_0) A_{N(2n-1),1} + a_{N(2n-1),2}(k_1 r_0) A_{N(2n-1),2}$$

$$+ a_{N(2n-1),3}(k_2 r_0) A_{N(2n-1),3} + a_{N(2n-1),4}(k_2 r_0) A_{N(2n-1),4} = -\frac{4}{\pi} V_0 \frac{(1+\nu_E) d_{13}}{2n-1},$$

$$c_{N(2n-1),1}(k_1 r_0) A_{N(2n-1),1} + c_{N(2n-1),2}(k_1 r_0) A_{N(2n-1),2}$$

$$+ c_{N(2n-1),3}(k_2 r_0) A_{N(2n-1),3} + c_{N(2n-1),4}(k_2 r_0) A_{N(2n-1),4} = 0,$$

$$u_{N(2n-1),1}(k_1 r_1) A_{N(2n-1),1} + u_{N(2n-1),2}(k_1 r_1) A_{N(2n-1),2}$$

$$+ u_{N(2n-1),3}(k_2 r_1) A_{N(2n-1),3} + u_{N(2n-1),4}(k_2 r_1) A_{N(2n-1),4} = 0,$$

$$l_{N(2n-1),3}(k_1 r_1) A_{N(2n-1),1} + l_{N(2n-1),4}(k_1 r_1) A_{N(2n-1),2}$$

$$+ l_{N(2n-1),3}(k_2 r_1) A_{N(2n-1),3} + l_{2N(2n-1),4}(k_2 r_1) A_{N(2n-1),4} = 0. \tag{17}$$

The formulas to determine the resonant frequencies for the boundary conditions (14) follow from the existence condition for the nontrivial solutions of the homogeneous ($V_0 = 0$) systems of equations (17):

$$\begin{vmatrix} a_{m1}(k_1 r_0) & a_{m2}(k_1 r_0) & a_{m3}(k_2 r_0) & a_{m4}(k_2 r_0) \\ c_{m3}(k_1 r_0) & c_{m3}(k_1 r_0) & c_{m1}(k_2 r_0) & c_{m2}(k_2 r_0) \\ u_{m1}(k_1 r_1) & u_{m2}(k_1 r_1) & u_{m3}(k_2 r_1) & u_{m4}(k_2 r_1) \\ l_{m1}(k_1 r_1) & l_{m2}(k_1 r_1) & l_{m3}(k_2 r_1) & l_{m4}(k_2 r_1) \end{vmatrix} = 0. \tag{18}$$

The boundary conditions (14) yield block systems of algebraic equations for the dimensionless constants $A_{N(2n-1),i}$ ($n = 1, 2, \ldots$):

$$a_{N(2n-1),1}(k_1 r_0) A_{N(2n-1),1} + a_{N(2n-1),2}(k_1 r_0) A_{N(2n-1),2}$$

$$+ a_{N(2n-1),3}(k_2 r_0) A_{N(2n-1),3} + a_{N(2n-1),4}(k_2 r_0) A_{N(2n-1),4} = -\frac{4}{\pi} V_0 \frac{(1+\nu_E) d_{13}}{2n-1},$$

$$c_{N(2n-1),1}(k_1 r_0) A_{N(2n-1),1} + c_{N(2n-1),2}(k_1 r_0) A_{N(2n-1),2}$$

$$+ c_{N(2n-1),3}(k_2 r_0) A_{N(2n-1),3} + c_{N(2n-1),4}(k_2 r_0) A_{N(2n-1),4} = 0,$$

$$a_{N(2n-1),1}(k_1 r_1) A_{N(2n-1),1} + a_{N(2n-1),2}(k_1 r_1) A_{N(2n-1),2}$$

$$+ a_{N(2n-1),3}(k_2 r_1) A_{N(2n-1),3} + a_{N(2n-1),4}(k_2 r_1) A_{N(2n-1),4} = -\frac{4}{\pi} V_0 \frac{(1+\nu_E) d_{13}}{2n-1},$$

$$c_{N(2n-1),1}(k_1 r_1) A_{N(2n-1),1} + c_{N(2n-1),2}(k_1 r_1) A_{N(2n-1),2}$$

$$+ c_{N(2n-1),3}(k_2 r_1) A_{N(2n-1),3} + c_{N(2n-1),4}(k_2 r_1) A_{N(2n-1),4} = 0. \tag{19}$$

The resonant frequencies can be determined by equating the fourth-order determinants of the homogeneous (at $V_0 = 0$) systems of algebraic equations (20) to zero:

TABLE 1

k	$N = 0$, $\overline{\omega}_{0,k}$	$N = 1$, $\overline{\omega}_{1,k}$	$N = 2$, $\overline{\omega}_{2,k}$	$N = 3$, $\overline{\omega}_{3,k}$
1	0.77405	1.2108	1.85491	2.31119
2	2.7658	2.70754	2.87211	3.50261
3	4.24997	4.61023	5.4193	6.40551
4	7.211044	7.05481	6.88761	6.87753
5	7.93913	8.24799	8.85883	9.53872
6	10.14251	10.15934	10.23068	10.43583
7	13.06489	13.04537	13.09362	13.22587

$$\begin{vmatrix} a_{m1}(k_1 r_0) & a_{m2}(k_1 r_0) & a_{m3}(k_2 r_0) & a_{m4}(k_2 r_0) \\ c_{m1}(k_1 r_0) & c_{m2}(k_1 r_0) & c_{m3}(k_2 r_0) & c_{m4}(k_2 r_0) \\ a_{m1}(k_1 r_1) & a_{m2}(k_1 r_1) & a_{m3}(k_2 r_1) & a_{m4}(k_2 r_1) \\ c_{m1}(k_1 r_1) & c_{m2}(k_1 r_1) & c_{m3}(k_2 r_1) & c_{m4}(k_2 r_1) \end{vmatrix} = 0. \qquad (20)$$

In (16)–(20), $m = N(2n-1)$ ($n = 1, 2, \ldots$, N is the number of radial cuts in the electrode coating).

The following general properties of the theoretical frequency spectrum of a plate with different number (N) of radial cuts can be found from the boundary conditions (12)–(14), formulas (6) and (7), and the frequency equations (16), (18), and (20):

if $N = 1$ (two electrodes), then resonances occur at frequencies $f_{1,k}$, $f_{3,k}$, $f_{5,k}$, …;

if $N = 2$ (four electrodes), then $f_{2,k}$, $f_{6,k}$, $f_{10,k}$, …;

if $N = 3$ (six electrodes), then $f_{3,k}$, $f_{9,k}$, $f_{15,k}$, …;

if $N = 4$ (eight electrodes), then $f_{4,k}$, $f_{12,k}$, $f_{20,k}$, …;

if $N = 5$ (ten electrodes), then $f_{5,k}$, $f_{15,k}$, $f_{25,k}$, …;

if $N = 6$ (12 electrodes), then $f_{6,k}$, $f_{18,k}$, $f_{30,k}$, …;

if $N = 7$ (14 electrodes), then $f_{7,k}$, $f_{21,k}$, $f_{35,k}$, …;

if $N = 8$ (16 electrodes), then $f_{8,k}$, $f_{24,k}$, $f_{40,k}$, … .

In the notation of frequencies $f_{m,k}$, the subscript "m" is the harmonic number with respect to the azimuth θ (circumferential mode number) and the subscript "k" is the root sequence number of the frequency equation.

3. Analysis of Numerical Results. The dimensionless resonant frequencies $\overline{\omega} = \sqrt{(1-\nu_E^2)s_{11}^E\rho}\,\omega r_1$ calculated from Eqs. (17), (19), and (21) for TsTS-19 piezoceramic [2] with characteristics $r_0 / r_1 = 0.4$, $\rho = 7740$ kg/m^3, $s_{11}^E = 15.2 \cdot 10^{-12}$ m^2/N, $s_{12}^E = -5.8 \cdot 10^{-12}$ m^2/N, $d_{31} = -125 \cdot 10^{-12}$ C/N are summarized in Tables 1, 2, and 3, respectively.

When $N = 0$ (axisymmetric vibrations) and one (inner or outer) of the edges is clamped (boundary conditions (12) and (13)), the second, fifth, and seventh frequencies represent radial vibrations (10) and the first, third, fourth, and sixth frequencies represent azimuthal vibrations (11). When the edges are free, the first, third, and sixth frequencies represent radial vibrations and the second, fourth, fifth, and seventh frequencies represent azimuthal vibrations. When one edge is clamped (boundary conditions (12) and (13)), the natural frequencies approach each other with increasing mode number. This is also true for the boundary conditions (14) (free edges), but the associated frequencies are lower.

Depending on the boundary conditions for the piezoceramic plate and the number N of cuts in its electrode coating, the first natural frequencies, both quasiradial and quasiazimuthal, for conditions (13) and (14) are considerably different (sometimes

TABLE 2

k	$N = 0$, $\overline{\omega}_{0,k}$	$N = 1$, $\overline{\omega}_{1,k}$	$N = 2$, $\overline{\omega}_{2,k}$	$N = 3$, $\overline{\omega}_{3,k}$
1	2.31578	2.46586	2.70391	3.14207
2	3.20884	3.3252	3.93292	4.80765
3	4.81907	5.20747	6.05161	6.743159
4	7.56991	7.377518	7.278965	7.747007
5	8.04387	8.391521	8.976372	9.558131
6	10.40329	10.43448	10.56156	10.88514
7	13.20124	13.1704	13.22125	13.38097

TABLE 3

k	$N = 0$, $\overline{\omega}_{0,k}$	$N = 1$, $\overline{\omega}_{1,k}$	$N = 2$, $\overline{\omega}_{2,k}$	$N = 3$, $\overline{\omega}_{3,k}$
1	1.42334	1.6265	0.69281	1.54389
2	3.31746	3.85103	2.34721	3.18735
3	5.49151	5.20302	4.85053	4.97488
4	6.05803	6.53165	5.05288	6.17078
5	8.896337	8.88278	7.294018	7.983204
6	10.59329	10.72654	8.93175	9.28126
7	11.76887	11.81621	11.049	11.42551

two- or three-fold) from those for condition (15). As the frequency number increases, the difference decreases to approximately 10% for the seventh frequency.

With increase in the number N of cuts in the electrode coating of the plate (even with only one edge clamped), the frequencies corresponding to small k become higher, and the frequency spectrum becomes more crowded in the high-frequency range.

It is of interest to analyze the dependence of the frequency spectrum on the geometry of the ring. Tables 4, 5, and 6 give the values of the first frequency as a function of the ratio r_0 / r_1 for different values of N and the following boundary conditions: clamped edge–free edge, free edge–clamped edge, and free edge–free edge, respectively.

The tables indicate that the frequencies strongly depend on the ring geometry and the number of cuts. The frequency is maximum for $N = 1$ and decreases with increasing radius of the hole for $N = 2, 3, 4$. With such a geometry of the ring, the frequency of vibrations increase with the number of cuts. As the ratio r_0 / r_1 is increased, the natural frequencies determined from (16) and (18) can differ severalfold from those determined from (20). The greater the ratio r_0 / r_1, the more the difference.

TABLE 4

r_0 / r_1	$N = 1$, $\overline{\omega}_{1,1}$	$N = 2$, $\overline{\omega}_{2,1}$	$N = 3$, $\overline{\omega}_{3,1}$	$N = 4$, $\overline{\omega}_{4,1}$
0.1	0.71165	1.34364	2.00921	2.61817
0.2	0.87091	1.45357	2.03302	2.62141
0.3	1.02743	1.62725	2.1194	2.6486
0.4	1.21076	1.85491	2.31119	2.75133
0.5	1.45919	2.12615	2.64278	3.01045
0.6	1.84263	2.46673	3.10797	3.52854
0.7	2.5143	3.02277	3.6822	4.32559
0.8	3.91731	4.2612	4.77424	5.39814
0.9	8.23944	8.40371	8.67026	9.02956

TABLE 5

r_0 / r_1	$N = 1$, $\overline{\omega}_{1,1}$	$N = 2$, $\overline{\omega}_{2,1}$	$N = 3$, $\overline{\omega}_{3,1}$	$N = 4$, $\overline{\omega}_{4,1}$
0.1	1.95992	2.75578	3.79396	4.56191
0.2	2.06778	2.50631	3.55341	4.4994
0.3	2.23762	2.50406	3.22194	4.15698
0.4	2.46665	2.70391	3.14207	3.81339
0.5	2.7481	3.08122	3.36569	3.78359
0.6	3.11977	3.60247	3.93426	4.18378
0.7	3.74549	4.21597	4.81566	5.20081
0.8	5.0915	5.42034	5.92438	6.5546
0.9	9.3562	9.51646	9.77759	10.13163

Conclusions. Nonaxisymmetric planar electroelastic vibrations can be excited in thin piezoceramic ring plates with radially cut electrode coating. The general solution to the relevant problem has been found. The natural frequency spectra for lower circumferential harmonics have been numerically analyzed for three types of boundary conditions, different number of radial cuts in the electrode coating, and different ratios of inner and outer radii of the plate. The dependence of the quasiradial and quasiazimuthal natural frequencies on the frequency number and the number of cuts in the electrode coating has been established. It has been established that the natural frequencies of the plate with one edge clamped are higher than those of the

TABLE 6

r_0 / r_1	$N = 1$, $\overline{\omega}_{1,1}$	$N = 2$, $\overline{\omega}_{2,1}$	$N = 3$, $\overline{\omega}_{3,1}$	$N = 4$, $\overline{\omega}_{4,1}$
0.1	1.56127	1.23169	2.00392	2.618
0.2	1.58725	1.05552	1.95461	2.61131
0.3	1.61479	0.86562	1.79267	2.55171
0.4	1.62666	0.69304	1.54389	2.35656
0.5	1.61051	0.54047	1.27326	2.04243
0.6	1.56808	0.4038	1.00296	1.67866
0.7	1.50873	0.28052	0.73471	1.28545
0.8	1.44201	0.17266	0.4703	0.8614
0.9	1.37384	0.07913	0.22072	0.41876

plate with all edges free. The natural frequencies of a plate with such boundary conditions can be changed by varying the geometry of the plate or the number of cuts.

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