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NATURAL MODES OF VIBRATION OF PIEZOELECTRIC CIRCULAR PLATES WITH RADIALLY CUT ELECTRODES

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The general solution to the problem of the nonaxisymmetric electromechanical vibrations of a piezoceramic ring plate is obtained. The spectra of natural frequencies and modes for the first circumferential harmonics are numerically determined and analyzed for plates with radially cut electrode coating and the following boundary conditions: clamped edge–free edge, free edge–clamped edge, free edge–free edge

Keywords: piezoceramic ring plate, radially cut electrode, nonaxisymmetric electromechanical vibrations, spectra of natural frequencies

Introduction. Thin piezoelectric planar transducers with thickness polarization are used in devices of various functionality [2–4, 6–8, 10–12]. Disk- and ring-shaped vibrators with solid electrodes on the faces undergo axisymmetric vibrations [4, 9]. The vibrations will be nonaxisymmetric with respect to the circumferential coordinate if the electroelastic sectors of a ring plate with radially cut electrodes are excited in antiphase. The circumferential vibration modes are a priori determined by the number of radial cuts in the electrodes [2]. There have been no systematic theoretical studies of the frequency spectrum and radial vibration modes. Theses aspects are addressed in the present paper.

1. Problem Formulation and General Solution. Consider a thin piezoelectric plate of thickness h. To describe the plate, we will use a cylindrical coordinate system r, θ, z with the plane $z = 0$ coinciding with the midsurface of the plate. If a thickness-polarized thin piezoceramic plate with electroded faces $z = \pm h/2$ is in plane stress state ($u_r(r,\theta,t)$, $u_\theta(r,\theta,t)$, $\sigma_{zz} = \sigma_{z\theta} = \sigma_{zr} = 0$, $E_r = E_\theta = 0$, $E_z(r,\theta,t)$), then the formulas below follow from the general constitutive equations [2, 5, 7, 10, 13]:

$$\sigma_{rr} = \frac{1}{(r,\theta,t)s_{11}^E}\left[\frac{\partial u_r}{\partial r} + \nu_E\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) - (1+\nu_E)d_{31}E_z\right],$$

$$\sigma_{\theta\theta} = \frac{1}{(1-\nu_E^2)s_{11}^E}\left(\nu_E\,\frac{\partial u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) - (1+\nu_E)d_{31}E_z),$$

$$\sigma_{r\theta} = \frac{1}{2(1+\nu_E)s_{11}^E}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right), \tag{1}$$

where we have used the formulas for strains and $s_{66}^E = 2(s_{11}^E - s_{12}^E)$, $\nu_E = -s_{12}^E / s_{11}^E$. If the thickness acceleration is neglected, the three equations of mechanical vibrations degenerate into two:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} = \rho\frac{\partial^2 u_r}{\partial t^2},$$

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$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{2\sigma_{r\theta}}{r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}=\rho\frac{\partial^2 u_\theta}{\partial t^2}. \tag{2}$$

After simple transformations in (1) and (2), we arrive at the vibration equations for displacements:

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}+\frac{1-\nu_E}{2}\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{1+\nu_E}{2}\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta}$$

$$+\frac{3-\nu_E}{2}\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}-(1+\nu_E)d_{31}\frac{\partial E_z}{\partial r}=(1-\nu_E^2)s_{11}^E\rho\frac{\partial^2 u_r}{\partial t^2},$$

$$\frac{1+\nu_E}{2}\frac{1}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{3-\nu_E}{2}\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1-\nu_E}{2}\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\right)$$

$$+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}-(1+\nu_E)d_{31}\frac{1}{r}\frac{\partial E_z}{\partial\theta}=(1-\nu_E^2)s_{11}^E\rho\frac{\partial^2 u_\theta}{\partial t^2}. \tag{3}$$

The solution of the system of equations (3) can be represented [3] in the form

$$u_r=\frac{\partial\Phi}{\partial r}+\frac{1}{r}\frac{\partial\Psi}{\partial\theta},$$

$$u_\theta=\frac{1}{r}\frac{\partial\Phi}{\partial\theta}-\frac{\partial\Psi}{\partial r}. \tag{4}$$

The functions $\Phi(r,\theta,t)$ and $\Psi(r,\theta,t)$ determined from the following wave equations satisfy Eqs. (3):

$$\Delta\Phi-(1+\nu_E)d_{31}E_z=(1-\nu_E^2)s_{11}^E\rho\frac{\partial^2\Phi}{\partial t^2},$$

$$\Delta\Psi=2(1+\nu_E)s_{11}^E\rho\frac{\partial^2\Psi}{\partial t^2}. \tag{5}$$

The electric potential for a plate with solid electrodes on the faces $z=\pm h/2$ is given by $\varphi=h^{-1}zV_0(t)$, the edge effect being neglected. This potential corresponds, according to [2, 5, 6, 9], to an electric field with $E_r=E_\theta=0, E_z=h^{-1}V_0(t)$; hence, the term $(1+\nu_E)d_{31}E_z$ in Eq. (5) should be omitted, considering (3).

The following expressions for stresses in terms of the potentials Φ and Ψ can be derived from (1), (4):

$$\sigma_{rr}=\frac{1}{(1-\nu_E^2)s_{11}^E}\left[\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)\right)+\nu_E\left(\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)\right)-(1+\nu_E)d_{13}E_z\right],$$

$$\sigma_{\theta\theta}=\frac{1}{(1-\nu_E^2)s_{11}^E}\left[\nu_E\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)\right)+\left(\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)\right)-(1+\nu_E)d_{13}E_z\right],$$

$$\sigma_{r\theta}=\frac{1}{2(1+\nu_E)s_{11}^E}\left(\frac{2}{r}\frac{\partial^2\Phi}{\partial r\partial\theta}-\frac{2}{r^2}\frac{\partial\Phi}{\partial\theta}-\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right). \tag{6}$$

The homogeneous boundary conditions for displacements and stresses (at $r=r_0$ and $r=r_1$) in a circular piezoceramic plate of radius r_1 with a hole of radius r_0 are taken one from each of the following two pairs ($j=0,1$):

$$u_r(r_j,\theta,t)=0\ \wedge\ \sigma_{rr}(r_j,\theta,t)=0,$$

$$u_\theta(r_j,\theta,t)=0 \ \wedge \ \sigma_{r\theta}(r_j,\theta,t)=0. \tag{7}$$

The initial conditions for steady-state harmonic vibrations are not formulated.

Consider a circular piezoceramic plate $r_0 < r < r_1$ with electrode coating on its faces $z=\pm h/2$ cut into $2N$ sectors. Adjacent sectors are connected in antiphase so that $E_{za}=(-1)^{n-1}V_0/h$, $n=1,\dots,2N$. If vibrations are harmonic, $f(r,\theta,t)=\mathrm{Re}\, f^a(r,\theta)\exp i\omega t$, where ω is the angular frequency, then a candidate solution to Eqs. (5) (the term $(1+\nu)d_{31}E_z$ in the first equation should be equated to zero [3]) in polar coordinates r,θ can be chosen in the form of series:

$$\Phi(r,\theta,t)=R^2\ \mathrm{Re}\sum_m \{A_{m,1}J_m(k_1r)+A_{m,2}Y_m(k_1r)\}\sin m\theta\exp i\omega t,$$

$$\Psi(r,\theta,t)=R^2\ \mathrm{Re}\sum_m \{A_{m,3}J_m(k_2r)+A_{m,4}Y_m(k_2r)\}\cos m\theta\exp i\omega t, \tag{8}$$

where $J_m(k_jr)$ and $Y_m(k_jr)$ are mth-order cylindrical functions of the first and second kinds [1]; $k_1^2=(1-\nu_E^2)s_{11}^E\rho\,\omega^2$, $k_2^2=2(1+\nu_E)s_{11}^E\rho\omega^2$; $A_{m,i}$ are dimensionless constants.

From (4), (6), and (8), we can find [5, 12, 13] the displacements

$$\begin{aligned} u_r &= R\,\mathrm{Re}\sum_m [u_{m1}(k_1r)A_{m,1}+u_{m2}(k_1r)A_{m,2}-u_{m3}(k_2r)A_{m,3}-u_{m4}(k_2r)A_{m,4}]\sin m\theta\exp i\omega t,\\ u_\theta &= R\,\mathrm{Re}\sum_m [l_{m3}(k_1r)A_{m,1}+l_{m4}(k_1r)A_{m,2}+l_{m1}(k_2r)A_{m,3}+l_{m2}(k_2r)A_{m,4}]\cos m\theta\exp i\omega t \end{aligned} \tag{9}$$

and the stresses

$$\sigma_{rr}(r,\theta,t)=-\mathrm{Re}\frac{1}{(1-\nu_E^2)s_{11}^E}\left\{\sum_m\Big(a_{m1}(k_1r)A_{m,1}+a_{m2}(k_1r)A_{m,2}+a_{m3}(k_2r)A_{m,3}+a_{m4}(k_2r)A_{m,4}\Big)\right.$$

$$\left.+\sin m\theta+\frac{4}{\pi}V_0(1+\nu_E)d_{13}\sum_{n=1}^{\infty}\frac{\sin N(2n-1)\theta}{2n-1}\right\}\exp i\omega t,$$

$$\sigma_{\theta\theta}(r,\theta,t)=-\mathrm{Re}\frac{1}{(1-\nu_E^2)s_{11}^E}\left\{\sum_m\Big(b_{m1}(k_1r)A_{m,1}+b_{m2}(k_1r)A_{m,2}+b_{m3}(k_2r)A_{m,3}+b_{m4}(k_2r)A_{m,4}\Big)\right.$$

$$\left.\times\sin m\theta+\frac{4}{\pi}V_0(1+\nu_E)d_{13}\sum_{n=1}^{\infty}\frac{\sin N(2n-1)\theta}{2n-1}\right\}\exp i\omega t,$$

$$\sigma_{r\theta}(r,\theta,t)=\mathrm{Re}\frac{1}{(1+\nu_E^2)s_{11}^E}\sum_m\Big(c_{m1}(k_1r)A_{m,1}+c_{m2}(k_1r)A_{m,2}+c_{m3}(k_2r)A_{m,3}$$

$$+c_{m4}(k_2r)A_{m,4}\Big)\cos m\theta\exp i\omega t, \tag{10}$$

where

$$a_{m1}(k_1r)=\Big[(1-\nu_E)k_1rJ_{m-1}(k_1r)+\Big(k_1^2r^2-(1-\nu_E)m(m+1)\Big)J_m(k_1r)\Big]R^2\big/r^2,$$

$$a_{m2}(k_1r)=\Big[(1-\nu_E)k_1rY_{m-1}(k_1r)+\Big(k_1^2r^2-(1-\nu_E)m(m+1)\Big)Y_m(k_1r)\Big],$$

$$a_{m3}(k_2r)=(1-\nu_E)m\big[k_2rJ_{m-1}(k_2r)-(m+1)J_m(k_2r)\big]R^2\big/r^2,$$

$$a_{m4}(k_2 r) = (1-\nu_E)m\left[k_2 r Y_m(k_2 r) - (m+1)Y_m(k_2 r)\right]R^2 \,\big/\, r^2,$$

$$b_{m1}(k_1 r) = \left[-(1-\nu_E)k_1 r J_{m-1}(k_1 r) + \left(\nu_E k_1^2 r^2 + (1-\nu_E)m(m+1)\right)J_m(k_1 r)\right]R^2 \,\big/\, r^2,$$

$$b_{m2}(k_1 r) = \left[-(1-\nu_E)k_1 r Y_{m-1}(k_1 r) + \left(\nu_E k_1^2 r^2 + (1-\nu_E)m(m+1)\right)Y_m(k_1 r)\right]R^2 \,\big/\, r^2,$$

$$b_{m3}(k_2 r) = -(1-\nu_E)m\left[k_2 r J_{m-1}(k_2 r) - (m+1)J_m(k_2 r)\right]R^2 \,\big/\, r^2,$$

$$b_{m4}(k_2 r) = -(1-\nu_E)m\left[k_2 r Y_{m-1}(k_2 r) - (m+1)Y_m(k_2 r)\right]R^2 \,\big/\, r^2,$$

$$c_{m1}(k_1 r) = m\left[k_1 r J_{m-1}(k_1 r) - (m+1)J_m(k_1 r)\right]R^2 \,\big/\, r^2,$$

$$c_{m2}(k_1 r) = m\left[k_1 r Y_{m-1}(k_1 r) - (m+1)Y_m(k_1 r)\right]R^2 \,\big/\, r^2,$$

$$c_{m3}(k_2 r) = \left[k_2 r J_{m-1}(k_2 r) + \left(\frac{1}{2}k_2^2 r^2 - m(m+1)\right)J_m(k_2 r)\right]R^2 \,\big/\, r^2,$$

$$c_{m4}(k_2 r) = \left[k_2 r Y_{m-1}(k_2 r) + \left(\frac{1}{2}k_2^2 r^2 - m(m+1)\right)Y_m(k_2 r)\right]R^2 \,\big/\, r^2,$$

$$u_{m1}(k_1 r) = -m\frac{R}{r}J_m(k_1 r) + k_1 R J_{m-1}(k_1 r), \quad u_{m2}(k_1 r) = -m\frac{R}{r}Y_m(k_1 r) + k_1 R Y_{m-1}(k_1 r),$$

$$u_{m3}(k_2 r) = m\frac{R}{r}J_m(k_2 r), \quad u_{m4}(k_2 r) = m\frac{R}{r}Y_m(k_2 r),$$

$$l_{m1}(k_1 r) = u_{m3}(k_1 r),\ l_{m2}(k_1 r) = u_{m2}(k_1 r),\ l_{m3}(k_2 r) = -u_{m3}(k_2 r),\ l_{m4}(k_2 r) = -u_{m2}(k_2 r). \tag{11}$$

Since $E_z^a = (-1)^{n-1}V_0 h^{-1}$, $n = 1, 2, \ldots, 2N$, the electric-field strength $E_z = \operatorname{Re} E_z^a \exp i\omega t$ can be expanded into a Fourier series in the angular coordinate θ:

$$E_z^a = -\frac{2V_0}{\pi h}\sum_{n=1}^{\infty}\frac{\sin N(2n-1)\theta}{2n-1}, \tag{12}$$

the subscript $m = N(2n-1)$, $n = 1, 2, \ldots$, in (9) and (10).

In the resonance case, the concept of complex moduli [2, 4] has to be used, i.e., the material constants should be considered complex ($\tilde{s}_{ij}^E = s_{ij}^E - is_{ij}^{E\,\mathrm{Im}}$, $\tilde{d}_{ij} = d_{ij} - id_{ij}^{\mathrm{Im}}$, $\tilde{\varepsilon}_{ij}^T = \varepsilon_{ij}^T - i\varepsilon_{ij}^{T\,\mathrm{Im}}$).

To determine the resonance frequencies, it is possible to neglect the loss tangents as small and to use the real values of the material constants.

If $N = 0$, there occur electroelastic radial vibrations

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho\frac{\partial^2 u_r}{\partial t^2},$$

$$\sigma_{rr} = \frac{1}{(1-\nu_E^2)s_{11}^E}\left(\frac{\partial u_r}{\partial r} + \nu_E\frac{u_r}{r} - (1+\nu_E)d_{31}E_z\right) \tag{13}$$

and azimuthal vibrations

TABLE 1a

k	$N=0$, $\overline{\omega}_{0,k}$	$N=1$, $\overline{\omega}_{1,k}$	$N=2$, $\overline{\omega}_{2,k}$	$N=3$, $\overline{\omega}_{3,k}$
1	0.77405	1.2108	1.85491	2.31119
2	4.24997	4.61023	5.4193	6.40551
3	7.211044	8.24799	8.85883	9.53872
4	10.14251	10.15934	10.23068	10.43583
5	13.06489	13.04537	13.09362	13.22587
6	15.98328	16.01617	16.11943	16.30792
7	18.89954	18.97603	19.19561	19.54078

TABLE 1r

k	$N=0$, $\overline{\omega}_{0,k}$	$N=1$, $\overline{\omega}_{1,k}$	$N=2$, $\overline{\omega}_{2,k}$	$N=3$, $\overline{\omega}_{3,k}$
1	2.7658	2.70754	2.87211	3.50261
2	7.93913	7.05481	6.88761	6.87753
3	13.14434	13.28651	13.59398	14.02106
4	18.36553	18.37633	18.41591	18.49402
5	23.59297	23.65134	23.82034	24.08278
6	28.82342	28.86298	28.9791	29.16424
7	34.05554	34.10693	34.25091	33.47553

$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{2\sigma_{r\theta}}{r}=\rho\frac{\partial^2 u_\theta}{\partial t^2},$$

$$\sigma_{r\theta}=\frac{1}{2(1+\nu_E)s_{11}^E}\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right). \qquad (14)$$

The natural frequencies of electroelastic radial vibrations (13) are analyzed in [11, 12]. Azimuthal vibrations (14), which cannot be excited electrically, are addressed for a fuller analysis of the results.

For $N>0$, the frequencies of radial (problem (13)) and azimuthal (problem (14)) vibrations will be called quasiradial and quasiazimuthal, respectively.

2. Analysis and Comparison of the Results. Consider a ring plate with the inner edge ($r=r_0$) clamped and the outer edge ($r=r_1$) free:

$$u_r(r_0,\theta,t)=0, \qquad u_\theta(r_0,\theta,t)=0,$$

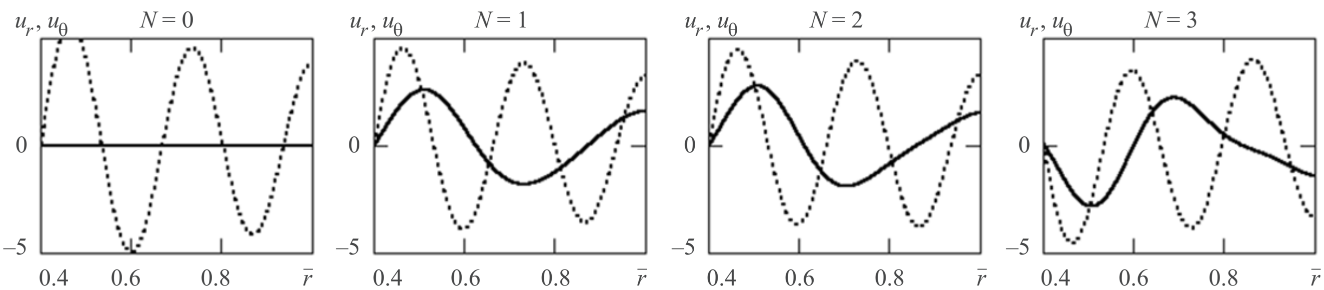

Fig. 1a

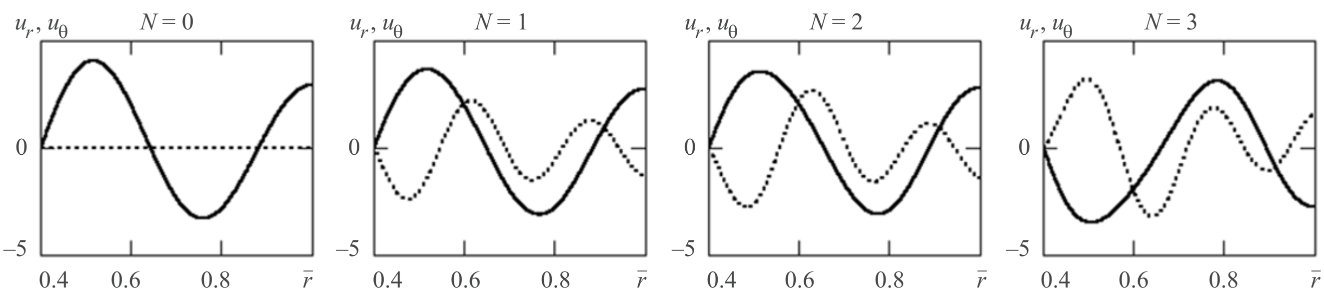

Fig. 1r

$$\sigma_{rr}(r_1, \theta, t) = 0, \qquad \sigma_{r\theta}(r_1, \theta, t) = 0. \tag{15}$$

Using expressions (9), (10) and boundary conditions (15), we obtain block systems of algebraic equations for the dimensionless constants $A_{N(2n-1),\,i}$, $n = 1, 2, \ldots$:

$$u_{N(2n-1),1}(k_1 r_0) A_{N(2n-1),1} + u_{N(2n-1),2}(k_1 r_0) A_{N(2n-1),2}$$
$$+ u_{N(2n-1),3}(k_2 r_0) A_{N(2n-1),3} + u_{N(2n-1),4}(k_2 r_0) A_{N(2n-1),4} = 0,$$
$$l_{N(2n-1),1}(k_1 r_0) A_{N(2n-1),1} + l_{N(2n-1),2}(k_1 r_0) A_{N(2n-1),2}$$
$$+ l_{N(2n-1),3}(k_2 r_0) A_{N(2n-1),3} + l_{N(2n-1),4}(k_2 r_0) A_{N(2n-1),4} = 0,$$
$$a_{N(2n-1),1}(k_1 r_1) A_{N(2n-1),1} + a_{N(2n-1),2}(k_1 r_1) A_{N(2n-1),2}$$
$$+ a_{N(2n-1),3}(k_2 r_1) A_{N(2n-1),3} + a_{N(2n-1),4}(k_2 r_1) A_{N(2n-1),4} = -\frac{4}{\pi} V_0 \frac{(1+\nu_E) d_{13}}{2n-1},$$
$$c_{N(2n-1),1}(k_1 r_1) A_{N(2n-1),1} + c_{N(2n-1),2}(k_1 r_1) A_{N(2n-1),2}$$
$$+ c_{N(2n-1),3}(k_2 r_1) A_{N(2n-1),3} + c_{N(2n-1),4}(k_2 r_1) A_{N(2n-1),4} = 0, \tag{16}$$

where $m = N(2n-1)$, $n = 1, 2, \ldots, N$ is the number of radial cuts in the electrode coating.

The resonance frequencies can be determined by equating the fourth-order determinants of the homogeneous (at $V_0 = 0$) systems of algebraic equations (16) to zero.

Tables 1r and 1a summarize the results of analyzing the frequency equations derived by equating the determinant of the system of equations (16) to zero. In the tables, the dimensionless resonant frequencies are denoted by $\overline{\omega} = \sqrt{(1-\nu_E^2) s_{11}^E \rho}\, \omega r_1$. The results have been obtained for different values of N and the first vibration modes (k is the harmonic number). Here and later on, the letter "r" in the number of the tables and figures refers to quasiradial frequencies, and the letter "a" to quasiazimuthal frequencies. The input data: $r_0 / r_1 = 0.4$ and $\rho = 7740$ kg/m^3, $s_{11}^E = 15.2 \cdot 10^{-12}$ m^2/N, $s_{12}^E = -5.8 \cdot 10^{-12}$ m^2/N, $d_{31} = -125 \cdot 10^{-12}$ C/N, which corresponds to TsTS-19 piezoceramics [6].

TABLE 2a

k	$N=0$, $\overline{\omega}_{0,k}$	$N=1$, $\overline{\omega}_{1,k}$	$N=2$, $\overline{\omega}_{2,k}$	$N=3$, $\overline{\omega}_{3,k}$
1	2.31583	2.46586	2.70391	3.14207
2	4.81907	5.20747	6.05161	6.74316
3	7.56991	7.37752	7.27896	7.74701
4	10.40329	10.43448	10.56156	10.88514
5	13.26914	13.42252	13.72799	14.13504
6	16.15099	16.18638	16.29824	16.50383
7	19.04174	18.41827	18.46417	18.55208

TABLE 2r

k	$N=0$, $\overline{\omega}_{0,k}$	$N=1$, $\overline{\omega}_{1,k}$	$N=2$, $\overline{\omega}_{2,k}$	$N=3$, $\overline{\omega}_{3,k}$
1	3.20884	3.3252	3.93292	4.80765
2	8.043868	8.39152	8.97637	9.55813
3	13.20124	13.1704	13.22125	13.38097
4	18.40479	19.11545	19.32803	19.66145
5	23.62303	23.68015	23.84421	24.09559
6	28.84782	28.88754	29.00366	29.18763
7	34.07609	34.13	34.27815	34.49971

The quasiazimuthal vibration modes at frequencies $\overline{\omega}_{0,5}$ = 13.06489, $\overline{\omega}_{1,5}$ = 13.04537, $\overline{\omega}_{2,5}$ = 13.09362, $\overline{\omega}_{3,5}$ = 13.22587 are shown in Fig. 1a and the quasiradial modes at frequencies $\overline{\omega}_{0,3}=13.14434$, $\overline{\omega}_{1,3}=13.28651$, $\overline{\omega}_{2,3}=13.59398$, $\overline{\omega}_{3,3}=14.02106$ are shown in Fig. 1r. The frequencies of quasiazimuthal ($\omega_{j,5}$) and quasiradial ($\omega_{j,3}$) vibrations are close; therefore, this case is chosen for the analysis of vibration modes. Figures 1r and 1a, and the figures that follow, show curves of u_r (solid lines) and $u_{\theta r}$ (dashed lines) versus $\bar{r}=r/r_1$ for different values of N ($N=0, \ldots, 3$).

If the inner edge ($r=r_0$) is free and the outer edge ($r=r_1$) is clamped, then

$$\sigma_{rr}(r_0,\theta,t)=0, \quad \sigma_{r\theta}(r_0,\theta,t)=0,$$
$$u_r(r_1,\theta,t)=0, \quad u_\theta(r_1,\theta,t)=0. \tag{17}$$

Using expressions (9) and (10), we obtain the following systems of algebraic equations for the constants $A_{N(2n-1),i}$ ($n=1, 2, \ldots$):

$$a_{N(2n-1),1}(k_1 r_0)A_{N(2n-1),1}+a_{N(2n-1),2}(k_1 r_0)A_{N(2n-1),2}$$

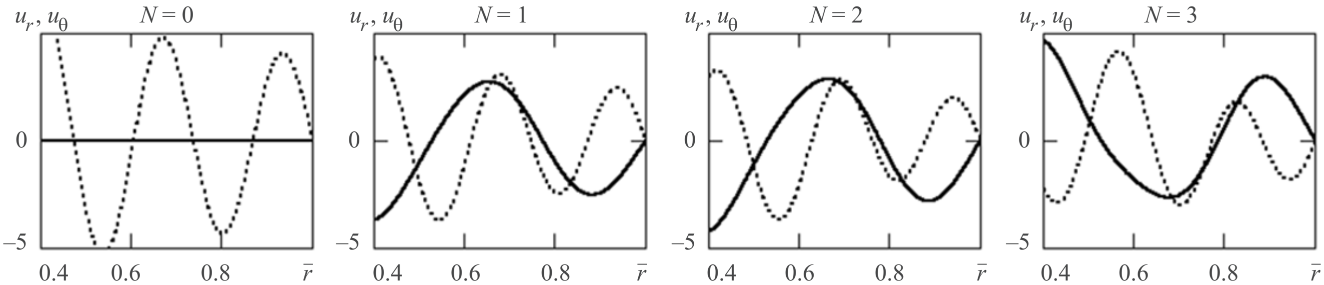

Fig. 2a

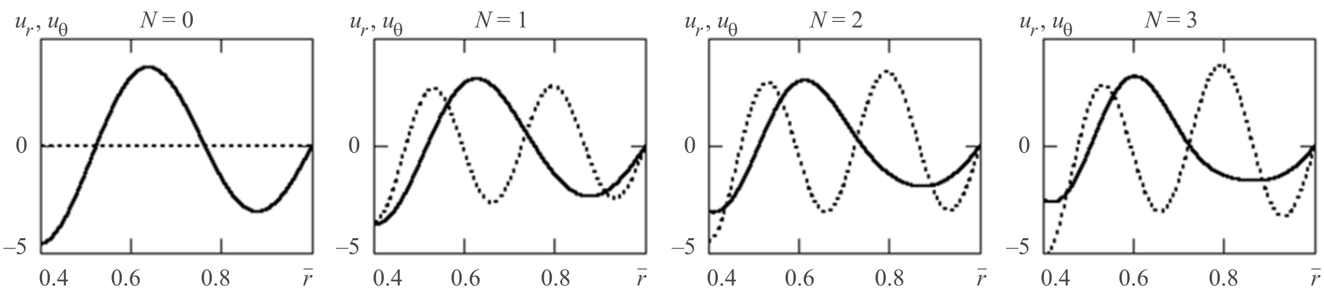

Fig. 2r

$$+a_{N(2n-1),3}(k_2 r_0)A_{N(2n-1),3} + a_{N(2n-1),4}(k_2 r_0)A_{N(2n-1),4} = -\frac{4}{\pi}V_0\frac{(1+\nu_E)d_{13}}{2n-1},$$

$$c_{N(2n-1),1}(k_1 r_0)A_{N(2n-1),1} + c_{N(2n-1),2}(k_1 r_0)A_{N(2n-1),2}$$

$$+c_{N(2n-1),3}(k_2 r_0)A_{N(2n-1),3} + c_{N(2n-1),4}(k_2 r_0)A_{N(2n-1),4} = 0,$$

$$u_{N(2n-1),1}(k_1 r_1)A_{N(2n-1),1} + u_{N(2n-1),2}(k_1 r_1)A_{N(2n-1),2}$$

$$+u_{N(2n-1),3}(k_2 r_1)A_{N(2n-1),3} + u_{N(2n-1),4}(k_2 r_1)A_{N(2n-1),4} = 0,$$

$$l_{N(2n-1),3}(k_1 r_1)A_{N(2n-1),1} + l_{N(2n-1),4}(k_1 r_1)A_{N(2n-1),2}$$

$$+l_{N(2n-1),3}(k_2 r_1)A_{N(2n-1),3} + l_{2N(2n-1),4}(k_2 r_1)A_{N(2n-1),4} = 0. \qquad (18)$$

The natural frequencies are found by equating the fourth-order determinants of the homogeneous systems of algebraic equations (18) to zero. The numerical results are presented in Tables 2r and 2a.

The quasiazimuthal vibration modes at frequencies $\overline{\omega}_{0,5}$ = 13.29614, $\overline{\omega}_{1,5}$ = 13.42252, $\overline{\omega}_{2,5}$ = 13.72799, $\overline{\omega}_{3,5}$ = 14.13504 are shown in Fig. 2a and the quasiradial modes at frequencies $\overline{\omega}_{0,3} = 13.20124$, $\overline{\omega}_{1,3} = 13.1704$, $\overline{\omega}_{2,3} = 13.22125$, $\overline{\omega}_{3,3} = 13.38097$ are shown in Fig. 2r.

The formulas to determine the resonant frequencies for the boundary conditions (14) follow from the existence condition for the nontrivial solutions of the homogeneous ($V_0 = 0$) systems of equations (18).

If the inner ($r = r_0$) and outer ($r = r_1$) edges are free, then

$$\sigma_{rr}(r_0,\theta,t) = 0, \qquad \sigma_{r\theta}(r_0,\theta,t) = 0,$$

$$\sigma_{rr}(r_1,\theta,t) = 0, \qquad \sigma_{r\theta}(r_1,\theta,t) = 0. \qquad (19)$$

Using expressions (9), (10) and boundary conditions (15), we obtain block systems of algebraic equations for the dimensionless constants $A_{N(2n-1),i}$ ($n = 1, 2, \ldots$):

$$a_{N(2n-1),1}(k_1 r_0)A_{N(2n-1),1} + a_{N(2n-1),2}(k_1 r_0)A_{N(2n-1),2}$$

TABLE 3a

k	$N=0$, $\overline{\omega}_{0,k}$	$N=1$, $\overline{\omega}_{1,k}$	$N=2$, $\overline{\omega}_{2,k}$	$N=3$, $\overline{\omega}_{3,k}$
1	3.31746	3.85103	2.34721	3.18735
2	6.05804	6.53165	5.05288	6.17078
3	8.89634	8.88278	7.29402	7.983204
4	11.76887	11.81621	11.04868	11.42551
5	14.65643	14.68479	14.77914	14.96509
6	17.5518	17.59723	17.73907	17.99138
7	20.45171	20.42391	20.3894	20.39869

TABLE 3r

k	$N=0$, $\overline{\omega}_{0,k}$	$N=1$, $\overline{\omega}_{1,k}$	$N=2$, $\overline{\omega}_{2,k}$	$N=3$, $\overline{\omega}_{3,k}$
1	1.42346	1.6265	0.69281	1.54389
2	5.49151	5.20302	4.85053	4.97488
3	10.59329	10.72654	8.93175	9.28126
4	15.78762	15.85037	16.02459	16.2763
5	21.00334	21.10521	21.36017	21.7125
6	26.22731	26.21529	26.24087	26.30856
7	31.45534	31.47436	31.52862	31.61186

$$+a_{N(2n-1),3}(k_2r_0)A_{N(2n-1),3}+a_{N(2n-1),4}(k_2r_0)A_{N(2n-1),4}=-\frac{4}{\pi}V_0\frac{(1+\nu_E)d_{13}}{2n-1},$$

$$c_{N(2n-1),1}(k_1r_0)A_{N(2n-1),1}+c_{N(2n-1),2}(k_1r_0)A_{N(2n-1),2}$$
$$+c_{N(2n-1),3}(k_2r_0)A_{N(2n-1),3}+c_{N(2n-1),4}(k_2r_0)A_{N(2n-1),4}=0,$$
$$a_{N(2n-1),1}(k_1r_1)A_{N(2n-1),1}+a_{N(2n-1),2}(k_1r_1)A_{N(2n-1),2}$$
$$+a_{N(2n-1),3}(k_2r_1)A_{N(2n-1),3}+a_{N(2n-1),4}(k_2r_1)A_{N(2n-1),4}=-\frac{4}{\pi}V_0\frac{(1+\nu_E)d_{13}}{2n-1},$$

$$c_{N(2n-1),1}(k_1r_1)A_{N(2n-1),1}+c_{N(2n-1),2}(k_1r_1)A_{N(2n-1),2}$$
$$+c_{N(2n-1),3}(k_2r_1)A_{N(2n-1),3}+c_{N(2n-1),4}(k_2r_1)A_{N(2n-1),4}=0. \qquad (20)$$

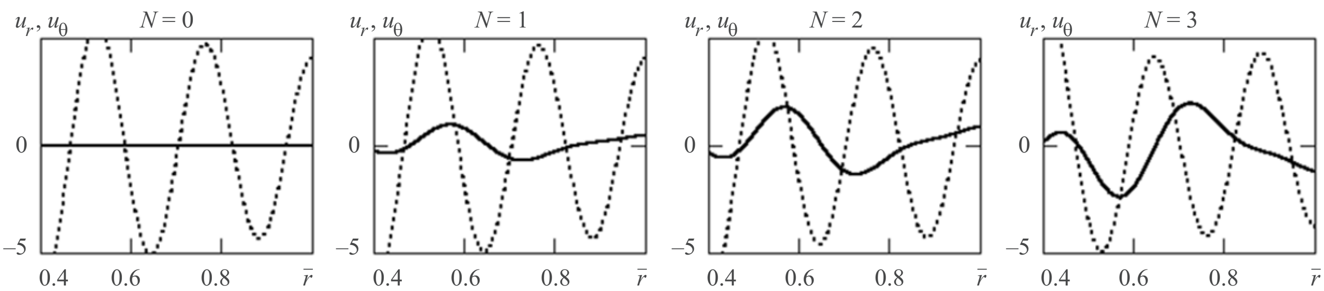

Fig. 3a

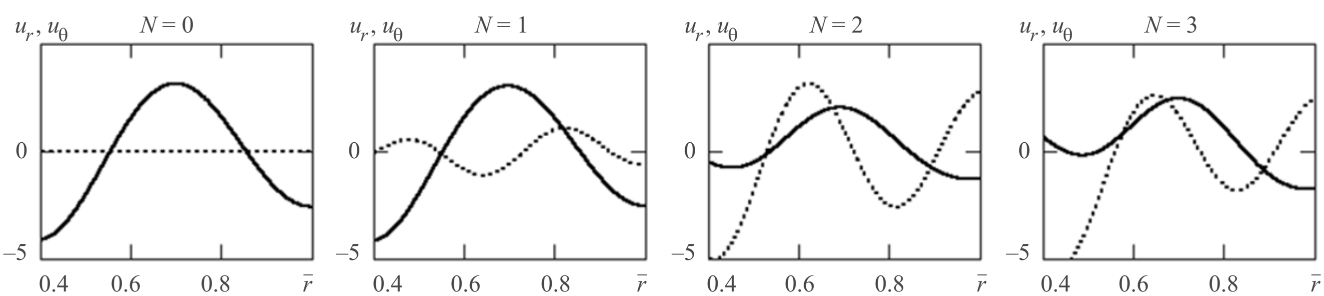

Fig. 3r

The formulas to determine the resonant frequencies for the boundary conditions (19) follow from the existence condition for the nontrivial solutions of the homogeneous ($V_0 = 0$) systems of equations (20).

The numerical results are presented in Tables 3r and 3a.

The quasiazimuthal vibration modes at frequencies $\overline{\omega}_{0,5} = 14.65643$, $\overline{\omega}_{1,5} = 14.68479$, $\overline{\omega}_{2,5} = 14.77914$, $\overline{\omega}_{3,5} = 14.96509$ are shown in Fig. 3a and the quasiradial modes at frequencies $\overline{\omega}_{0,3} = 10.59329$, $\overline{\omega}_{1,3} = 10.72654$, $\overline{\omega}_{2,3} = 8.93175$, $\overline{\omega}_{3,3} = 9.28126$ are shown in Fig. 3r.

The following general properties of the theoretical frequency spectrum of a plate with different number (N) of radial cuts can be found from the boundary conditions (15), (17), (19), formulas (9), (10), and the frequency equations derived from systems (16), (18), and (20):

if $N = 1$ (two electrodes), then $f_{1,k}, f_{3,k}, f_{5,k}, \ldots$; if $N = 2$ (four electrodes), then $f_{2,k}, f_{6,k}, f_{10,k}, \ldots$; if $N = 3$ (six electrodes), then $f_{3,k}, f_{9,k}, f_{15,k}, \ldots$; if $N = 4$ (eight electrodes), then $f_{4,k}, f_{12,k}, f_{20,k}, \ldots$; if $N = 5$ (ten electrodes), then $f_{5,k}, f_{15,k}, f_{25,k}, \ldots$; if $N = 6$ (12 electrodes), then $f_{6,k}, f_{18,k}, f_{30,k}, \ldots$; if $N = 7$ (14 electrodes), then $f_{7,k}, f_{21,k}, f_{35,k}, \ldots$; if $N = 8$ (16 electrodes), then $f_{8,k}, f_{24,k}, f_{40,k}, \ldots$. In the notation of frequencies $f_{m,k}$, the subscript "m" is the harmonic number with respect to the azimuth θ (circumferential mode number) and the subscript "k" is the root sequence number of the frequency equation.

Conclusions. As the number of cuts of the electrode coating is increased, the quasiradial and quasiazimuthal frequencies for small k increase if one of the edges of the plate is clamped. This is not so for the plate with free edges.

For the boundary conditions (15) and (17), the difference between the quasiradial and quasiazimuthal frequencies for equal k and different N tends to zero with increase in k.

For $k \le 2$, the frequencies corresponding to different N can differ several-fold. As k is increased, the relative difference of frequencies corresponding to different N decreases to a fraction of a percent even at $k \ge 6$, irrespective of the nature of vibrations.

If one of the edges is clamped (boundary conditions (15), (17)), two quasiazimuthal frequencies appear between the quasiradial frequencies ω_k and ω_{k+1}. If all the edges are free, this conclusion holds true beginning from $k = 2$.

With increase in k, one of the quasiazimuthal frequencies tends to a quasiradial frequency, while the vibration modes are qualitatively and quantitatively different. This effect is the strongest for the frequencies $\omega_{j,5}$ and $\omega_{j,3}$ of quasiazimuthal and quasiradial vibrations, respectively.

The quasiazimuthal frequencies of the plate with free edges are higher than those for the plate with one edge clamped. The effect is opposite for quasiradial vibrations.

Although the quasiazimuthal and quasiradial frequencies tend to each other with increase in the number of cuts, the vibration modes become more and more different. The number of zeros of the function u_r is the same for the boundary conditions (15) and (17). This conclusion also applies to the function u_θ. Each quasiradial frequency has a close quasiazimuthal frequency, both corresponding to the same number of nodal points, irrespective of the value of *k*. The number of nodal points does not depend on the number of cuts. The frequency increases with *N*, and the difference of frequencies is considerable for the first modes and decreases with increase in *k*.

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