DOI 10.1007/s10778-014-0660-8
*International Applied Mechanics, Vol. 50, No. 6, November, 2014*

# STUDYING THE AXISYMMETRIC THERMOVISCOELASTOPLASTIC DEFORMATION OF LAYERED SHELLS TAKING INTO ACCOUNT THE THIRD DEVIATORIC STRESS INVARIANT

**M. E. Babeshko and Yu. N. Shevchenko**

**A procedure for the numerical analysis of the thermoviscoelastoplastic stress–strain state of thin layered shells of revolution under axisymmetric loading is proposed. Constitutive equations that incorporate the third deviatoric stress invariant are used to describe the inelastic deformation of isotropic materials. Numerical results are analyzed**

**Keywords:** thermoviscoelastoplastic stress–strain state, constitutive equations, third deviatoric stress invariant, method of successive approximations, shell of revolution

**Introduction.** Methods for solving boundary-value problems of thermoviscoplasticity are intensively developed in the contemporary literature. Some of them [4, 12, 13, etc.] are based on the classical constitutive equations [1, 9–11] describing the thermoviscoplastic deformation of isotropic materials. Methods based on constitutive equations that describe in some way the dependence of material properties on the type of loading were proposed in a few publications. For example, the methods outlined in [17, 18] employ equations of viscoplasticity [16] for materials with different behavior in tension and compression.

Here axisymmetric problems of thermoviscoplasticity for thin shells are solved using experimentally validated equations of thermoviscoplasticity [14, 15]. These equations describe the thermoviscoelastoplastic deformation of isotropic materials along paths of small curvature taking into account the dependence of material properties on the stress mode, which is characterized by the stress mode angle [1] and is calculated using the second and third deviatoric stress invariants. The components of the stress tensors and the linear strains (which will be referred to as strains for brevity) are related assuming that the strains have elastic and inelastic components and that the stress deviators and plastic strain differentials are coaxial. The equations include two nonlinear functions found experimentally. One of these functions relates the first invariants of the stress and strain tensors, while the other function relates the second invariants of the respective deviatoric tensors. These functions are individualized in two series of base tests on tubular specimens under proportional loading at several constant values of the stress mode angle and several temperatures. The first series of tests involves instantaneous deformation of specimens (i.e., the loading rate does not affect the form of the functions). The second series includes creep tests at the same initial loading rate as in the tests of the first series. If the first invariants of the stress and strain tensors are in linear relationship and the relationship between the second invariants of the deviatoric stress and strain tensors is independent of the stress mode and determined from simple-tension tests, then the constitutive equations go over into the widely used equations describing deformation along paths of small curvature [4], which coincide with the equations of incremental plasticity [1, 9, etc.] associated with the von Mises yield criterion.

Expanding upon [5, 8], we will outline a method to solve axisymmetric problems of thermoviscoplasticity for thin layered shells allowing for the dependence of the material properties on temperature and stress mode and using the constitutive equations [14, 15].

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S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterova St., Kyiv, Ukraine 03057, e-mail: plast@inmech.kiev.ua. Translated from Prikladnaya Mekhanika, Vol. 50, No. 6, pp. 14–26, November–December 2014. Original article submitted September 12, 2013.

**1. Problem Formulation and Basic Equations.** Consider a shell of revolution with isotropic layers. Being initially ($t = t_0$) in stress/strain-free state at temperature $T = T_0$, the shell is subjected to axisymmetric nonuniform heating and mechanical loads except torsion. The layers are perfectly bonded. We choose a curvilinear orthogonal coordinate system $s$, $\theta$, $\varsigma$ fixed to the undeformed continuous coordinate surface, where $s$ ($s_a \le s \le s_b$) and $\theta$ ($0 \le \theta \le 2\pi$) are the meridional and circumferential coordinates, and $\varsigma$ ($\varsigma_0 \le \varsigma \le \varsigma_l$) is the normal (to the coordinate surface) coordinate; $\varsigma_0$ corresponds to the inner face of the first (inner) layer, and $\varsigma_l$ to the outer face of the last (outer) layer; $l$ is the number of layers of thickness $h_i = \varsigma_i - \varsigma_{i-1}$, $i = 1, 2, \dots, l$. The coordinate surface is chosen to be the midsurface or one of the faces of the layers.

The temperature field of the shell is assumed known from the solution of the heat-conduction problem or from somewhere else. It is assumed that the loads cause the shell to deform within and beyond the elastic limit, the creep strains are commensurable with the elastic and plastic strains, and elastic unloading may occur in the plastic range. In formulating the boundary-value problem, we will follow Cauchy's approach [14, 15], writing all the equations in the chosen coordinate system fixed to the undeformed coordinate surface. We will use a geometrically linear quasistatic problem formulation and the Kirchhoff–Love hypothesis for the whole laminate to determine the stress–strain state (SSS) of the shell. To describe the deformation of the materials, we will use equations of thermoviscoplasticity [14, 15] incorporating the stress mode and the equations of deformation along paths of small curvature [4] disregarding the stress mode and linearized by the method of additional stresses. The SSS of the shell is determined by the method of successive approximations.

To solve the problem, we divide the loading period into small time intervals with endpoints as close as possible to the onsets of unloading. Within each time interval, we will use the differential equilibrium equations for an element of an axisymmetrically loaded (without torsion) shell of revolution and the kinematic and constitutive equations. The equilibrium equations are as follows [3]:

$$\frac{d(rN_s)}{ds} - \cos\varphi N_\theta + \frac{r}{R_s} Q_s + rq_s = 0,$$

$$\frac{d(rQ_s)}{ds} - \sin\varphi N_\theta - \frac{r}{R_s} N_s + rq_\varsigma = 0,$$

$$\frac{d(rM_s)}{ds} - \cos\varphi M_\theta - rQ_s = 0, \tag{1.1}$$

where $N_s, N_\theta, Q_s$ are forces; $M_s$ and $M_\theta$ are moments; $\vartheta_s$ is the angle of rotation of the normal to the coordinate surface in the meridional plane; $r$ is the parallel radius; $R_s$ is the radius of curvature of a meridian of the coordinate surface; $\pi - \varphi$ is the angle between the normal to the coordinate surface and the $z$-axis, which is the axis of revolution of the shell; $q_s$, $q_\varsigma$ are the distributed loads referred to the coordinate surface.

The kinematic equations are as follows [3]:

$$\varepsilon_s = \frac{du}{ds} + \frac{w}{R_s}, \qquad \varepsilon_\theta = \frac{\cos\varphi}{r} u + \frac{\sin\varphi}{r} w,$$

$$\kappa_s = \frac{d\vartheta_s}{ds}, \qquad \kappa_\theta = \frac{\cos\varphi}{r}\vartheta_s, \qquad \vartheta_s = -\frac{dw}{ds} + \frac{u}{R_s}, \tag{1.2}$$

where $\varepsilon_s, \varepsilon_\theta, \kappa_s, \kappa_\theta$ are the strains and changes of curvature of the coordinate surface of the shell in the directions $s$ and $\theta$; $u$ and $w$ are displacements of its particles in the directions $s$ and $\varsigma$. We will use the following formulas to transform between the strains $\varepsilon_s$ and $\varepsilon_\theta$ and the strains $\varepsilon_{ss}$ and $\varepsilon_{\theta\theta}$ at an arbitrary point of the shell, assuming that $\varsigma / R_s, \varsigma / R_\theta$ (where $R_\theta$ is the radius of curvature of the coordinate surface in the circumferential direction) can be neglected compared with unity:

$$\varepsilon_{ss} = \varepsilon_s + \varsigma\kappa_s, \qquad \varepsilon_{\theta\theta} = \varepsilon_\theta + \varsigma\kappa_\theta. \tag{1.3}$$

The components of the stress ($\sigma_{ss}$, $\sigma_{\theta\theta}$) and strain ($\varepsilon_{ss}$, $\varepsilon_{\theta\theta}$, $\varepsilon_{\varsigma\varsigma}$) tensors at an arbitrary point of the shell at the $k$th step of loading are related by Hooke's law with additional stresses:

$$\sigma_{ss} = A_{11}\varepsilon_{ss} + A_{12}\varepsilon_{\theta\theta} - A_{1D}, \qquad \sigma_{\theta\theta} = A_{12}\varepsilon_{ss} + A_{22}\varepsilon_{\theta\theta} - A_{2D}, \tag{1.4}$$

$$\varepsilon_{\varsigma\varsigma} = -\frac{\nu}{1-\nu}(\varepsilon_{ss} + \varepsilon_{\theta\theta}) - \frac{1-2\nu}{1-\nu}(e_{ss}^{(\mathrm{n})} + e_{\theta\theta}^{(\mathrm{n})}) + \frac{1+\nu}{1-\nu}(\varepsilon_T + \varepsilon_0^{(\mathrm{p})} + \varepsilon_0^{(\mathrm{c})}), \tag{1.5}$$

$$A_{11} = A_{22} = \frac{E}{1-\nu^2}, \qquad A_{12} = \nu A_{11}, \tag{1.6}$$

$$A_{1D} = A_{11}[e_{ss}^{(\mathrm{n})} + \nu e_{\theta\theta}^{(\mathrm{n})} + (1+\nu)(\varepsilon_T + \varepsilon_0^{(\mathrm{p})} + \varepsilon_0^{(\mathrm{c})})],$$

$$A_{2D} = A_{11}[e_{\theta\theta}^{(\mathrm{n})} + \nu e_{ss}^{(\mathrm{n})} + (1+\nu)(\varepsilon_T + \varepsilon_0^{(\mathrm{p})} + \varepsilon_0^{(\mathrm{c})})], \tag{1.7}$$

where $E$ and $\nu$ are the temperature-dependent elastic modulus and Poisson's ratio; $E = 2G(1+\nu)$, $\varepsilon_T = \alpha_T(T - T_0)$, $G$ is the shear modulus; $\alpha_T$ is the coefficient of linear thermal expansion;

$$\varepsilon_0^{(\mathrm{p})} = \frac{\varepsilon_{ss}^{(\mathrm{p})} + \varepsilon_{\theta\theta}^{(\mathrm{p})} + \varepsilon_{\varsigma\varsigma}^{(\mathrm{p})}}{3}, \qquad \varepsilon_0^{(\mathrm{c})} = \frac{\varepsilon_{ss}^{(\mathrm{c})} + \varepsilon_{\theta\theta}^{(\mathrm{c})} + \varepsilon_{\varsigma\varsigma}^{(\mathrm{c})}}{3},$$

$e_{ss}^{(\mathrm{n})}$ $(s \to \theta)$ are the components of the plastic-strain deviator determined as the sums of their increments $\Delta_k e_{ss}^{(\mathrm{n})}$ $(s \to \theta)$:

$$e_{ss}^{(\mathrm{n})} = \sum_{i=1}^{k} \Delta_i e_{ss}^{(\mathrm{n})} \qquad (s \to \theta). \tag{1.8}$$

To determine $\varepsilon_0^{(\mathrm{p})}$ and $\varepsilon_0^{(\mathrm{c})}$, we will use the relationship between the first invariants of the stress and strain tensors, $\sigma_0 = (\sigma_{ss} + \sigma_{\theta\theta})/3$ and $\varepsilon_0 = (\varepsilon_{ss} + \varepsilon_{\theta\theta} + \varepsilon_{\varsigma\varsigma})/3$,

$$\sigma_0 = F_1(\varepsilon_0^*, T, \omega_\sigma), \tag{1.9}$$

$$\varepsilon_0^* = \varepsilon_0 - \varepsilon_T - \varepsilon_0^{(\mathrm{c})}, \tag{1.10}$$

$$\varepsilon_0^{(\mathrm{c})} = \sum_{i=1}^{k} \Delta_i \varepsilon_0^{(\mathrm{c})}, \tag{1.11}$$

$$\omega_\sigma = \frac{1}{3}\arccos\left[-\frac{3\sqrt{3}}{2}\frac{I_3(D_\sigma)}{S^3}\right] \qquad (0 \le \omega_\sigma \le \pi/3), \tag{1.12}$$

$$S = [(\sigma_{ss}^2 + \sigma_{\theta\theta}^2 - \sigma_{ss}\sigma_{\theta\theta})/3]^{1/2}, \tag{1.13}$$

where $\omega_\sigma$ is the stress mode angle; $I_3(D_\sigma)$ is the third invariant of the deviatoric stress tensor; $S$ is the shear-stress intensity. The increment $\Delta_k e_{ss}^{(\mathrm{n})}$ at an arbitrary $k$th step of loading is defined by

$$\Delta_k e_{ss}^{(\mathrm{n})} = \left\langle \frac{2\sigma_{ss} - \sigma_{\theta\theta}}{3S} \right\rangle_k \Delta_k \Gamma^{(\mathrm{n})} \qquad (s \to \theta), \tag{1.14}$$

where $\Delta_k \Gamma^{(\mathrm{n})}$ is the increment of plastic shear strain intensity,

$$\Delta_k \Gamma^{(\mathrm{n})} = \Delta_k \Gamma^{(\mathrm{p})} + \Delta_k \Gamma^{(\mathrm{c})}, \tag{1.15}$$

$\Delta_k \Gamma^{(\mathrm{p})}$ and $\Delta_k \Gamma^{(\mathrm{c})}$ are the increments of instantaneous plastic-shear-strain and creep-strain intensities. The angular brackets in (1.14) denote averaging over a step of loading. To determine $\Delta_k \Gamma^{(\mathrm{p})}$, we assume that

$$S = F_2(\Gamma^*, T, \omega_\sigma), \tag{1.16}$$

where $\Gamma^*$ is the intensity of instantaneous shear strains,

$$\Gamma^* = \frac{S}{2G} + \Gamma^{(p)}, \qquad \Gamma^{(p)} = \sum_{i=1}^{k} \Delta_i \Gamma^{(p)}. \tag{1.17}$$

The functions $F_1$ (1.9) and $F_2$ (1.16) are determined from the first series of base tests on tubular specimens under proportional loading, as described in [5, 8, 14, 15]. To determine $\Delta_k \varepsilon_0^{(c)}$ and $\Delta_k \Gamma^{(c)}$, we use the data of the second series of base (creep) tests. The following approximating expressions are proposed in [15]:

$$\dot{\Gamma}^{(c)}(S, T, \omega_\sigma) = \exp(c_2 \ln(c_1 S) + c_3 + c_4 T + c_5 \omega_\sigma + c_6 \omega_\sigma^2), \tag{1.18}$$

$$\dot{\varepsilon}_0^{(c)}(\sigma_0, T, \omega_\sigma) = \exp(d_2 \ln(d_1 \sigma_0) + d_3 + d_4 T + d_5 \omega_\sigma + d_6 \omega_\sigma^2), \tag{1.19}$$

where $c_i, d_i, i = 1, \ldots, 6$, are coefficients found from the best fit of expressions (1.18) and (1.19) to the test data. Then we have

$$\Delta_k \varepsilon_0^{(c)} = \dot{\varepsilon}_0^{(c)} \Delta_k t, \qquad \Delta_k \Gamma^{(c)} = \dot{\Gamma}^{(c)} \Delta_k t, \tag{1.20}$$

where $\Delta_k t = t_k - t_{k-1}$ is the duration of a step.

The constitutive equations (1.4), (1.5) differ by the presence of $\varepsilon_0^{(p)}$ and $\varepsilon_0^{(c)}$ in (1.5) and (1.7) from the equations of deformation along paths of small curvature [4] where the stress mode is disregarded. If the function $F_1$ (1.9) has the form $\sigma_0 = K(\varepsilon_0 - \varepsilon_T)$, $K = E/(1-2\nu)$, then $\varepsilon_0^{(p)} = 0$ and $\varepsilon_0^{(c)} = 0$. Expressions (1.5) and (1.7) depend only on the plastic strains (1.8), which are calculated using the function $F_2(\Gamma^*, T, \omega_\sigma)$ (1.16). If function (1.16) is independent of the stress mode and determined from uniaxial instantaneous tension tests (i.e., $\omega_\sigma = \pi/3$) and corresponding creep curves, then Eqs. (1.4)–(1.7) go over into the equations of deformation along paths of small curvature [4].

We will use formulas (1.4) to relate the forces, moments, and strains of the coordinate surface of the shell. Substituting (1.4) into the expressions for forces and moments and replacing the strains at an arbitrary point of the shell with their expressions (1.3), we obtain the formulas

$$\begin{aligned} N_s &= C_{11}^{(0)} \varepsilon_s + C_{12}^{(0)} \varepsilon_\theta + C_{11}^{(1)} \kappa_s + C_{12}^{(1)} \kappa_\theta - N_{1D}^{(0)}, \\ N_\theta &= C_{12}^{(0)} \varepsilon_s + C_{22}^{(0)} \varepsilon_\theta + C_{12}^{(1)} \kappa_s + C_{22}^{(1)} \kappa_\theta - N_{2D}^{(0)}, \\ M_s &= C_{11}^{(1)} \varepsilon_s + C_{12}^{(1)} \varepsilon_\theta + C_{11}^{(2)} \kappa_s + C_{12}^{(2)} \kappa_\theta - N_{1D}^{(1)}, \\ M_\theta &= C_{12}^{(1)} \varepsilon_s + C_{22}^{(1)} \varepsilon_\theta + C_{12}^{(2)} \kappa_s + C_{22}^{(2)} \kappa_\theta - N_{2D}^{(1)}, \end{aligned} \tag{1.21}$$

$$C_{mn}^{(j)} = \sum_{i=1}^{l} \int_{\varsigma_{i-1}}^{\varsigma_i} A_{mn}^{(i)} \varsigma^j d\varsigma, \quad N_{mD}^{(j)} = \sum_{i=1}^{l} \int_{\varsigma_{i-1}}^{\varsigma_i} A_{mD}^{(i)} \varsigma^j d\varsigma \quad (m, n = 1, 2,\ j = 0, 1, 2). \tag{1.22}$$

Thus, we have all the equations (equilibrium, kinematic, constitutive) needed to determine the axisymmetric SSS of a layered shell taking the stress mode into account.

**2. Governing Equations.** Equations (1.21) with (1.1) and (1.2) constitute a system of 12 equations. Let us choose the functions $N_s, Q_s, M_s, u, w, \vartheta_s$ as basic unknowns, for which various boundary conditions can be formulated. Expressing the other unknown functions in terms of the basic unknown functions, we reduce the above system to a system of six ordinary differential equations:

$$\frac{d\vec{Y}}{ds} = P(s)\vec{Y} + \vec{f}(s) \tag{2.1}$$

with the boundary conditions

$$B_1\vec{Y}(s_a) = \vec{b}_1, \quad B_2\vec{Y}(s_b) = \vec{b}_2, \tag{2.2}$$

where $\vec{Y} = \{N_s, Q_s, M_s, u, w, \vartheta_s\}$ is the column vector of unknown functions; $P(s)$ is the matrix of the system; $\vec{f}(s)$ is the column vector of additional terms; $B_1$ and $B_2$ are given matrices; $\vec{b}_1$ and $\vec{b}_2$ are given column vectors of boundary conditions. The elements of the matrix $P(s)$ are defined by the formulas

$$p_{11} = -\frac{\cos\varphi}{r}(1+\lambda_1), \quad p_{13} = -\frac{\cos\varphi}{r}\lambda_2, \quad p_{14} = \frac{\cos^2\varphi}{r^2}(\lambda_1 C_{12}^{(0)} + C_{22}^{(0)} + \lambda_2 C_{12}^{(1)}),$$

$$p_{15} = p_{14}\tan\varphi, \quad p_{16} = -\frac{\cos^2\varphi}{r^2}(\lambda_3 C_{12}^{(0)} + \lambda_4 C_{12}^{(1)} - C_{22}^{(1)}), \quad p_{21} = \frac{1}{R_s} - \frac{\lambda_1 \sin\varphi}{r},$$

$$p_{22} = -\frac{\cos\varphi}{r}, \quad p_{23} = -\frac{\sin\varphi}{r}\lambda_2, \quad p_{24} = p_{15}, \quad p_{25} = p_{15}\tan\varphi, \quad p_{26} = p_{16}\tan\varphi,$$

$$p_{31} = -p_{22}\lambda_3, \quad p_{32} = -1, \quad p_{33} = -\frac{\cos\varphi}{r}(1-\lambda_4),$$

$$p_{34} = \frac{\cos^2\varphi}{r^2}(\lambda_1 C_{12}^{(1)} + C_{22}^{(1)} + \lambda_2 C_{12}^{(2)}), \quad p_{35} = p_{34}\tan\varphi,$$

$$p_{36} = -\frac{\cos^2\varphi}{r^2}(\lambda_3 C_{12}^{(1)} - C_{22}^{(2)} + \lambda_4 C_{12}^{(2)}), \quad p_{41} = \frac{C_{11}^{(2)}}{\delta}, \quad p_{42} = 0, \quad p_{43} = -\frac{C_{11}^{(1)}}{\delta},$$

$$p_{44} = -\lambda_1 p_{22}, \quad p_{45} = -p_{21}, \quad p_{46} = -p_{31},$$

$$p_{51} = p_{52} = p_{53} = 0, \quad p_{54} = -p_{12}, \quad p_{55} = 0, \quad p_{56} = -1,$$

$$p_{61} = p_{43}, \quad p_{62} = 0, \quad p_{63} = \frac{C_{11}^{(0)}}{\delta}, \quad p_{64} = -p_{13}, \quad p_{65} = -p_{23}, \quad p_{66} = \lambda_4 p_{22}, \tag{2.3}$$

$$\lambda_1 = (C_{11}^{(1)}C_{12}^{(1)} - C_{12}^{(0)}C_{11}^{(2)})/\delta, \quad \lambda_2 = (C_{11}^{(1)}C_{12}^{(0)} - C_{12}^{(1)}C_{11}^{(0)})/\delta,$$

$$\lambda_3 = (C_{11}^{(2)}C_{12}^{(1)} - C_{12}^{(2)}C_{11}^{(1)})/\delta, \quad \lambda_4 = (C_{11}^{(0)}C_{12}^{(2)} - C_{12}^{(1)}C_{11}^{(2)})/\delta,$$

$$\delta = C_{11}^{(0)}C_{11}^{(2)} - (C_{11}^{(1)})^2.$$

The components of the vector $\vec{f}(s)$ are defined by

$$f_1 = -\frac{\cos\varphi}{r}[\lambda_1 N_{1D}^{(0)} + \lambda_2 N_{1D}^{(1)} + N_{2D}^{(0)}] - q_s,$$

$$f_2 = -\frac{\sin\varphi}{r}[\lambda_1 N_{1D}^{(0)} + \lambda_2 N_{1D}^{(1)} + N_{2D}^{(0)}] - q_\varsigma,$$

$$f_3 = \frac{\cos\varphi}{r}[\lambda_3 N_{1D}^{(0)} + \lambda_4 N_{1D}^{(1)} - N_{2D}^{(0)}],$$

$$f_4 = \frac{C_{11}^{(2)} N_{1D}^{(0)} - C_{11}^{(1)} N_{1D}^{(1)}}{\delta}, \qquad f_5 = 0, \qquad f_6 = -\frac{C_{11}^{(1)} N_{1D}^{(0)} - C_{11}^{(0)} N_{1D}^{(1)}}{\delta}. \tag{2.4}$$

As is seen from (2.3) and (2.4), the matrix of the system depends on the thermal, mechanical, and geometric characteristics of the shell, while the components of the vector $\vec{f}(s)$ additionally depend on the plastic strains, which should be corrected using functions (1.9) and (1.16) during successive approximations.

**3. Problem-Solving Algorithm.** Let we know the shape of the meridian, the number and thicknesses of the layers, boundary conditions, magnitude and law of variation in the load as well as the functions $F_1$ (1.9) and $F_2$ (1.16) for several constant values of temperature and angle $\omega_\sigma$, Poisson's ratio, and the coefficient of linear thermal expansion depending on temperature. Also, the coefficients of (1.18), (1.19) must be set. To solve the problem, it is necessary to divide the loading process into small steps. It is convenient that the loads at the first step be such that the body deforms elastically. Then we can set $e_{ss}^{(\mathrm{n})} = e_{\theta\theta}^{(\mathrm{n})} = 0$, $\varepsilon_0^{(\mathrm{p})} = \varepsilon_0^{(\mathrm{c})} = 0$ in Eqs. (1.5), (1.7) and solve the thermoelastic problem (2.1), (2.2) for the shell to obtain its SSS at the first step. The process of successive approximations at the second and subsequent steps will use the displacements, strains, and stresses at the end of the previous step. Initiating the process of successive approximations at the second or the $k$th step, we assume that $(\sigma_{ss})_{k-1}$, $(\sigma_{\theta\theta})_{k-1}$, $(e_{ss}^{(\mathrm{n})})_{k-1}$, $(e_{\theta\theta}^{(\mathrm{n})})_{k-1}$, $(\Gamma^{(\mathrm{p})})_{k-1}$, $(\varepsilon_0^{(\mathrm{p})})_{k-1}$, $(\varepsilon_0^{(\mathrm{c})})_{k-1}$ are known. Next, we use $(e_{ss}^{(\mathrm{n})})_{k-1}$, $(e_{\theta\theta}^{(\mathrm{n})})_{k-1}$ and $(\varepsilon_0^{(\mathrm{p})})_{k-1}$, $(\varepsilon_0^{(\mathrm{c})})_{k-1}$ to calculate $A_{1D}$ and $A_{2D}$ (1.7) and to solve the boundary-value problem (2.1), (2.2) for loads and boundary conditions corresponding to the end of the current step. After determining the SSS, we can find $\varepsilon_0$ and $\omega_\sigma$ (1.12) and $\varepsilon_0^*$ (1.10). By linear interpolation in temperature and angle $\omega_\sigma$, we draw the curve of $F_1$ to find $\sigma_0$ corresponding to $\varepsilon_0^*$ and calculate $\varepsilon_0^{(\mathrm{p})} = \varepsilon_0^* - \sigma_0 / K$. The values of $\sigma_0$, $\omega_\sigma$ and temperature at the end of the step are used to determine $\Delta\varepsilon_0^{(\mathrm{c})} = \dot{\varepsilon}_0^{(\mathrm{c})} \cdot \Delta t$ and $\varepsilon_0^{(\mathrm{c})} = (\varepsilon_0^{(\mathrm{c})})_{k-1} + \Delta\varepsilon_0^{(\mathrm{c})}$. Next we calculate $\Delta_k \Gamma^{(\mathrm{p})}$. To this end, we use linear interpolation in temperature and angle $\omega_\sigma$ to draw the curve of $F_2$ to determine $S^{(\mathrm{d})}$ corresponding to

$$\Gamma^* = \Gamma_{k-1}^{(\mathrm{p})} + \Delta_k \Gamma^{(\mathrm{p})} + \frac{S}{2G}, \tag{3.1}$$

where $S$ is given by (1.13). In the general case, at the $M$th iteration of the $k$th step, we have

$$\Delta_k \Gamma^{(\mathrm{p})} = \sum_{m=1}^{M-1} \Delta_m \Gamma^{(\mathrm{p})} + \frac{S - S^{(\mathrm{d})}}{2G}. \tag{3.2}$$

Then we find $\dot{\Gamma}^{(\mathrm{c})}$ (1.18) using the values of $S^{(\mathrm{d})}$, $\omega_\sigma$, and temperature at the end of the step and calculate $\Delta_k \Gamma^{(\mathrm{c})}$ (1.20). Next we determine $\Delta_k \Gamma^{(\mathrm{n})}$ (1.15), the increments of the components of the plastic strain deviator (1.14), and the values of these components (1.8). After correcting $A_{1D}$ and $A_{2D}$ (1.7) using the corrected values of $\varepsilon_0^{(\mathrm{p})}$, $\varepsilon_0^{(\mathrm{c})}$, $e_{ss}^{(\mathrm{n})}$, $e_{\theta\theta}^{(\mathrm{n})}$, we can find the next approximation of the solution of the boundary-value problem (2.1), (2.2). Thus, each approximation at the current step of loading requires the solutions found in the previous approximation and at the previous step. The process of successive approximations is terminated once

$$\left| \frac{\sigma_0 - F_1(\varepsilon_0^*, T, \omega_\sigma)}{K} \right| \le \delta_1, \qquad \left| \frac{S - F_2(\Gamma^*, T, \omega_\sigma)}{2G} \right| \le \delta_2, \tag{3.3}$$

where $\delta_1$ and $\delta_2$ are measures of accuracy with which the calculated values of $\omega_\sigma$, $\varepsilon_0^*$, $\sigma_0$, and $T$ satisfy Eq. (1.9) and the values of $\omega_\sigma$, $S$, $\Gamma^*$, and $T$ satisfy Eq. (1.16). Computational experience suggests that $\delta_1 < \delta_2$.

The above algorithm is used only if elements of the body undergo active loading. To identify whether the process is active loading or unloading in each element where plastic deformation occurs ($\Gamma^{(\mathrm{p})} > 0$) at an arbitrary step, it is necessary to test the following condition after finding the first approximation:

TABLE 1

| Step number | $N_s^*$, mN/m | $q_\varsigma^*$, MPa | $T$, °C | $t$, min |
|---|---|---|---|---|
| 1 | 1641 | 8.26 | 590 | 0 |
| 2 | 1753 | 8.43 | 606 | 1.5 |
| 3 | 1874 | 8.79 | 615 | 3.3 |
| 4 | 1914 | 9.03 | 633 | 4.3 |
| 5 | 2006 | 9.30 | 642 | 5.5 |
| 6 | 2114 | 9.72 | 654 | 6.7 |
| 7 | 2184 | 10.39 | 660 | 8 |
| 8 | 2268 | 11.31 | 674 | 9.3 |
| 9 | 2287 | 11.65 | 682 | 10.8 |
| 10 | 2289 | 11.81 | 691 | 11.1 |

$$\Delta\Gamma^{(p)} \geq 0. \tag{3.4}$$

If condition (3.4) is satisfied, then the process is active loading; otherwise, unloading. Then we should set $\Delta\Gamma^{(p)} = 0$, $\Delta\Gamma^{(c)} = 0$ in this element and use $e_{ss}^{(n)}$, $e_{\theta\theta}^{(n)}$ and $\varepsilon_0^{(p)}$, $\varepsilon_0^{(c)}$ corresponding to the end of the previous step to determine $A_{1D}$ and $A_{2D}$ needed to solve the boundary-value problem in the subsequent approximations of the step. Whether the steps of loading are properly chosen is checked by repeating calculations with smaller steps. The steps are reduced until the values obtained at the end of the process with different number of steps differ by less than a specified small amount.

**4. Numerical Results.** Let us discuss the solutions of some problems found with the above approach. We will consider an example to analyze the practical convergence of the method of successive approximations in the presence of creep strains. Consider a thin cylindrical shell (made of Kh18N10T material) with a mid-surface radius of 10 cm, a length of 20 cm, and a thickness of 1 cm. At the initial time $t = t_0 = 0$, the shell is in stress/strain-free state at temperature $T = T_0 = 20$ °C. Let us determine its thermoviscoelastoplastic SSS after it is first uniformly heated to temperature $T = 580$ °C and then subjected to a tensile force $N_s$, uniform internal pressure $q_\varsigma$, and further uniform heating. Table 1 collects values of loads and temperature at certain time points throughout 10 steps of loading. These data represents the test on a tubular specimen described in [15].

The material properties are represented by tables of the functions $F_1$ (1.9) and $F_2$ (1.16) [5, 8] for $T = 500$ °C, $T = 700$ °C; $\omega_\sigma = 0$, $\omega_\sigma = \pi/6$, $\omega_\sigma = \pi/3$, $\nu = 0.27$, $\alpha_T = 0.1 \cdot 10^{-4}$ °C$^{-1}$.

By symmetry, we may consider just half the shell. The boundary conditions at an arbitrary $k$th step $(1 \leq k \leq 10)$ are given by $s = s_a = 0$, $u = Q_s = \vartheta_s = 0$, $s = s_b = 10$ cm, $N_s = N_{sk}^*$, $Q_s = \vartheta_s = 0$, where the values of $N_{sk}^*$ and $q_{\varsigma k}^*$ for each step are summarized in Table 1. Under such loading conditions, the stress–strain state of the shell is homogeneous. To solve the problem, we use the method of successive approximations described above. Note that this problem can be solved in a different way. The problem is statically determinate, the stresses in the shell are determined by the loads, and the strains can be found from the constitutive equations. The strains determined from the constitutive equations are considered exact, while the strains calculated with the above algorithm for different values of $\delta_2$ (3.3) are considered approximate. Some calculated results are presented in Tables 2–4.

TABLE 2

| Step number | $\varepsilon_{ss} \cdot 10^5$ | | | |
|---|---|---|---|---|
| | Exact value | Approximate value $\delta_2 = 1 \cdot 10^{-4}$ | Approximate value $\delta_2 = 1 \cdot 10^{-7}$ | Stress mode disregarded |
| 1 | 1484 | 1469 | 1484 | 1464 |
| 2 | 1606 | 1592 | 1605 | 1577 |
| 3 | 2001 | 1910 | 2000 | 1807 |
| 4 | 2349 | 2260 | 2348 | 2009 |
| 5 | 2797 | 2697 | 2795 | 2266 |
| 6 | 3658 | 3511 | 3657 | 2578 |
| 7 | 4380 | 4220 | 4378 | 2783 |
| 8 | 6199 | 5838 | 6195 | 3118 |
| 9 | 7180 | 6792 | 7176 | 3231 |
| 10 | 7667 | 7271 | 7663 | 3307 |

TABLE 3

| Step number | $\varepsilon_{\theta\theta} \cdot 10^5$ | | | |
|---|---|---|---|---|
| | Exact value | Approximate value $\delta_2 = 1 \cdot 10^{-4}$ | Approximate value $\delta_2 = 1 \cdot 10^{-7}$ | Stress mode disregarded |
| 1 | 605 | 605 | 605 | 598 |
| 2 | 623 | 623 | 623 | 612 |
| 3 | 625 | 628 | 625 | 615 |
| 4 | 635 | 638 | 635 | 627 |
| 5 | 632 | 635 | 632 | 626 |
| 6 | 613 | 618 | 613 | 624 |
| 7 | 617 | 622 | 617 | 625 |
| 8 | 675 | 684 | 675 | 639 |
| 9 | 746 | 755 | 746 | 649 |
| 10 | 792 | 800 | 792 | 661 |

TABLE 4

| Step number | $\varepsilon_{\varsigma\varsigma}\cdot 10^5$ | | | |
|---|---|---|---|---|
| | Exact value | Approximate value $\delta_2 = 1\cdot 10^{-4}$ | Approximate value $\delta_2 = 1\cdot 10^{-7}$ | Stress mode disregarded |
| 1 | –285 | –270 | –285 | –280 |
| 2 | –361 | –347 | –361 | –355 |
| 3 | –712 | –624 | –711 | –556 |
| 4 | –1003 | –917 | –1002 | –713 |
| 5 | –1400 | –1303 | –1399 | –938 |
| 6 | –2173 | –2031 | –2172 | –1207 |
| 7 | –2806 | –2652 | –2804 | –1391 |
| 8 | –4428 | –4077 | –4425 | –1691 |
| 9 | –5286 | –4908 | –5283 | –1789 |
| 10 | –5709 | –5323 | –5705 | –1848 |

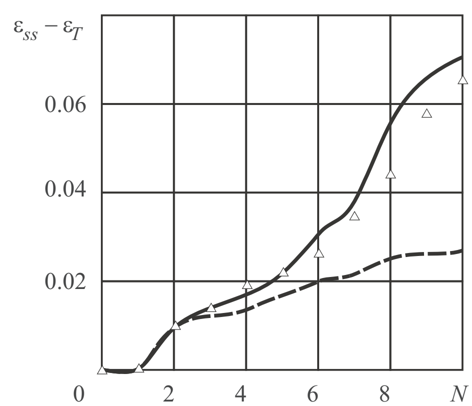


Fig. 1

Column 1 indicates step numbers, column 2 contains the exact values of $\varepsilon_{ss}$, $\varepsilon_{\theta\theta}$, and $\varepsilon_{\varsigma\varsigma}$, and columns 3 and 4 approximate values obtained by the method of successive approximations for $\delta_2 = 1\cdot 10^{-4}$ and $\delta_2 = 1\cdot 10^{-7}$. Column 5 collects the values of strains obtained using the theory of deformation along paths of small curvature regardless of the stress mode. Tables 2–4 indicate that the approximate values tend to the exact values with decrease in $\delta_2$ (3.3), and at $\delta_2 = 1\cdot 10^{-7}$, the approximate and exact values of strains differ only in the fifth decimal place. Thus, the algorithm allows obtaining results with prescribed accuracy.

Comparing the strains calculated from the constitutive equations allowing for the stress mode and the strains calculated from the theory of deformation along paths of small curvature disregarding the stress mode, we see that the values of $\varepsilon_{ss}$ and $\varepsilon_{\varsigma\varsigma}$ at the end of the process found in the former case are approximately 2 and 3 times, respectively, greater than in the latter case.

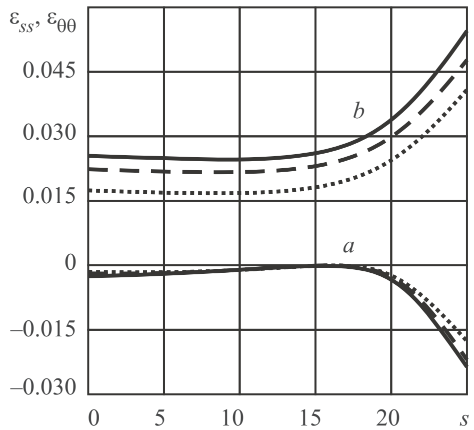


Fig. 2

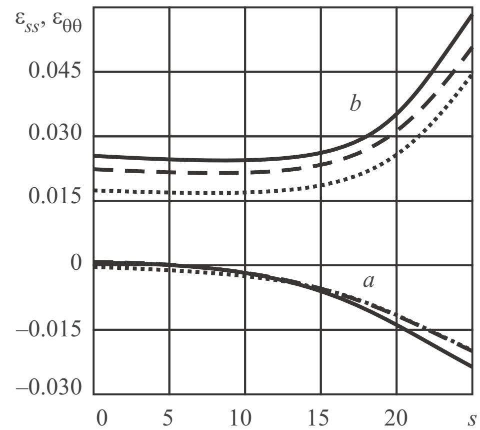


Fig. 3

Also, the creep-strain intensity at the end of the process is 13% of the total-strain intensity in the former case and is negligible ($1\cdot 10^{-5}$) compared with the instantaneous-strain intensity in the latter case. Thus, the results obtained using the constitutive equations incorporating the stress mode and using the equations describing deformation along paths of small curvature regardless of the stress mode differ both quantitatively and qualitatively.

The calculated strains are compared with the experimental data from [5]. Figure 1 shows the strain $\varepsilon_{ss} - \varepsilon_T$ versus the step number $N$ calculated with (solid line) and without (dashed line) regard to the stress mode and their experimental values (triangles). It can be seen that the strains at the end of the process calculated with and without regard to the stress mode are higher by approximately 10% and lower by more than 50%, respectively, than the experimental values. The strains $\varepsilon_{\theta\theta} - \varepsilon_T$ are considerably (by an order of magnitude) less than the strains $\varepsilon_{ss} - \varepsilon_T$. They range from –0.001 to 0.002 in both experiment and calculations; therefore, they are omitted here. Thus, the calculations confirm that the constitutive equations [14, 15], which incorporate the stress mode, adequately describe the thermoviscoplastic deformation of Kh18N10T material.

Now consider a two-layer shell whose mid-surface meridian is a circular arc of radius $R_s = 40$ cm symmetric about the axis $s = 0$; the length of half-meridian $s_b - s_a = 25$ cm, the parallel radius $r = r_0 = 20$ cm at $s = s_a = 0$, the thickness of the layers $h_1 = h_2 = 1$ cm. The shell, which is initially in stress/strain-free state at temperature $T_0 = 20$ °C, is subjected to increasing uniform internal pressure and steady-state temperature field varying only across the thickness of the shell. The internal pressure changes from 28 to 54 MPa. The temperature of the shell $T(\varsigma_0) = 20$ °C, $T(\varsigma_1) = 81$ °C, $T(\varsigma_2) = 200$ °C. These values are chosen [2] so that the steady-state temperature field of the shell is similar to that of a two-layer sphere of radius $R_s$. The inner layer of the shell is made of Kh18N10T material whose properties depending on temperature and stress mode are given in [5, 8, 14, 15]. The outer layer is made of 30KhGSA material whose properties depending on temperature but not on the stress mode are given in [4]. A half of the shell is considered. Its edge is hinged and free in the radial direction: $s_a = 0$, $u = 0$, $Q_s = 0$, $\vartheta_s = 0$, $s_b = 25$ cm; $M_s = 0$, $u\sin\theta - w\cos\theta = 0$, $N_s\cos\theta + Q_s\sin\theta = 0$, $\theta = \pi - (s_b - s_a)/R_s$.

The loading period is divided into 51 steps. Three cases are examined: (i) the material properties of the layers depend on the temperature and stress mode of the inner layer; (ii) the material properties of the layers depend on the temperature alone; (iii) the material properties depend only on the stress mode of the inner layer.

An analysis shows that both materials deform beyond the elastic limit at the end of the loading process and unloading does not occur. Figures 2–5 show some calculated results for cases (i), (ii), and (iii) by solid, dashed, and dotted curves, respectively. These figures show the variation in the strains $\varepsilon_{ss}$, $\varepsilon_{\theta\theta}$ (Figs. 2 and 3) and the stresses $\sigma_{ss}$, $\sigma_{\theta\theta}$ (MPa) (Figs. 4 and 5) along the meridian $s$ (cm) for $\varsigma = -h_1$ (Figs. 2 and 4) and $\varsigma = h_2$ (Figs. 3 and 5), lines $a$ and $b$ corresponding to $\varepsilon_{ss}$, $\sigma_{ss}$ and $\varepsilon_{\theta\theta}$, $\sigma_{\theta\theta}$, respectively. It can be seen that allowing for the stress mode increases the maximum meridional and circumferential strains by more than 20% (Fig. 3) and 15%, respectively, decreases the stresses in the inner layer by approximately 12%, and slightly increases the stresses in the outer layer. It should be noted that allowing for the dependence of the material properties of the inner layer on the stress mode affects not only the SSS in this layer, but also the SSS in the outer layer whose properties do not depend on the stress mode. Also, allowing for the temperature dependence of the material properties changes the maximum strains and stresses by approximately 40% and 10%, respectively.

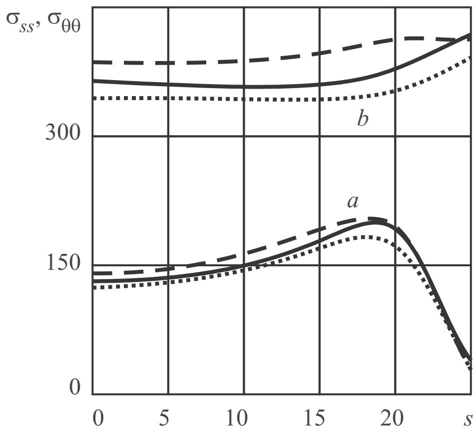


Fig. 4

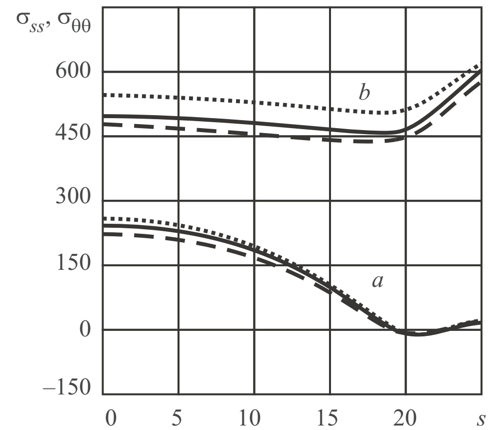


Fig. 5

Thus, allowing for the dependence of the material properties of the layers on temperature and one of the layers on the stress mode has a strong effect on the SSS of the shell.

**Conclusions.** The basic results of this study are the following.

1. We have developed a procedure to determine the thermoelastoplastic SSS of layered shells made of isotropic materials with properties dependent on temperature and stress mode under axisymmetric loading that induces creep deformation. The procedure has been tested by solving a test problem.

2. It has been shown that using the method of successive approximations makes it possible to solve the boundary-value problem with prescribed accuracy. The results obtained with this procedure are in good agreement with the experimental data reported in [15].

3. The thermoplastic problem for a two-layer barrel-shaped shell has been solved. The dependence of the material properties of the layers on temperature and stress mode has a strong effect on the SSS of the shell.

## REFERENCES


1. L. M. Kachanov, *Fundamentals of the Theory of Plasticity*, Dover, New York (2004).
2. I. A. Motovilovets and V. I. Kozlov, *Thermoelasticity*, Vol. 1 of the six-volume series *Mechanics of Coupled Fields in Structural Members* [in Russian], Naukova Dumka, Kyiv (1987).
3. V. V. Novozhilov, *Theory of Elasticity*, Pergamon Press, Oxford (1961).
4. Yu. N. Shevchenko, M. E. Babeshko, and R. G. Terekhov, *Thermoviscoelastoplastic Processes of Combined Deformation of Structural Members* [in Russian], Naukova Dumka, Kyiv (1992).
5. M. E. Babeshko and Yu. N. Shevchenko, "Describing the thermoelastoplastic deformation of compound shells under axisymmetric loading with allowance for the third invariant of stress deviator," *Int. Appl. Mech.*, **46**, No. 12, 1362–1371 (2010).
6. J. Betten, "Mathematical modeling of material behaviour under creep conditions," *Appl. Mech. Rev.*, **54**, No. 2, 107–132 (2001).
7. G. G. Chen and T. R. Hsu, "A mixed explicit-implicit (EL) algorithms for creep stress analysis," *Int. J. Numer. Meth. Eng.*, **26**, No. 2, 511–524 (1988).
8. A. Z. Galishin and Yu. N. Shevchenko, "Determining the axisymmetric thermoelastoplastic state of thin shells with allowance for the third invariant of the deviatoric stress tensor," *Int. Appl. Mech.*, **49**, No. 6, 675–684 (2013).
9. R. Hill, *The Mathematical Theory of Plasticity*, Clarendon Press, Oxford (1950).
10. S. Murakami and N. Ohno, "A continuum theory of creep and creep damage," in: *Creep in Structures*, Springer-Verlag, Berlin (1981), pp. 422–444.
11. F. K. G. Odquist, *Mathematical Theory of Creep and Creep Rupture*, Clarendon Press, Oxford (1966).

12. Yu. N. Shevchenko and M. E. Babeshko, “The thermoviscoelastoplastic state of laminated shells of revolution under axisymmetric loading,” in: *Proc. 6th Int. Congr. on Thermal Stresses* (May 26–29, 2005, Vienna, Austria), 1, Vienna (2005), pp. 237–230.
13. A. Yu. Shevchenko, M. V. Banyas, and I. K. Senchenkov, “A variant of the equations of nonisothermal plastic flow,” *Int. Appl. Mech.*, **48**, No. 5, 602–607 (2012).
14. Yu. N. Shevchenko, R. G. Terekhov, and N. N. Tormakhov, “Elastoplastic deformation of elements of an isotropic solid along paths of small curvature: Constitutive equations incorporating the stress mode,” *Int. Appl. Mech.*, **43**, No. 6, 621–630 (2007).
15. Yu. N. Shevchenko and N. N. Tormakhov, “Thermoviscoplastic deformation along paths of small curvature: Constitutive Equations including the third deviatoric stress invariant,” *Int. Appl. Mech.*, **48**, No. 6, 688–699 (2012).
16. A. Zolochevsky, “Creep of isotropic and anisotropic materials with different behaviour in tension and compression,” in: M. Zyczkowski (ed.), *Creep in Structures*, Springer-Verlag, Berlin (1991), pp. 217–220.
17. A. Zolochevsky, A. Galishin, S. Sklepus, and G. Z. Voyiadjis, “Analysis of creep deformation and creep damage in thin-walled branches shells from materials with different behavior in tension and compression,” *Int. J. Solids Struct.*, **44**, No. 16, 5075–5100 (2007).
18. A. Zolochevsky, S. Sklepus, T. H. Hyde, A. A. Becker, and S. Paravali, “Numerical modeling of creep and creep damage in thin plates of arbitrary shape from materials with different behavior in tension and compression under plane stress condition,” *Int. J. Numer. Meth. Eng.*, **80**, No. 11, 1406–1436 (2009).