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THERMOVISCOPLASTICITY THEORY INCORPORATING THE THIRD DEVIATORIC STRESS INVARIANT

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The applicability of three versions of constitutive equations to the description of the thermoviscoplastic deformation of isotropic materials is discussed. It is shown that the thermoviscoplastic equations incorporating the third deviatoric stress invariant are in good agreement with experiment

Keywords: thermoviscoplasticity, isotropic material, three versions of constitutive equations, third deviatoric stress invariant, experimental data

Introduction. The characteristics of some initially isotropic polycrystalline materials under creep are known [3, 10, 16] to depend on the stress mode. Constitutive equations describing the isothermal deformation of such materials were derived in [3, 6, 9–11, 16, 17] using data of reference tension, compression, and torsion tests. Here we will use the constitutive equations from [15] to describe the thermoviscoplastic deformation of such materials. These equations generalize the thermoplastic equations proposed in [14] and describe the thermoviscoelastoplastic deformation of isotropic materials along paths of small curvature, taking into account the dependence of the material properties on temperature and stress mode. The equations from [15] incorporate the stress mode angle [1], which is expressed in terms of the second and third deviatoric stress invariants, and relate the components of the engineering stress and strain tensors. It was assumed that the strains are represented as the sum of elastic and inelastic components and that the stress deviator and the deviator of inelastic strain differentials are coaxial. The equations include two nonlinear functions found experimentally. One of these functions relates the first invariants of the stress and strain tensors, while the other function relates the second invariants of the respective deviatoric tensors. These functions are individualized in two series of reference tests on tubular specimens under proportional loading at several constant values of the stress mode angle and several temperatures. The first series of tests involves instantaneous deformation of specimens (i.e., the loading rate does not affect the form of the functions). The second series includes creep tests at the same initial loading rate as in the tests of the first series. If the first invariants of the stress and strain tensors are in linear relationship and the relationship between the second invariants of the stress and strain deviators is independent of the stress mode, then the constitutive equations go over into the equations describing deformation along paths of small curvature [2], which coincide with the widely used [2, 12, 13, etc.] equations of incremental plasticity [1, 2, 8, 18, etc.] associated with the von Mises yield criterion.

The assumption of the coaxiality beteewn the stress deviators and the deviators of inelastic strain differentials and the nonlinear functions appearing in the constitutive equations of thermoviscoplasticity were experimentally validated in [14, 15].

Expanding upon [14, 15], we will validate the equations of thermoviscoplasticity by comparing experimental values of the strain components with their values calculated from these equations, given stress values. Moreover, we will also validate the version of these equations from [4] in the case where the first stress and strain invariants are in linear relationship.

1. Thermoviscoplastic Equations. The constitutive equations [15] that describe nonisothermal deformation along paths of small curvature, relate the components of the stress σ_{ij} and strain ε_{ij} tensors, and incorporate the stress mode are

$$\sigma_{ij} = 2G\varepsilon_{ij} + (K - 2G)\varepsilon_0 \delta_{ij} - \sigma_{ij}^{(\mathrm{d})}, \tag{1.1}$$

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$$\sigma_{ij}^{(\mathrm{d})} = 2G\left[e_{ij}^{(\mathrm{n})} + \frac{1+\nu}{1-2\nu}(\varepsilon_T + \varepsilon_0^{(\mathrm{p})} + \varepsilon_0^{(\mathrm{c})})\delta_{ij}\right], \tag{1.2}$$

where $K = E/(1-2\nu)$, $E = 2G(1+\nu)$, $\varepsilon_T = \alpha_T(T-T_0)$; E, ν and α_T are, respectively, the elastic modulus, Poisson's ratio, and thermal linear expansion coefficient, each depending on temperature T; T_0 is the initial temperature; $e_{ij}^{(\mathrm{n})}$ are the components of the inelastic-strain deviator; $\varepsilon_0^{(\mathrm{p})} = \varepsilon_{ii}^{(\mathrm{p})}/3$ and $\varepsilon_0^{(\mathrm{c})} = \varepsilon_{ii}^{(\mathrm{c})}/3$ are the first invariants of the plastic strain and creep tensors, respectively.

To use Eqs. (1.1), the loading process should be divided into steps. At the end of the Nth step of loading, the components of the inelastic-strain deviator are represented as the sum of their increments:

$$e_{ij}^{(\mathrm{n})} = \sum_{k=1}^{N} \Delta_k e_{ij}^{(\mathrm{n})}. \tag{1.3}$$

To determine $\varepsilon_0^{(\mathrm{p})}$ and $\varepsilon_0^{(\mathrm{c})}$, we will use the relationship between the first invariants of the stress and strain tensors, $\sigma_0 = \sigma_{ii}/3$ and $\varepsilon_0 = \varepsilon_{ii}/3$,

$$\sigma_0 = F_1(\varepsilon_0^*, T, \omega_\sigma), \tag{1.4}$$

$$\varepsilon_0^* = \varepsilon_0 - \varepsilon_T - \varepsilon_0^{(\mathrm{c})}, \tag{1.5}$$

$$\varepsilon_0^{(\mathrm{c})} = \sum_{k=1}^{N} \Delta_k \varepsilon_0^{(\mathrm{c})}, \tag{1.6}$$

$$\omega_\sigma = \frac{1}{3}\arccos\left[-\frac{3\sqrt{3}}{2}\frac{I_3(D_\sigma)}{S^3}\right] \quad (0 \le \omega_\sigma \le \pi/3), \tag{1.7}$$

where ω_σ is the stress mode angle; $I_3(D_\sigma) = |s_{ij}|$ is the third invariant of the stress deviator D_σ; $s_{ij} = \sigma_{ij} - \sigma_0\delta_{ij}$ are the components of the stress deviator; S is the shear-stress intensity,

$$S = (s_{ij}s_{ij}/2)^{1/2}. \tag{1.8}$$

The increment $\Delta_k e_{ij}^{(\mathrm{n})}$ at an arbitrary step of loading is defined by

$$\Delta_k e_{ij}^{(\mathrm{n})} = \left\langle \frac{s_{ij}}{S} \right\rangle_k \Delta_k \Gamma^{(\mathrm{n})}, \tag{1.9}$$

where $\Delta_k \Gamma^{(\mathrm{n})}$ is the increment of inelastic shear strain intensity,

$$\Delta_k \Gamma^{(\mathrm{n})} = \Delta_k \Gamma^{(\mathrm{p})} + \Delta_k \Gamma^{(\mathrm{c})}, \tag{1.10}$$

$\Delta_k \Gamma^{(\mathrm{p})}$ and $\Delta_k \Gamma^{(\mathrm{c})}$ are the increments of instantaneous plastic-shear-strain and creep-strain intensities. The angular brackets in (1.9) denote averaging over a step of loading. To determine $\Delta_k \Gamma^{(\mathrm{p})}$, we assume that

$$S = F_2(\Gamma^*, T, \omega_\sigma), \tag{1.11}$$

where Γ^* is the intensity of instantaneous shear strains,

$$\Gamma^* = \frac{S}{2G} + \Gamma^{(\mathrm{p})},$$

$$\Gamma^{(p)} = \sum_{k=1}^{N} \Delta_k \Gamma^{(p)}. \tag{1.12}$$

The functions F_1 (1.4) and F_2 (1.11) are determined from the first series of reference tests on tubular specimens under proportional loading, as described in [5, 7, 14, 15]. To determine $\Delta_k \varepsilon_0^{(c)}$ and $\Delta_k \Gamma^{(c)}$, we use the data of the second series of reference (creep) tests. The following approximating expressions are proposed in [15]:

$$\dot{\Gamma}^{(c)}(S, T, \omega_\sigma) = \exp\left(c_2 \ln(c_1 S) + c_3 + c_4 T + c_5 \omega_\sigma + c_6 \omega_\sigma^2\right), \tag{1.13}$$

$$\dot{\varepsilon}_0^{(c)}(\sigma_0, T, \omega_\sigma) = \exp\left(d_2 \ln(d_1 \sigma_0) + d_3 + d_4 T + d_5 \omega_\sigma + d_6 \omega_\sigma^2\right), \tag{1.14}$$

where c_i, d_i $(i = 1, \ldots, 6)$ are coefficients found from the best fit of expressions (1.13) and (1.14) to the test data. Then

$$\Delta_k \varepsilon_0^{(c)} = \dot{\varepsilon}_0^{(c)} \Delta_k t,$$

$$\Delta_k \Gamma^{(c)} = \dot{\Gamma}^{(c)} \Delta_k t, \tag{1.15}$$

where $\Delta_k t = t_k - t_{k-1}$ is the step duration.

Note that a simpler version [4] of these constitutive equations where the first stress and strain invariants are in linear relationship rather than (1.4) can be used.

2. Algorithm for the Analysis of the Deformation of a Tubular Specimen. Procedures for testing tubular specimens and plotting graphs of (1.4) and (1.11) are detailed in [4, 5, 7, 14, 15]. The tests were conducted at different constant and variable ratios between the tensile axial force and the internal pressure, different temperatures and loading rates, either strongly affecting or not the deformation curves. In these tests, the stress state of the specimen is characterized by the axial and circumferential stresses σ_{11}, σ_{22}, the radial stress being negligible compared with σ_{11} and σ_{22}, i.e., $\sigma_{33} = 0$. The strain state is characterized by the axial, circumferential, and radial strains ε_{11}, ε_{22}, ε_{33}, of which ε_{11} and ε_{22} are measured and ε_{33} is calculated, as described in [5, 15].

When determining the strain state of the specimen, we assume that the stresses are given. The elastic modulus, Poisson's ratio, and linear thermal expansion coefficient of the material must be known. Functions (1.4) and (1.11) are also assumed known from reference tests on tubular specimens proportionally loaded at $\omega_\sigma = 0, \pi/6, \pi/3$ and at several temperatures from the range considered (reference curves). At intermediate values of the angle and temperature, functions (1.4) and (1.11) are determined by linear interpolation from the reference curves. The coefficients in (1.13) and (1.14) must be known as well. Then from Eqs. (1.1) we derive expressions for the strains in the specimen:

$$\varepsilon_{11} = \frac{\sigma_{11} - \nu\sigma_{22}}{2G(1+\nu)} + \frac{\overset{*}{\sigma}_{11} - \nu\overset{*}{\sigma}_{22}}{2G(1+\nu)}, \qquad \varepsilon_{22} = \frac{\sigma_{22} - \nu\sigma_{11}}{2G(1+\nu)} + \frac{\overset{*}{\sigma}_{22} - \nu\overset{*}{\sigma}_{11}}{2G(1+\nu)},$$

$$\varepsilon_{33} = -\frac{\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22}) - \frac{1-2\nu}{1-\nu}(e_{11}^{(n)} + e_{22}^{(n)}) + \frac{1+\nu}{1-\nu}(\varepsilon_T + \varepsilon_0^{(p)} + \varepsilon_0^{(c)}), \tag{2.1}$$

$$\left(\overset{*}{\sigma}_{11} = \frac{2G}{1-\nu}\left[e_{11}^{(n)} + \nu e_{22}^{(n)} + (1+\nu)\left(\varepsilon_T + \varepsilon_0^{(p)} + \varepsilon_0^{(c)}\right)\right],\right.$$

$$\left.\overset{*}{\sigma}_{22} = \frac{2G}{1-\nu}\left[e_{22}^{(n)} + \nu e_{11}^{(n)} + (1+\nu)(\varepsilon_T + \varepsilon_0^{(p)} + \varepsilon_0^{(c)})\right]\right). \tag{2.2}$$

The following algorithm can be used to calculate the strains in the specimen. Divide the loading process into a number of steps. It is convenient that the first step be within the elastic region; then $e_{11}^{(n)} = e_{22}^{(n)} = 0$ and $\varepsilon_0^{(p)} = \varepsilon_0^{(c)} = 0$ in (2.1) and (2.2) and the strains at the first step can be found from the theory of elasticity. To determine the strains at the Nth step from the stresses, calculate $(\sigma_0)_N = ((\sigma_{11} + \sigma_{22})/3)_N$, $(S)_N$ (1.8) and $(\omega_\sigma)_N$ (1.7). Among the reference curves (1.4), find the curve

TABLE 1

t, min	T, °C	σ_{11}, MPa	σ_{22}, MPa	$\varepsilon_{11}\cdot 10^3$	$\varepsilon_{22}\cdot 10^3$	$\varepsilon_{33}\cdot 10^3$
0	590	164.1	82.6	6	–1	–4.2
1.5	606	175.3	84.3	10.1	–1	–8.1
3.3	615	187.4	87.9	14.1	–1	–11.9
4.3	633	191.4	90.3	19.2	–1	–16.8
5.5	642	200.6	93	22.2	–1	–19.5
6.7	654	211.4	97.2	26.3	0	–24.3
8	660	218.4	103.9	34.6	1	–33
9.3	674	226.8	113.1	43.9	2	–42.4
10.8	682	228.7	116.5	57.6	1	–53.7
11.1	691	228.9	118.1	65	2	–61.1

corresponding to $(\omega_\sigma)_N$ and the temperature T_N of the specimen. On this curve, find the value of $(\varepsilon_0^*)_N$ corresponding to $(\sigma_0)_N$ and calculate $(\varepsilon_0^{(p)})_N = (\varepsilon_0^*)_N - \dfrac{(\sigma_0)_N}{K(T_N)}$. Next, use the values of $(\sigma_0)_N$, $(\omega_\sigma)_N$, T_N and formulas (1.14), (1.15), (1.6) to determine $(\varepsilon_0^{(c)})_N$. Then, by linear interpolation of the reference curves (1.11), find the curve corresponding to T_N and $(\omega_\sigma)_N$, and calculate the value of $(\Gamma^*)_N$ corresponding to $(S)_N$. Use (1.12) to get $\Delta_N\Gamma^{(p)} = (\Gamma^*)_N - \dfrac{(S)_N}{2G(T_N)} - (\Gamma^*)_{N-1} + \dfrac{(S)_{N-1}}{2G(T_{N-1})}$. Use the values of $(S)_N$, $(\omega_\sigma)_N$, and T_N and formulas (1.13) and (1.15) to determine $\Delta_N\Gamma^{(c)}$ and $\Delta_N\Gamma^{(n)}$ (1.10). After that, calculate the increments of the inelastic components of the strain deviator (1.9) and the components themselves (1.3). Finally, calculate σ_{11}^* and σ_{22}^* (2.2) and use formulas (2.1) to determine the unknown strains.

3. Calculated Results. Table 1 collects the results of analyzing the nonisothermal deformation of the specimen using the above algorithm [15].

The specimen is made of Kh18N10T alloy with characteristics borrowed from [14, 15]. Tables 2 and 3 collect the values of functions (1.4) and (1.11), respectively, calculated as described in [4, 5, 7, 14, 15].

The calculations have been performed (i) taking the stress mode into account and using functions (1.4) and (1.11) (Tables 2 and 3); (ii) taking (1.11) into account and using the linear relationship between the first stress and strain invariants instead of (1.4), i.e., using a simplified version of the constitutive equations [4]; and (iii) using the theory of deformation along paths of small curvature and disregarding the stress mode [2]. The creep strains are described by expressions (1.13) and (1.14) with coefficients presented in [15].

Figures 1 and 2 show the calculated axial, ε_{11}, and radial, ε_{33}, strains, respectively, versus the number of steps of loading. The strain ε_{22} is not shown because it is much lower than ε_{11} and ε_{33} and varies from $-1\cdot 10^{-3}$ to $2\cdot 10^{-3}$. The solid, dotted, and dashed lines represent cases (i), (ii), and (iii), respectively, and the triangles stand for the experimental data.

The values of ε_{33} calculated by the procedure from [15] using the plastic incompressibility condition are considered as experimental values. Figures 1 and 2 demonstrate that the results of cases (i) and (ii) are different and are in good agreement with

TABLE 2

ε_0^*	T = 500 °C			T = 700 °C		
	$\omega_\sigma = 0$	$\omega_\sigma = \pi/6$	$\omega_\sigma = \pi/3$	$\omega_\sigma = 0$	$\omega_\sigma = \pi/6$	$\omega_\sigma = \pi/3$
0.0002	72	72	72	34	34	34
0.0006	166	131	88	70	60	50
0.001	178	139	100	82	68	62
0.0024	205	151	116	160	130	100
0.004	211	159	124	186	137	104
0.008	219	169	133	194	147	112
0.02	240	197	154	216	177	136

TABLE 3

Γ^*	T = 500 °C			T = 700 °C		
	$\omega_\sigma = 0$	$\omega_\sigma = \pi/6$	$\omega_\sigma = \pi/3$	$\omega_\sigma = 0$	$\omega_\sigma = \pi/6$	$\omega_\sigma = \pi/3$
0.0002	26	26	26	12	12	12
0.01	104	104	104	80	80	80
0.02	129	129	129	96	92	103
0.04	149	134	165	128	107	132
0.06	173	145	188	134	111	141
0.08	179	153	200	140	116	150
0.10	184	157	212	146	121	159
0.16	199	169	232	172	135	186

the experimental data. The difference between the maximum strains calculated in cases (i), (ii) and found experimentally reaches 12%. The results of case (iii) are in good agreement with the experimental data at the first six steps of loading when the strains do not exceed 3% (in absolute magnitude). The difference between the results of case (iii) and the experimental data increases with the strains, reaching more than 50% at the end of the process.

Conclusions. The thermoviscoplastic deformation of a tubular specimen made of Kh18N10T alloy has been analyzed numerically using three versions of the theory of plastic deformation along paths of small curvature. It has been shown that the strains calculated with the theory of thermoviscoplasticity [15] based on nonlinear relationships between the first stress and strain invariants and between the second invariants of the stress and strain deviators, which incorporate the stress mode, are in good agreement with the experimental strains. It has also been shown that the deformation process can be described well by a simpler version of the constitutive equations [4] in which the stress mode appears only in the relationship between the second

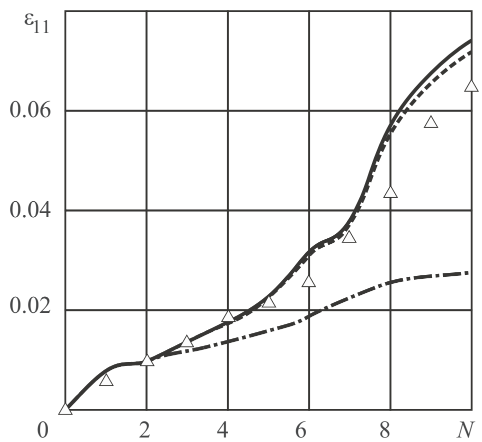

Fig. 1

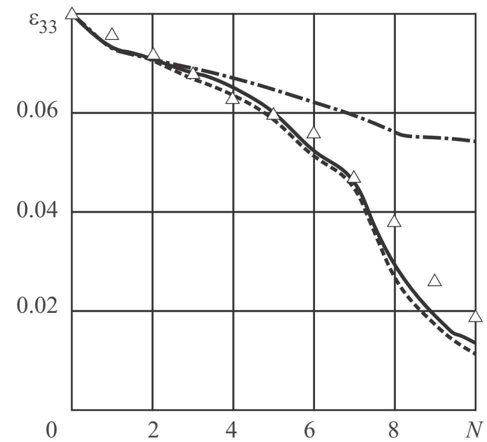

Fig. 2

invariants of the stress and strain deviators, while the first invariants of the stress and strain tensors are in conventional linear relationship. The results calculated with the constitutive equations [2] describing deformation along paths of small curvature regardless of the stress mode are in agreement with the experimental data only at the initial steps of loading when the strains are less than 3% and differ from the experimental data by more than 50% at the end of the process when the strains reach 7%.

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