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ON INERTIAL-NAVIGATION SYSTEM WITHOUT ANGULAR-RATE SENSORS

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Operating algorithms for autonomous inertial-navigation systems without angular-rate sensors are outlined. Systems with 6, 9, and 12 accelerometers are considered. Since six accelerometers are sufficient to measure the angular acceleration, using 9 or 12 accelerometers allows improving the accuracy of determining the angular-rate vector. For this purpose, the additional information provided by the extra accelerometers is used. Correction algorithms are presented. It is shown, by way of examples, that such systems may be effective at high angular rates, when using angular-rate sensors becomes problematic

Keywords: autonomic inertial navigation system, angular rate sensors, attitude determination, quaternion

Introduction. The conventional inertial-navigation systems (INSs) [1, 3] employ angular-rate sensors (ARSs) and accelerometers. Intensive research intended to design small and relatively cheap components is now underway [17]. It is of interest to develop INSs based on accelerometers alone, and some countries do research and development work in this field [7]. Naturally, such INSs cannot provide adequate accuracy of navigation over long periods of autonomous operation. It is, therefore, makes sense to integrate INS with the global positioning system (GPS) [10], i.e., to consider it as a component of a GPS/INS system [18]. This system can be used in relatively cheap unmanned aircraft [6].

Here, as in [14], we address the problem of developing INS without ARSs. It includes the problem of determining the angular rate from GPS measurements of the linear velocities of three points and the problem of determining the angular acceleration (integrating it yields the angular rate) with accelerometers. Below, as in [14], we will address two problems of determining the kinematic parameters of a rigid body. One problem is to use measured velocities of three points of the body to determine the angular-rate vector and velocities of the origin of the body-fixed frame of reference. The other problem is to use the measured acceleration of three points of the body and its known angular rate to determine the angular accelerations and accelerations of the origin of the moving frame of reference. Next we will consider onboard measuring systems with 6, 9, and 12 accelerometers. Various systems with six accelerometers (e.g., [14, 19]) allow determining angular accelerations, but are not capable of correcting the results of its integration without "external" sources of information. It will be shown that with nine accelerometers, it is possible to use the additional data provided by the three extra accelerometers to correct the errors of integration of the angular acceleration. However, such a measuring system does not always allow correcting errors of integration. A measuring system with 12 accelerometers is much more effective. The effectiveness of such an INS at high angular rates when, as indicated in [7], the use of ARSs may be problematic will be demonstrated by way of example.

Since issues of creating a GPS/INS system based of such INS was detailed in [14], this topic is omitted here.

Since the INS under consideration does not ensure high accuracy, in considering its operating algorithms, we will neglect, for simplicity, the rotation of the Earth and the Coriolis acceleration, though allowing for these factors does not involve major difficulties.

1. Basic Equations. The well-known equations related to the attitude-determination problem for a rigid body [4, 5, 13, 20] are presented below. Let us describe different ways to determine the attitude.

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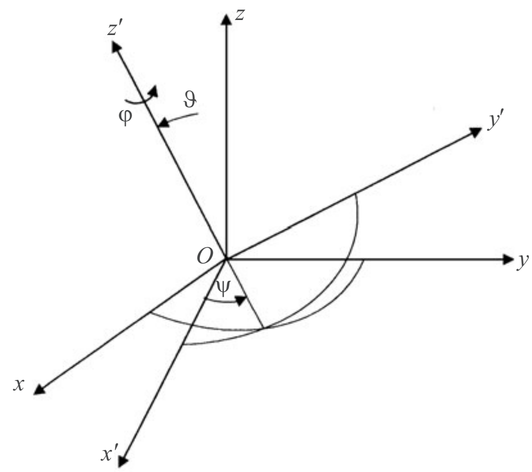

Fig. 1

The Euler angles ψ, ϑ, φ (precession, nutation, and intrinsic rotation) describe the orientation of a body, i.e., the transition of the body from the initial position defined by the axes of $Oxyz$ to the final position defined by the axes of $Ox'y'z'$ (Fig. 1). This transition can be carried out by rotating the body through an angle χ about the axis defined by angles α, β, γ. Therefore, the orientation of a body can be characterized by four Euler–Rodrigues parameters [4]: $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ (Euler's parameters [20]):

$$\lambda_1 = \cos\alpha \sin\chi/2, \quad \lambda_2 = \cos\beta \sin\chi/2, \quad \lambda_3 = \cos\gamma \sin\chi/2, \quad \lambda_0 = \cos\chi/2.$$

It is clear that

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1.$$

The Euler–Rodrigues parameters are expressed in terms of the Euler angles as follows:

$$\lambda_0 = \cos\frac{\vartheta}{2}\cos\frac{\varphi+\psi}{2}, \quad \lambda_1 = \sin\frac{\vartheta}{2}\cos\frac{\varphi-\psi}{2},$$

$$\lambda_2 = \sin\frac{\vartheta}{2}\sin\frac{\psi-\varphi}{2}, \quad \lambda_3 = \cos\frac{\vartheta}{2}\sin\frac{\varphi+\psi}{2}. \tag{1.1}$$

The orientation of a rigid body relative to a fixed coordinate frame $Oxyz$ can be defined by a coordinate transformation matrix A (direction cosine matrix between the fixed and moving coordinate frames); i.e., if m is some vector in the fixed frame, and its components k are the projections of this vector onto the axes of the moving frame ($Ox'y'z'$), then

$$k = Am. \tag{1.2}$$

This matrix can be represented in terms of the Euler–Rodrigues parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ as follows:

$$A(\lambda) = \begin{bmatrix} \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & 2(\lambda_1\lambda_2 + \lambda_0\lambda_3) & 2(\lambda_1\lambda_3 - \lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2 - \lambda_0\lambda_3) & \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 & 2(\lambda_2\lambda_3 + \lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3 + \lambda_0\lambda_2) & 2(\lambda_2\lambda_3 - \lambda_0\lambda_1) & \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 \end{bmatrix}. \tag{1.3}$$

The inverse formulas hold as well. For example, if $A = [a_{ij}]$, $ij = \overline{1,3}$, and $1 + a_{11} + a_{22} + a_{33} > 0$, then [5, 20] we have

$$\lambda_0 = \frac{1}{2}\sqrt{1 + a_{11} + a_{22} + a_{33}}, \quad \lambda_1 = \frac{a_{23} - a_{32}}{2\sqrt{1 + a_{11} + a_{22} + a_{33}}},$$

$$\lambda_2 = \frac{a_{31} - a_{13}}{2\sqrt{1 + a_{11} + a_{22} + a_{33}}}, \qquad \lambda_3 = \frac{a_{12} - a_{21}}{2\sqrt{1 + a_{11} + a_{22} + a_{33}}}, \qquad a = \sqrt{1 + a_{11} + a_{22} + a_{33}}. \tag{1.4}$$

The projections $\omega_1, \omega_2, \omega_3$ of the angular velocity vector of the body onto the body-fixed axes are expressed in terms of the Euler angles as follows [4]:

$$\omega_1 = \dot{\psi} \sin \vartheta \sin \varphi + \dot{\vartheta} \cos \varphi,$$

$$\omega_2 = \dot{\psi} \sin \vartheta \cos \varphi - \dot{\vartheta} \sin \varphi,$$

$$\omega_3 = \dot{\psi} \cos \vartheta + \varphi. \tag{1.5}$$

Measuring the projections of the angular velocity vector $\omega = [\omega_1 \ \omega_2 \ \omega_3]^T$ onto the body-fixed axes and knowing the initial position of the rigid body, we can find the vector (quaternion) of Euler–Rodrigues parameters $\lambda = [\lambda_0 \ \lambda_1 \ \lambda_2 \ \lambda_3]^T$ by integrating the kinematic equations

$$\dot{\lambda} = \frac{1}{2} \cdot \Omega \lambda, \qquad \Omega = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}, \qquad ||\lambda||^2 = \lambda^T \lambda = 1, \tag{1.6}$$

where $||\cdot||$ denotes spectral matrix norm; the superscript T denotes transposition.

If the frames $Oxyz$ and $Ox'y'z'$ are close (the Euler angles are small), we can use an approximate expression (say formula (26) in [20]) for the matrix A:

$$A \cong \begin{bmatrix} 1 & \mu_3 & -\mu_2 \\ -\mu_3 & 1 & \mu_1 \\ \mu_2 & -\mu_1 & 1 \end{bmatrix}, \tag{1.7}$$

where μ_1, μ_2, μ_3 are the small angles of rotation of $Oxyz$ about the x-, y-, z-axes, respectively.

2. Determining the Velocities. The problem of determining the angular rate of a rigid body and velocity of its point from measured velocities of its three points was addressed in [9, 16, etc.]. The problem is formulated as follows (Fig. 2). Given three vectors r_1, r_2, r_3 defining points at which the linear velocity is measured, use these measurements to determine the angular-rate vector ($\omega = [\omega_1 \ \ \omega_2 \ \ \omega_3]^T$) of the body and the linear velocity vector ($v_0 = [v_1 \ \ v_2 \ \ v_3]^T$) of the origin O_1 of the body-fixed frame of reference. Using the well-known formula (see, e.g., [4, formula (2.7.8)] and [9, formula (2)]) for the velocity of a rigid body's point defined by a vector r

$$v = v_0 + \omega \times r, \tag{2.1}$$

it is possible to write the following linear equations ([16, formula (6)] and [9, formula (4)]) relating the unknown components of the vectors ω, v and the measured velocities of points:

$$V = \Omega P + v_0 h^T, \tag{2.2}$$

$$\left(\Omega = \omega \times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \qquad P = [r_1 \ \ r_2 \ \ r_3], \qquad h = [1 \ \ 1 \ \ 1]^T, \right.$$

V is a matrix whose columns are the velocity vectors of the points defined by vectors r_1, r_2, r_3).

Let $\beta_1, \beta_2, \beta_3$ and $\gamma_1, \gamma_2, \gamma_3$ be the columns of the matrices P^T and V^T, i.e., $P^T = [\beta_1, \beta_2, \beta_3]$, $V^T = [\gamma_1, \gamma_2, \gamma_3]$. Then formula (2.2) can be written as a system of linear equations for ω, v_0:

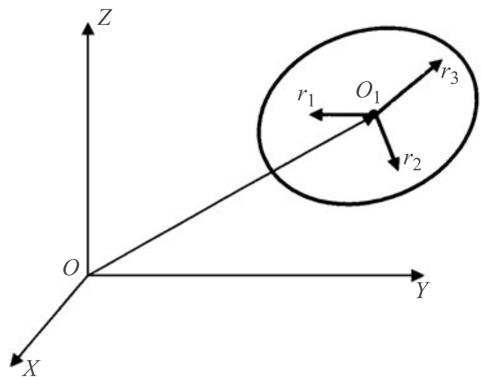

Fig. 2

$$A_v x = B, \quad x = \begin{bmatrix} \omega \\ v_0 \end{bmatrix}, \quad A_v = \begin{bmatrix} o & \beta_3 & \beta_2 & h & o & o \\ -\beta_3 & o & \beta_1 & o & h & o \\ \beta_2 & -\beta_1 & o & o & o & h \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}, \tag{2.3}$$

where o is a zero 3×1 matrix.

Since the velocity is measured with errors, we represent (2.3) in the form

$$A_v x = B_0 + n_v, \tag{2.4}$$

where n_v are the measurement errors; B_0 is the vector of exact values of the velocities of the points of interest.

3. Determining the Accelerations. Let us now address the problem of determining the angular acceleration of a body and the acceleration of its point from measurements of the acceleration of its three points. The problem can be formulated as follows. Let three vectors ρ_1, ρ_2, ρ_3 define points of a rigid body at which accelerometers are placed to measure the components of the acceleration vector of the point of interest. Given these measurements and angular-rate vector ($\omega = [\omega_1\ \omega_2\ \omega_3]^T$), determine the angular acceleration vector ($\varepsilon = [\varepsilon_1\ \ \varepsilon_2\ \ \varepsilon_3]^T = d\omega/dt$) and the acceleration vector ($w_0 = [w_1\ w_2\ w_3]^T$) of the origin of the body-fixed frame of reference. An analog of formula (2.1) for this problem is [4, formula (2.17.9)] for the acceleration (w) of a point of a rigid body defined by vector ρ:

$$w = w_0 + \varepsilon \times \rho + \omega \times (\omega \times \rho). \tag{3.1}$$

Denoting $U = [W_1\ W_2\ W_3]$, W_i are the acceleration vectors of the points defined by ρ_i $(i = 1, 2, 3)$, we can use (3.1) to write a formula analogous to (2.2):

$$U = \Omega^2 P_w + E P_w + w_0 h^T, \tag{3.2}$$

where $P_w = [\rho_1\ \ \rho_2\ \ \rho_3]$, $E = \varepsilon\times = \begin{bmatrix} 0 & -\varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{bmatrix}$; Ω, h are matrices similar to those appearing in (2.2)).

Formula (3.2) can be represented as a system of linear equations for ε, w_0. Let $\alpha_1, \alpha_2, \alpha_3$; $\delta_1, \delta_2, \delta_3$; $\sigma_1, \sigma_2, \sigma_3$ be the columns of the matrices U^T, P_w^T, $(\Omega^2 P_w)^T$, i.e.,

$$U^T = [\alpha_1\ \ \alpha_2\ \ \alpha_3]; \quad P_w^T = [\delta_1\ \delta_2\ \delta_3]; \quad (\Omega^2 P_w)^T = [\sigma_1\ \sigma_2\ \sigma_3].$$

Then formula (3.2) can be represented in a form similar to (2.3):

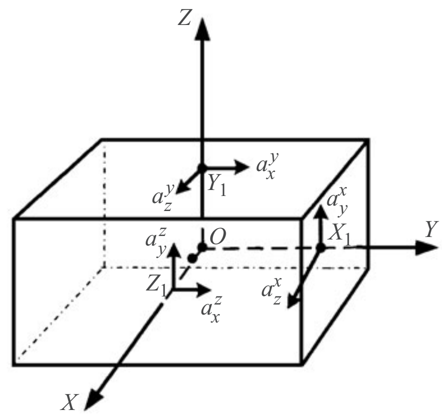

Fig. 3

$$A_w x = B_\omega + B_w, \qquad x = \begin{bmatrix} \varepsilon \\ w_0 \end{bmatrix}, \tag{3.3}$$

$$A_w = \begin{bmatrix} o & \delta_3 & \delta_2 & h & o & o \\ -\delta_3 & o & \delta_1 & o & h & o \\ \delta_2 & -\delta_1 & o & o & o & h \end{bmatrix}, \qquad B_\omega = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}, \qquad B_w = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix},$$

where o is a zero 3×1 matrix, as in (2.3).

As with (2.3), assuming that the readings of the accelerometers are inaccurate, we represent (3.3) as

$$A_w x = B_\omega + B_{wo} + n_w, \tag{3.4}$$

where n_w are the measurement errors; the components B_{wo} are exact values of accelerations.

Since the matrix A_w in (3.4) is 9×6, it is possible to exclude three rows in (3.4). To illustrate this statement, we will consider the arrangement of accelerometers in Fig. 3. Here X_1, Y_1, Z_1 are points on the axes *OX*, *OY*, *OZ* with two accelerometers at each. The orientation of their sensitive axes is indicated in the figure. For example, a_x^y denotes that this accelerometer measures the acceleration of the point Y_1 in the direction of the *OX*-axis. With such an arrangement of accelerometers, it is possible to deleted the first, fifth, and ninth rows in the system of nine equations (3.4), leaving six equations. Thus, if the angular-rate vector is known (vector B_ω), six accelerometers are sufficient to determine the vectors ε and w_0 (see example 1).

Note that another arrangement of six accelerometers that allows determining the angular acceleration vector ε as a linear combination of the accelerometers' readings was proposed in [19]. It is obvious, however, that the accuracy of the current value of the angular-rate vector determined by integrating the angular acceleration is strongly dependent on the accuracy of specifying the initial angular-rate vector. To weaken this dependence, it is reasonable to increase the number of accelerometers and to use the extra information to improve the accuracy of ω.

Let there be nine accelerometers. The six accelerometers in Fig. 3 are supplemented with three accelerometers placed at the point *O* and having sensitive axes directed along the axes *OX*, *OY*, *OZ*, respectively, i.e., these accelerometers measure the acceleration of the origin. The readings of these accelerometers are denoted by a_x^0, a_y^0, a_z^0. Let the distance from the origin to each of the points X_1, Y_1, Z_1 be equal to *L*.

Let $n_y^x = a_y^x - a_y^0$, $n_x^y = a_x^y - a_x^0$, $n_z^x = a_z^x - a_z^0$, $n_z^y = a_z^y - a_y^0$, $n_x^z = a_x^z - a_x^0$, $n_y^z = a_y^z - a_y^0$. With such an arrangement of accelerometers, the formulas below follow from (3.1) or (3.2):

$$2L\varepsilon_1 = n_z^y - n_y^z, \qquad 2L\varepsilon_2 = n_x^z - n_z^x, \qquad 2L\varepsilon_3 = n_y^x - n_x^y, \tag{3.5}$$

$$2L\omega_2\omega_3 = n_z^y + n_y^z, \qquad 2L\omega_1\omega_3 = n_x^z + n_z^x, \qquad 2L\omega_1\omega_2 = n_y^x + n_x^y. \tag{3.6}$$

Note that Eqs. (3.5) coincide with [1, formula (3.390)]. Thus, when there are nine accelerometers, formulas (3.6) define three more quantities: $\omega_1\omega_2, \omega_1\omega_3, \omega_2\omega_3$. This information is expedient to use to correct the errors of integration of the angular acceleration ε. If two of the three components of the vector ω are zero (rotation about a fixed axis), formulas (3.6) cannot be used to correct the errors of integration.

In this connection, it makes sense to supplement the nine accelerometers with another three ones that measure at the points X_1, Y_1, and Z_1 the accelerations along the axes *OX*, *OY*, and *OZ*, respectively. Note that this arrangement of accelerometers is the same as in [1, Fig. 3.7]. Denote the readings of these three accelerometers by a_x^x, a_y^y, a_z^z. Let $n_x^x = a_x^x - a_x^0$, $n_y^y = a_y^y - a_y^0$, $n_z^z = a_z^z - a_z^0$. For this measuring system of 12 accelerometers, formulas (3.5) and (3.6) should be supplemented with

$$2L\omega_1^2 = n_x^x - n_y^y - n_z^z, \quad 2L\omega_2^2 = -n_x^x + n_y^y - n_z^z, \quad 2L\omega_3^2 = -n_x^x - n_y^y + n_z^z. \tag{3.7}$$

Thus, when there are 12 accelerometers, formulas (3.6) and (3.7) should be used to correct the errors of integration of the angular acceleration.

4. Inertial-Navigation System. Since $w_0 = dv_0 / dt$, $\varepsilon = d\omega / dt$, system (3.4) can be considered as a system of nonlinear differential equations for ω, i.e., considering the accelerometers' readings (B_w) as known external stimuli, we can, given initial conditions, calculate $\omega(t)$ and $v_0(t)$ by integrating (3.4). Thus, this approach allows us to find the angular rate without the need to use ARSs. In this case, however, it is necessary to take into account the following facts.

The vector x appearing in (3.4) is set in the moving frame of reference. Since it is the position of the body in the inertial frame of reference that is of interest, it is expedient to project the second component of the vector x (vector w_0) onto the axes of the inertial frame of reference and to integrate it there to determine the velocity and coordinates of the origin of the moving frame. The first component of the vector x (vector ε) should be used to determine the current orientation of the body, which can be described by either the Euler–Rodrigues parameters (1.1) or the cosine matrix (1.2) (related by (1.3) and (1.4)). It is convenient to determine the Euler–Rodrigues parameters by integrating Eq. (1.6) in which the components of the vector ω are found by integrating Eq. (3.4). The matrix A used to project w_0 onto the inertial axes is found from (1.3). This matrix allows us to project the vector w_0 onto the inertial frame of reference and, as indicated above, to integrate it there to determine the current velocity and coordinates of the body.

Thus, the implementation of such an inertial system involves

(i) calculation of ω by integrating three differential equations (first three equations in (3.4) or Eqs. (3.5)),

(ii) determination of the quaternion λ defining, according to (1.3), the cosine matrix A (for projecting the acceleration vector w_0 onto the inertial frame of reference) by integrating system (1.6) (four equations), and

(iii) determination of the velocity and coordinates of the body by integrating six equations.

In other words, it is necessary to integrate a system of differential equations of the 13th order. The initial conditions for this system are the values of the following quantities at the initial time: coordinates ($r_0 = [x_0 \;\; y_0 \;\; z_0]^T$), initial orientation (quaternion $\bar{\lambda}$) or cosine matrix $A(\bar{\lambda})$, velocities ($\bar{v}_0 = [v_{x0} \;\; v_{y0} \;\; v_{z0}]^T$), angular-velocity vector ($\omega_0 = [\omega_{x0} \;\; \omega_{y0} \;\; \omega_{z0}]$).

Note that $\bar{v}_0$, ω_0 are determined from GPS measurements of the velocities of three points of the body using the algorithm described in Sec. 2.

Thus, the functioning of the INS under consideration involves the integration of nonlinear (in the case of six accelerometers) system of differential equations of the 13th order. For implementation purposes, it would be appropriate to examine the case of "discretization" where the INS sensors are read not continuously, but at regular time intervals Δt, i.e., with frequency $f = 1/\Delta t$. Then the required navigation parameters (cosine matrix $A(\lambda)$, velocity v, coordinates r) are calculated at time intervals Δt. Since various discretization procedures can be used to calculate the navigation parameters, we will dwell on each of them. Let us estimate the quaternions at times $t_i, t_i - t_{i-1} = \Delta t$, $i = 1, 2, 3, \ldots$ [2, 13]. Thus, let the quasicoordinates (components of the vector $\nabla\theta_i = \int_{t_i}^{t_i+\Delta t} \omega dt$) be known on the time interval Δt. After the solution $\delta\lambda(t_i)$ of Eq. (1.6) subject to the initial condition $[1 \;\; 0 \;\; 0 \;\; 0]^T$ on the time interval Δt is expressed in terms of these quasicoordinates (i.e., calculating the quaternion corresponding to a small-angle rotation of the rigid body in time Δt), the orientation of the body can be described by successive multiplication of $\delta\lambda(t_i)$ "elementary" quaternions:

$$\lambda(t_i)=\lambda(t_{i-1})\delta\lambda(t_i),$$

$$\delta\lambda(t_i)=\left[\delta\lambda_0(t_i)\quad \delta\lambda_1(t_i)\quad \delta\lambda_2(t_i)\quad \delta\lambda_1(t_i)\right]^T. \tag{4.1}$$

The procedure has the following matrix form:

$$\lambda(t_i)=\begin{bmatrix}\delta\lambda_0(t_1) & -\delta\lambda_1(t_1) & -\delta\lambda_2(t_1) & -\delta\lambda_3(t_1)\\ \delta\lambda_1(t_1) & \delta\lambda_0(t_1) & \delta\lambda_3(t_1) & -\delta\lambda_3(t_1)\\ \delta\lambda_2(t_1) & -\delta\lambda_3(t_1) & \delta\lambda_0(t_1) & \delta\lambda_1(t_1)\\ \delta\lambda_3(t_1) & \delta\lambda_2(t_1) & -\delta\lambda_2(t_1) & \delta\lambda_0(t_1)\end{bmatrix}\begin{bmatrix}\lambda_0(t_{i-1})\\ \lambda_1(t_{i-1})\\ \lambda_2(t_{i-1})\\ \lambda_3(t_{i-1})\end{bmatrix}. \tag{4.2}$$

Expressions for the quaternions $\delta\lambda(t_i)$ in terms of the quasicoordinates $\nabla\theta_i$ that, depending on complexity, provide a certain degree of approximation are presented in [2, 13].

Next we will use the following approximation of the quaternion $\delta\lambda(t_i)$ [13, formula (2.6)]:

$$\delta\lambda(t_i)=\begin{bmatrix}1-\|\nabla\theta_i\|^2/12\\ \nabla\theta_i/2-\left(\nabla\theta_i\times\nabla\theta_{i-1}\right)/24\end{bmatrix}. \tag{4.3}$$

As in [13], to calculate $\nabla\theta_i$, we will use a quadratic spline-approximation of the angular-rate vector $\omega(t)$. For example, if $\omega(t_{i-2})$, $\omega(t_{i-1})$, $\omega(t_i)$ are known, then

$$\nabla\theta_i=\frac{\Delta t}{12}(5\omega(t_i)+8\omega(t_{i-1})-\omega(t_{i-2})), \tag{4.4}$$

$$\omega(t_i)=\omega(t_{i-1})+\frac{\varepsilon(t_i)+\varepsilon(t_{i-1})}{2}\Delta t, \tag{4.5}$$

where $\varepsilon(t_i)$ is the angular-acceleration vector determined from the accelerometers' readings at the time t_i (in (3.2), the elements of the matrix Ω are calculated using the components of the vector $\omega(t_{i-1})$). Using (4.1), (4.2), and the estimate of the quaternion $\delta\lambda(t_i)$ obtained from (4.4), we find the quaternion $\lambda(t_i)$ and, according to (1.3), the matrix $A(\lambda(t_i))$.

Using the matrix $A(\lambda(t_i))$ to project the vector $w_0(t_i)$ defined by (3.3) onto the body-fixed axes, we obtain estimates of the velocity $v(t_i)$ and coordinates $r(t_i)$ [13].

In this connection, we will restrict ourselves to the estimation of the accuracy of the matrix $A(\lambda)$.

5. Improving the Accuracy of ω. Let us consider how to use formulas (3.6) for improving the accuracy of ω in the cases of nine and 12 accelerometers. In the latter case, not only formulas (3.6) but also formulas (3.7) are used. Let us now consider the case of nine accelerometers whose readings give the angular acceleration vector ε (formulas (3.5)) and the components of the vector $\Omega_n=[\omega_2\omega_3\quad \omega_1\omega_3\quad \omega_2\omega_1]^T$ (formulas (3.6)). Assuming that $\Delta\omega_i=\omega(t_i)-\omega(t_{i-1})$ is small, we can write

$$\Omega_n=H\Delta\omega_i+\Omega_{n0},\qquad H=\begin{bmatrix}0 & \omega_3 & \omega_2\\ \omega_3 & 0 & \omega_1\\ \omega_2 & \omega_1 & 0\end{bmatrix},\qquad \Omega_{n0}=[\omega_2\omega_3\quad \omega_1\omega_3\quad \omega_1\omega_2]^T, \tag{5.1}$$

where Ω_n is defined by (3.6); the components of the vector ω appearing in H and Ω_{n0} are equal to the components of the vector $\omega(t_{i-1})$. Otherwise, with the assumption of smallness of $\Delta\omega_i$, we have the standard problem of weighed least-squares estimation of parameters [8]. Namely, if there is an initial estimate $\Delta\overline{\omega}_i=[\varepsilon(t_i)+\varepsilon(t_{i-1})]\Delta t/2$, then, according to (5.1), the vector z is observed:

$$z=\Omega_n-\Omega_{n0}=H\Delta\omega+\nu, \tag{5.2}$$

where ν is the vector of measurement errors. The quantity $\Delta\widehat{\omega}_i$ is estimated as follows [8, formula (12,2, 7)]:

$$\Delta\widehat{\omega}_i = \Delta\overline{\omega}_i + PH^T R^{-1}(z - H\Delta\overline{\omega}_i), \qquad P^{-1} = M^{-1} + H^T R^{-1} H, \tag{5.3}$$

where M is the covariance matrix of errors of the estimate $\Delta\overline{\omega}_i$; R is the covariance matrix of measurement errors ν in (5.2). Finally, the vector ω at time t_i is defined by

$$\omega(t_i) = \omega(t_{i-1}) + \Delta\widehat{\omega}_i, \tag{5.4}$$

where $\Delta\widehat{\omega}_i$ follows from (5.3).

Note that the matrix P^{-1} may be ill-conditioned; therefore, it may be reasonable to use the approach from [12, 13] to obtain the matrices P appearing in (5.3).

Since the matrices M and R are symmetric and positive definite, they can be represented in the form $M = m^2$, $R = r^2$, i.e., $m = M^{1/2}$, $r = R^{1/2}$. Then the matrix P^{-1} can be expressed as

$$P^{-1} = [m^{-1} \quad H^T r^{-1}]\,[m^{-1} \quad H^T r^{-1}]^T. \tag{5.5}$$

Using QR-decomposition, we transform the matrix $[m^{-1} \quad H^T r^{-1}]^T$ as

$$[m^{-1} \; H^T r^{-1}]^T = Q[\rho \;\; 0]^T, \tag{5.6}$$

where Q is an orthogonal matrix; ρ is an invertible matrix.

Considering that $Q^T Q = I$ and substituting (5.6) into (5.5), we obtain $P^{-1} = \rho^T \rho$ or $P = \rho^{-1}\rho^{-T}$. Thus, expression (5.3) can be represented as

$$\Delta\widehat{\omega}_i = \Delta\overline{\omega}_i + \rho^{-1}\rho^{-T} H^T R^{-1}(z - H\Delta\overline{\omega}_i). \tag{5.7}$$

If $M = \mu^2 I$, $R = \gamma^2 I$, then formula (5.7) can be represented as

$$\Delta\widehat{\omega}_i = \Delta\overline{\omega}_i + \rho^{-1}\rho^{-T} H^T (z - H\Delta\overline{\omega}_i), \tag{5.8}$$

where ρ is determined by the QR-decomposition of the following matrix:

$$[\lambda I \;\; H^T]^T, \qquad \lambda = \gamma / \mu. \tag{5.9}$$

Note that the correction algorithm described above can also be applied to 12 accelerometers. In this case, the matrix H and vector Ω_{n0} appearing in (5.1) have the form

$$H = \begin{bmatrix} 0 & \omega_3 & \omega_2 & 2\omega_1 & 0 & 0 \\ \omega_3 & 0 & \omega_1 & 0 & 2\omega_2 & 0 \\ \omega_2 & \omega_1 & 0 & 0 & 0 & 2\omega_3 \end{bmatrix}^T,$$

$$\Omega_{n0} = \left[\omega_2\omega_3 \quad \omega_1\omega_3 \quad \omega_1\omega_2 \quad \omega_1^2 \quad \omega_2^2 \quad \omega_3^2\right]^T.$$

Here, as with nine accelerometers, the components H and Ω_{n0} are determined by the components of the vector $\omega(t_{i-1})$. The components of the vector Ω_n are defined by (3.6) and (3.7).

6. Examples.

Example 1 [14]. Let us address the following navigation problem. Let a frame of reference $Oxyz$ be fixed to the Earth's surface. In this frame, the body (to which the frame $Ox'y'z'$ is fixed) circles in the plane xy with velocity $\overline{v} = 30$ m/sec and period T = 60 sec. During motion, its orientation (frame $Ox'y'z'$) is described by the following time-dependent Euler angles: $\psi = \dot{\psi}t$, $\dot{\psi} = 2\pi / T$, $\varphi = 0$, $\vartheta = 0$. According to (1.5), the projections of the angular rate onto the moving axes are $\omega_1 = \omega_2 = 0$, $\omega_3 = \dot{\psi}$. The initial attitude of the body, according to (1.1), is specified by the quaternion $\lambda = [1 \;\; 0 \;\; 0 \;\; 0]^T$ and, hence, according to (1.3), by a

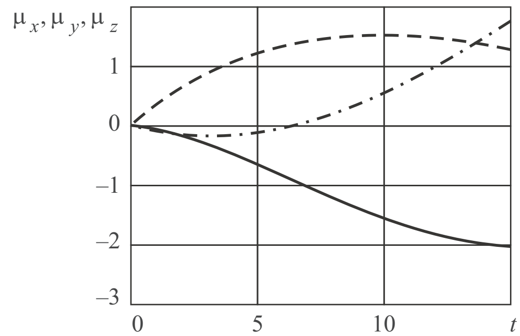

Fig. 4

cosine matrix, which is a unit matrix. At $t = 0$, the body is on the y-axis at a distance $R_0 = T\bar{v} / 2\pi$, its velocity (vector v_0 appearing in (2.2)) and acceleration w_0 in (3.2) are defined by $v_0 = [-\bar{v} \;\; 0 \;\; 0]^T$, $w_0 = \left[0 \;\; -\dfrac{\bar{v}^2}{R_0} + g\right]^T$; where $g = 9.81$ m/sec^2 is the acceleration of gravity. The accelerometers are placed on the coordinate axes as shown in Fig. 3, where $L = 0.1$ m. The errors of the accelerometers (n_w in (3.4)) are modeled by uniformly distributed noncorrelated random numbers with zero expectation and variance $\sigma_w = 10^{-3}$ m/sec^2.

Thus, the above initial conditions and the assumptions on the errors of accelerometers allow us to model the operation of an inertial-navigation system without angular-rate sensors. To illustrate the algorithm described in Sec. 2, we will consider a situation where the initial (angular and linear) velocities of the body are determined from GPS-measured velocities of three points of the body at $t = 0$. In this connection, we assume that the matrix P in (2.2) has the form

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix P^{-1} does not exist, it is impossible to use the algorithm [16].

The measurement errors (the components of the vector n_v in (2.4)) are assumed noncorrelated uniformly distributed numbers with zero expectation and variance $\sigma_v = 10^{-1}$ m/sec. Using these initial data and the algorithm of Sec. 2, we obtain the following estimates:

$$\omega(0) = \left[-10^{-3} \quad 6.4 \cdot 10^{-3} \quad 0.1024\right]^T,$$

$$v_0(0) = \left[-29.9955 \quad -3.4 \cdot 10^{-3} \quad 0.0101\right]^T. \tag{6.1}$$

These estimates are used as the initial conditions in modeling an inertial-navigation system ($\omega(0)$ is measured in 1/sec and $v_0(0)$ is measured in m/sec). The initial values of the other parameters, namely, the parameters defining the initial position and attitude of the body are their exact values at $t = 0$.

The operation of the system for 15 sec was simulated using the ode45.m and rand.m Matlab routines to integrate a system of differential equations and to generate random numbers.

Figure 4 presents quantities (obtained by modeling attitude errors) μ_x, μ_y, μ_z (in degrees), which are off-diagonal elements of the matrix $A^T(\bar{\lambda})A(\lambda)$ approximated as (1.7)

$$A^T(\bar{\lambda})A(\lambda) \cong \begin{bmatrix} 1 & \mu_z & -\mu_y \\ -\mu_z & 1 & \mu_x \\ \mu_y & -\mu_x & 1 \end{bmatrix},$$

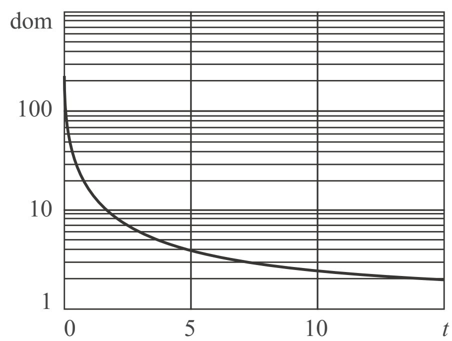

Fig. 5

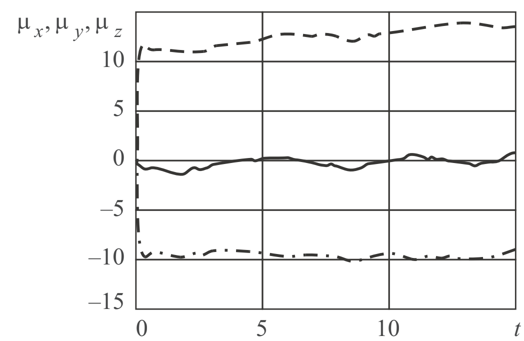

Fig. 6

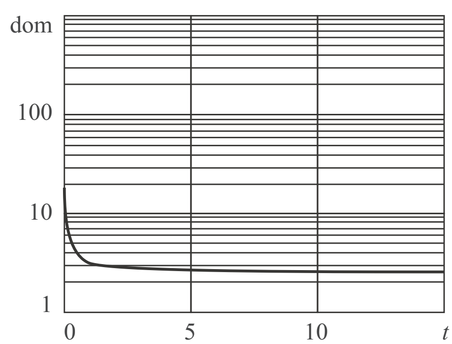

Fig. 7

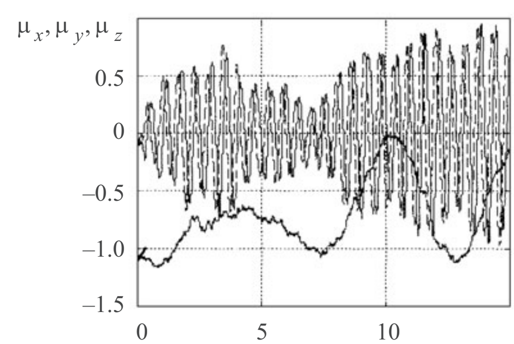

Fig. 8

where λ is the quaternion obtained by integration; $\overline{\lambda}$ is its exact value; $A(\overline{\lambda}), A(\lambda)$ are the exact and estimated (by integration) values of the cosine matrix.

In Fig. 4 and others, the solid line corresponds to μ_x, the dashed line to μ_y, and the dash-and-dot line to μ_z. Note that the results in Fig. 4 correspond to the results in [14, Figs. 9–11]. It should also be noted that Fig. 4 was obtained using the ode45.m Matlab routine for the integration of a system of nonlinear differential equations.

Example 2. Let a body have nine accelerometers, of which six accelerometers are arranged as shown in Fig. 3 and the other three are at the origin (see Sec. 3 for details). The origin of the instrument frame of reference fixed to the moving frame of reference ($Ox'y'z'$) is defined by a vector $R = [0\ 1\ 0]^T$. The orientation of the moving frame is specified by the following time-dependent Euler angles: $\psi(0)=0,\ \theta(0)=\pi/4,\ \varphi(0)=0,\ \dot{\psi}=1,\ \dot{\theta}=0,\ \dot{\varphi}=10.$ The projections of the angular rate onto the moving axes are defined by (1.5). Note that the modulus of angular-rate vector exceeds 600 deg/sec in this example, while in example 1 it is 6 deg/sec. In this connection, we assume that $\Delta t = 10^{-3}$ sec. The initial orientation (quaternion) is defined by (1.1). As in example 1, we obtain $\sigma_w = 10^{-3}$ m/sec^2, $L = 0.1$ m. The error of the initial angular rate (analog of (6.1)) is modeled as follows. The initial value ($\tilde{\omega}(0)$) is given by

$$\tilde{\omega}(0) = \omega(0)/2, \tag{6.2}$$

where $\omega(0)$ is the exact value defined by (5).

Let $\lambda = 0.1$ in (5.9). The errors of the kinematic parameters obtained in modeling of motion for 15 sec are presented in Figs. 5 and 6.

Figure 5 gives the values (in deg/sec) of

$$\mathrm{dom}(t_k) = \left(\sum_{i=1}^{k} ||\omega(t_i) - \tilde{\omega}(t_i)|| \right) \Big/ k,$$

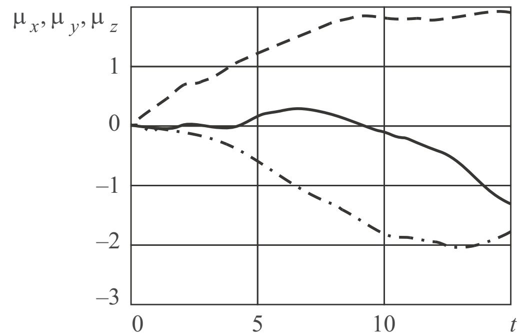

Fig. 9

where $\tilde{\omega}(t_i)$ is the estimate of the angular rate obtained from (5.4); $\omega(t_i)$ is the exact value of the angular-rate vector. In other words, dom characterizes the accuracy of estimating the current angular-rate vector obtained with (5.4) (dom is measured in deg/sec in Fig. 5). Figure 6 gives attitude errors (notation is the same as in Fig. 4). It may be concluded that the correction algorithm (5.4) makes it possible to increase considerably the accuracy of the current value of ω and, hence, attitude. For example, according to Fig. 5 the error of ω at the 15th second is of the order of 2 deg/sec, whereas, according to (6.2), the error of the initial value of ω is of the order of 300 deg/sec. As a consequence, the attitude is determined more accurately (Fig. 6). For example, at the initial stage, the attitude error caused by an inaccurate initial value of ω increases differently. After the reduction in the error of the current value of ω, the attitude errors do not undergo significant changes. While the error of the initial angular rate is of the order of 300 deg/sec, the attitude error at the 15th second is of the order of 10 deg.

Example 3. Let us consider a measuring system containing 12 accelerometers (see Sec. 3 for its description). We will keep the initial data of Example 2 and change only the values of σ_w and λ. Let $\sigma_w = 10^{-1}$ m/sec^2, $\lambda = 700$, i.e., the accuracy of accelerometers is reduced by two orders of magnitude. The results of simulation are presented in Figs. 7 and 8 (the notation being the same as in Figs. 5 and 6).

These results are indicative of much higher efficiency of the measuring system with 12 accelerometers. For example, according to Fig. 7, despite the fact that the measurement errors of the accelerations are higher by two orders of magnitude, the error of the current value of ω at the 15th second is of almost the same order as in Example 2. Thus, in such a system, the error of the initial value of ω decreases much more rapidly and, consequently, the attitude errors are reduced (compare Figs. 6 and 8).

Example 4. We continue the discussion of Example 1. Keeping all the initial data of Example 1, we assume that the measuring system has 12 accelerometers (as in Example 3). Let $\mu = 7$ in (5.9). The results of simulation are presented in Fig. 9.

Comparing Figs. 9 and 4 reveals identical accuracy of determining the attitude. Unlike Example 1, however, the estimate ω is obtained using the finite-difference scheme (4.5) rather than the integration of a nonlinear differential equation (ode45.m).

Conclusions. Operating algorithms for autonomous inertial-navigation systems without angular-rate sensors have been outlined. Systems containing 6, 9, and 12 accelerometers have been considered. Since six accelerometers are sufficient to measure the angular acceleration, systems with 9 or 12 accelerometers can use the extra accelerometers to improve the accuracy of the angular-rate vector. Correction algorithms have been presented. It has been shown, by way of examples, that such systems may be effective if the vehicle is moving with high angular rate at which the use of angular-rate sensors becomes problematic.

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