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NONSTATIONARY DEFORMATION OF AN ELECTROELASTIC NONCLOSED CYLINDRICAL SHELL UNDER MECHANICAL AND ELECTRIC LOADING

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A numerical–analytic solution describing the nonstationary vibrations of an infinitely long nonclosed cylindrical electroelastic shell with hinged ends is found. The direct and inverse piezoelectric effects are considered. The dynamic processes are modeled using the linear theory of thin electroelastic shells based on the generalized Kirchhoff–Love hypotheses. To satisfy the boundary conditions, additional loads are introduced. The Laplace transform is used to reduce the problem to a system of Volterra equations. The numerical results are plotted and analyzed

Keywords: nonclosed infinitely long cylindrical shell, nonstationary vibrations, Laplace transform

Introduction. Devices that have piezoceramic transducers as components are of wide use [15]. Of the great variety of configurations, transducers in the form of thin-walled cylindrical shells are widely used and their dynamic behavior is of great interest [4, 5, 7, 18]. Of particular interest is the vibrations caused by impulsive electromechanical loads [1, 8, 16, 17]. There are still open questions on the behavior of piezoceramic elements with various design features and various boundary conditions.

The present paper addresses the deformation of a piezoelectric cylindrical strip with hinged longitudinal edges under either mechanical or electric impulsive loading. Among the relevant publications, noteworthy are [6, 11–14] which study the coupled electroelastic processes in piezoelectric transducers of similar shape.

1. Problem Formulation. Consider an infinitely long open circular cylindrical shell with a central angle of $2\theta_0$ (Fig. 1). The shell consists of perfectly bonded thin electroelastic (inner) and elastic (outer) layers of thickness h_p and h_m, the radius of the interface being R_1. The electroelastic layer is polarized throughout the thickness and coated with continuous, infinitely thin electrodes, of which the inner one is grounded, and an electric potential V is applied to the outer electrode. Uniform pressure p_0 (Fig. 1) acts on a domain with a central angle of $2\theta_2$ symmetric about the edges. The longitudinal edges of the shell are hinged.

Our goal here is to determine the transient characteristics of a bimorph transducer operating in the modes of inverse piezoelectric effect and direct piezoelectric effect.

The initial conditions are zero (the transducer is at rest until $t=0$).

2. Equations of Motion. With the Kirchhoff–Love hypotheses generalized to electromecanics, the initial system of equations of motion of a bimorph cylindrical piezoelectric transducer is as follows [8]:

$$\frac{\partial^2 w}{\partial t^2}+\delta\frac{\partial^4 w}{\partial\theta^4}+w-\delta\frac{\partial^3 u_0}{\partial\theta^3}+\frac{\partial u_0}{\partial\theta}=\gamma_0 q_z-\gamma_1\left(V_\theta-\frac{a_\mathrm{p}}{R_0}\frac{\partial^2 V_\theta}{\partial\theta^2}\right),$$

$$\frac{\partial^2 u_0}{\partial t^2}-(1+\delta)\frac{\partial^2 u_0}{\partial\theta^2}+\delta\frac{\partial^3 w}{\partial\theta^3}-\frac{\partial w}{\partial\theta}=\gamma_0 q_\theta+\gamma_1\left(1+\frac{a_\mathrm{p}}{R_0}\right)\frac{\partial V_\theta}{\partial\theta}, \tag{1}$$

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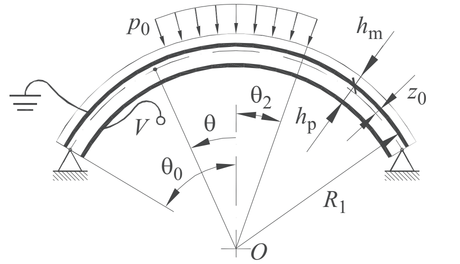

Fig. 1

where w and u_0, q_z and q_θ are the normal and tangential displacements of the datum surface [9] and external load, respectively; V_θ is the function describing the profile of the electric potential applied to the outer conductive coating; $a_p = z_0 - h_p / 2$ is the distance between the datum surface with radius of curvature $R_0 = R_1 - z_0$ and the mid-surface of the electroelastic layer.

If the distance z_0 between the datum surface and the interface (Fig. 1) is equal to $(c_{1p} h_p^2 - c_{1m} h_m^2)/2D_N$, then the stresses and strains are in the simplest relationship [9]:

$$N_\theta = \frac{D_N}{R_0}\left(\frac{\partial u_0}{\partial\theta} + w\right) - e_1 V_\theta,$$

$$M_\theta = \frac{\overline{D}}{R_0^2}\left(\frac{\partial u_0}{\partial\theta} - \frac{\partial^2 w}{\partial\theta^2}\right) - e_1 a_p V_\theta. \tag{2}$$

Electric-flux density does not depend on the thickness coordinate and is expressed as

$$D_z = \frac{e_1}{R_0}\left(1 + \frac{a_p}{R_0}\right)\frac{\partial u_0}{\partial\theta} + \frac{e_1}{R_0}\left(w - \frac{a_p}{R_0}\frac{\partial^2 w}{\partial\theta^2}\right) + \frac{\gamma_1}{\gamma_2}\frac{V_\theta}{R_0}. \tag{3}$$

The constant coefficients are determined from the formulas

$$\gamma_0 = R_0^2 / D_N, \quad \gamma_1 = -\gamma_0 e_1 / R_0, \quad \gamma_2 = -e_1^2 h_p / D_N \varepsilon_3, \quad \delta = \overline{D} / D_N R_0^2,$$

$$\rho_h = \rho_p h_p + \rho_m h_m, \quad D_N = c_{1p} h_p + c_{1m} h_m, \quad \overline{D} = c_{1p} J_p + c_{1m} J_m + \Delta D,$$

$$\Delta D = (e_1^2 / \varepsilon_3) J_{p0}, \quad J_p = (z_0^3 - (z_0 - h_p)^3)/3, \quad J_m = ((z_0 + h_m)^3 - z_0^3)/3,$$

$$J_{p0} = h_p^3 / 12, \quad \overline{\nu} = \nu + (1-\nu)\Delta D / \overline{D}, \quad c_{1j} = 1/s_{11}^j (1-\nu^2),$$

$$e_1 = c_{1p} d_{31}(1+\nu), \quad \varepsilon_3 = \varepsilon_{33}^T (1 - 2d_{31} e_1 / \varepsilon_{33}^T),$$

s_{11}^j, ν are the elastic compliances and Poisson's ratio of the materials ($j = m, p$); d_{31}, ε_{33}^T are the piezoelectric modulus and permittivity of the piezoceramics.

The equations of motion (1) are supplemented with zero initial conditions:

$$w\big|_{t=0} = \frac{\partial w}{\partial t}\bigg|_{t=0} = u_0\big|_{t=0} = \frac{\partial u_0}{\partial t}\bigg|_{t=0} = 0$$

and mechanical hinged-boundary conditions

$$w\big|_{\theta=\pm\theta_0} = 0, \quad u_0\big|_{\theta=\pm\theta_0} = 0, \quad M_\theta\big|_{\theta=\pm\theta_0} = 0. \tag{4}$$

The electric boundary conditions depend on how the piezoceramic layer is electroded and how electric energy is supplied (picked up). If the continuous electrodes are connected to a generator or short-circuited, then the function

$$V_\theta(\theta,t)=V(t) \tag{5′}$$

in (1) assumed given. If the electrodes are open-circuited or connected to a electronic device of infinitely high impedance, then it is necessary that the running current of displacement through the mid-surface of a piezolayer be equal to zero, i.e., $\frac{\partial}{\partial t}\int_{-\theta_0}^{\theta_0} D_z d\theta=0$. Then we obtain from (3) the following expression for the unknown potential difference:

$$V_\theta=V=-\frac{\gamma_2}{\gamma_1}\frac{1}{2\theta_0}\int_{-\theta_0}^{\theta_0}\left[\left(1+\frac{a_p}{R_0}\right)\frac{\partial u_0}{\partial\theta}+\left(w-\frac{a_p}{R_0}\frac{\partial^2 w}{\partial\theta^2}\right)\right]d\theta. \tag{5″}$$

Equations (1), (4), (5′), (5″) constitute a closed-form system describing the coupled vibrations of the bimorph electroelastic transducer. Note that the time t is normalized to $R_0\sqrt{\rho_h/D_N}$, while the displacements w and u_0 to R_0.

3. Problem-Solving Method. If the longitudinal edges of the shell are hinged, then the functions w and u_0 can be represented as follows, taking into account symmetry about the section $\theta=0$ (Fig. 1):

$$w=\sum_{k=1}^{\infty}c_k(t)\cos KN\theta,$$

$$u_0=\sum_{k=1}^{\infty}b_k(t)\sin KN\theta, \tag{6}$$

where c_k, b_k are unknown coefficients; $\theta\in[-\theta_0,\theta_0]$, $K=(2k-1)/2$, $N=\pi/\theta_0$.

The electric potential is expanded into a Fourier series of even functions:

$$V_\theta=\sum_{k=1}^{\infty}v_k(t)\cos KN\theta. \tag{7}$$

Its coefficients can be expressed in terms of the function V: $v_k=V\cdot 2\sin KN\theta_0/KN\theta_0$ according to Eqs. (5′) and (5″).

Series (6) and (7) do not allow the complete satisfaction of conditions (4). Therefore, to satisfy the equality $u_0|_{\theta=\pm\theta_0}=0$, we supplement the given mechanical load p_0 with additional unknown tangential forces q_1 distributed over small strips of width $\Delta\theta$ near the sections $\theta=\pm\theta_0$. Such an approach is used, for example, in [3] in studying the unsteady axisymmetric deformation of a mechanically loaded elastic plate.

The components of the mechanical load

$$q_z=-p_0(t)H(\theta_2-|\theta|),$$
$$q_\theta=-q_1(t)\cdot\mathrm{sign}(\theta)\cdot H(|\theta|-(\theta_0-\Delta\theta)),$$

where H is the Heaviside function, can also be represented similarly to (6):

$$q_z=\sum_{k=1}^{\infty}f_k(t)\cos KN\theta, \qquad q_\theta=\sum_{k=1}^{\infty}g_k(t)\sin KN\theta \tag{8}$$

$$(f_k=-p_0\cdot 2\sin KN\theta_2/KN\theta_0, \qquad g_k=-N_0\cdot 2\sin KN\theta_0/R_0\theta_0).$$

Here the passage to the limit to obtain the concentrated forces $N_0=\lim_{\Delta\theta\to 0}q_1R_0\Delta\theta$ in the sections $\theta=\pm\theta_0$ has already been done.

Substituting series (6)–(8) into the Laplace-transformed system (1), we obtain an algebraic system of equations for the Laplace transforms of the coefficients c_k and b_k:

$$b_k^L \xi_k^{(1)} + c_k^L (s^2 + \xi_k^{(2)}) = \gamma_0 f_k^L - \gamma_1 \xi_k^{(4)} v_k^L,$$

$$b_k^L (s^2 + \xi_k^{(3)}) + c_k^L \xi_k^{(1)} = \gamma_0 g_k^L - \gamma_1 \xi_k^{(5)} v_k^L, \tag{9}$$

where s is the Laplace parameter; $\xi_k^{(1)} = KN(1+\delta K^2 N^2)$, $\xi_k^{(2)} = 1+\delta K^4 N^4$, $\xi_k^{(3)} = K^2 N^2 (1+\delta)$, $\xi_k^{(4)} = 1+(a_\text{p} / R_0) K^2 N^2$, $\xi_k^{(5)} = KN(1+a_\text{p} / R_0)$.

The solution of system (9) becomes

$$c_k^L = D_k^c / D_k, \quad b_k^L = D_k^b / D_k, \tag{10}$$

$$D_k(s) = s^4 + s^2 \lambda_k^{(1)} + \lambda_k^{(2)},$$

$$D_k^c = \gamma_0 f_k^L (s^2 + \lambda_k^{(5)}) - \gamma_0 g_k^L \xi_k^{(1)} - \gamma_1 v_k^L \xi_k^{(4)} (s^2 + \lambda_k^{(3)}),$$

$$D_k^b = -\gamma_0 f_k^L \xi_k^{(1)} + \gamma_0 g_k^L (s^2 + \lambda_k^{(4)}) - \gamma_1 v_k^L \xi_k^{(5)} (s^2 + \lambda_k^{(7)}),$$

and

$$\lambda_k^{(1)} = \xi_k^{(2)} + \xi_k^{(3)}, \quad \lambda_k^{(2)} = \xi_k^{(2)} \xi_k^{(3)} - \xi_k^{(1)2}, \quad \lambda_k^{(3)} = \xi_k^{(3)} - \xi_k^{(1)} \xi_k^{(6)},$$

$$\lambda_k^{(4)} = \xi_k^{(2)}, \quad \lambda_k^{(5)} = \xi_k^{(3)}, \quad \lambda_k^{(7)} = \xi_k^{(2)} - \xi_k^{(1)} / \xi_k^{(6)}, \quad \xi_k^{(6)} = \xi_k^{(5)} / \xi_k^{(4)}.$$

Next, using the standard rules of operational calculus, we recover the original functions:

$$c_k = -\gamma_0 \frac{2\sin KN\theta_2}{KN\theta_0} p_0 * I_k^{(5)} + \gamma_0 \xi_k^{(1)} \frac{2\sin KN\theta_0}{R_0 \theta_0} N_0 * I_k^{(1)} - \gamma_1 \xi_k^{(4)} \frac{2\sin KN\theta_0}{KN\theta_0} V * I_k^{(3)},$$

$$b_k = \gamma_0 \xi_k^{(1)} \frac{2\sin KN\theta_2}{KN\theta_0} p_0 * I_k^{(1)} - \gamma_0 \frac{2\sin KN\theta_0}{R_0 \theta_0} N_0 * I_k^{(4)} - \gamma_1 \xi_k^{(5)} \frac{2\sin KN\theta_0}{KN\theta_0} V * I_k^{(7)}, \tag{11}$$

where $f * I = \int_0^t f(\tau) I(t-\tau) d\tau$; the subintegral functions I have the form:

$$I_k^{(r)} = \sum_{j=1}^{2} \frac{\beta_k^{(r,j)}}{\alpha_k^{(j)}} \sin(\alpha_k^{(j)} t) \quad (r = \overline{1, 3, 5, 7}),$$

$$\beta_k^{(1,j)} = \frac{1}{\alpha_k^{(\iota)2} - \alpha_k^{(j)2}}, \quad \beta_k^{(r,j)} = \frac{\lambda_k^{(r)} - \alpha_k^{(j)2}}{\alpha_k^{(\iota)2} - \alpha_k^{(j)2}} \quad (\iota, j = 1, 2;\ \iota \ne j;\ r \ne 1);$$

$\alpha_k^{(j)}$ are the modules of the imaginary roots of the biquadratic equation $D_k(s) = 0$.

In the case of electromechanical excitation of the transducer, the functions p_0 and V appearing in (11) are assumed given, and the unknown force N_0 can be found from the condition

$$u_0 \big|_{\theta=\theta_0} = \sum_{k=1}^{\infty} b_k \sin KN\theta_0 = 0, \tag{12}$$

which is a Volterra equation of the first kind.

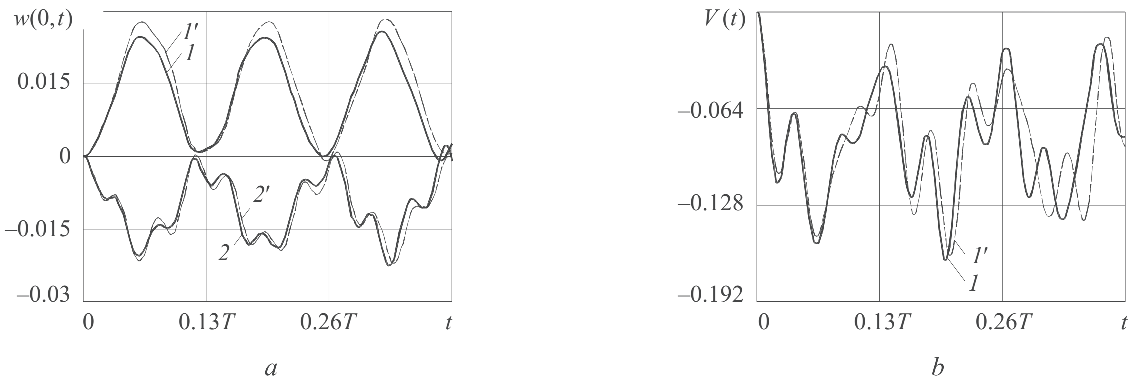

Fig. 2

If the electroelastic layer is in the mode of direct piezoelectric effect ("idling"), then both N_0 and V are unknown. To calculate them, we set up a system of integral equations in which one equation is (12) and the other equation is derived by substituting (11) into (8) in view of (6):

$$V = -\frac{\gamma_2}{\gamma_1} \sum_{k=1}^{\infty} c_k \xi_k^{(4)} \frac{\sin KN\theta_0}{KN\theta_0}. \tag{13}$$

The system of integral equations (12), (13) is solved numerically using special regularizing algorithms [10] stable against computational errors.

After finding the values of p_0, N_0, and V, we substitute them into (11) and calculate the components of the displacement vector (6).

4. Analysis of the Numerical Results. The numerical analysis was made for a nonclosed shell with circumference length $\hat{l}_0 = 2R_1\theta_0 = 104.7$ mm consisting of a PZT-5 piezoceramic layer (h_p = 2 mm) and a BT-6 titanium alloy layer (h_m = h_p/2) with the following material characteristics: ρ_p = 7600 kg/m^3, s_{11}^p = 15.4·10^{-12} m^2/N, ν = 0.331, d_{13} = –178·10^{-12} C/N, ε_{33}^T = 1750·ε_0, ε_0 = 8.85·10^{-12} F/m, ρ_m = 4450 kg/m^3, s_{11}^m = 8.85·10^{-12} m^2/N.

For numerical purposes, 40 terms were retained in series (6)–(8), which ensures an approximation error of no greater than 3%. The accuracy of calculation was also controlled by varying the time step length during the numerical solution of the integral equations (12), (13).

As a test example, we consider a nonclosed bimorph shell with relatively large radius $R_1 = 50$ m subject to pressure step $p_0(t) = 10^4 \cdot H(t)$ N/m^2. The electrodes are short-circuited ($V = 0$), and the area of application of the mechanical load is defined by a central angle $2\theta_2 = \theta_0$. It is clear that with R_1 and $\hat{l}_0 / (h_\text{p} + h_\text{m})$, the transducer is similar to an asymmetric bimorph strip of width $\bar{l}_0 = \hat{l}_0$ modeled in [2]. The normal displacements (omitted here) in the section equidistant from the edges calculated by formulas (6) and (11) and by the procedure outlined in [2] are in good agreement and can be described, with adequate accuracy, by the equation $w = w_\text{st}(1 - \cos(2\pi t/T))$, where $w_\text{st} = -0.0478$ mm is the static deflection in the same section induced by mechanical load $p_0 = 10^4$ N/m^2 on the area $\theta \in [-\theta_2, \theta_2]$, $T = 2.03$ msec is the period of the lower natural mode of the bimorph strip at $V = 0$ [2].

The greater the curvature of the transducer, the less the maximum radial displacement in the section $\theta = 0$. In particular, for $R_1 = 0.05$ m ($\theta_0 = \pi/3$) and the mode of direct piezoelectric effect, the amplitude $w(0,t)$ decreases by 98.7%, while the period of the principal vibration mode by 87.5% (curve *1*, Fig. 2*a*). The potential difference between the electrodes of the piezoceramic layer in this case is represented by curve *1* in Fig. 2*b*.

Note that in Fig. 2, the function $w(0,t)$ is normalized to w_st, and $V(t)$ to the potential difference V_st between the open-circuited electrodes of the bimorph strip under static mechanical load p_0 ($V_\text{st} \approx 62.5$ V).

In Fig. 2*a*, solid curve *2* illustrates the reaction of a transducer with $R_1 = 0.05$ m to instantaneous constant electric potential $V(t) = -V_\text{st} \cdot H(t)$. Curve *2* suggests that the electric excitation of the transducer causes the displacement $w(0,t)$ to vary with time in a more complicated manner (than curve *1*). Vibrations with higher frequency are represented by the first term in (6), (7). For the chosen boundary and loading conditions, the shape of the curves on which these vibrations are imposed are mainly determined by the terms with $k = 2$.

To validate the results (Fig. 2), the problems were solved using a finite-element software. The numerical solutions are shown by dashed curves with primed number. A possible cause of the disagreement is that the electroelastic transducer model is based on the assumption of infinite shear stiffness. Generally, however, it may be stated that the curves of radial displacement (Fig. 2*a*) and potential difference (Fig. 2*b*) are in satisfactory qualitative and quantitative agreement.

Conclusions. A solution describing the nonstationary deformation of an electroelastic two-layer transducer in the form of a nonclosed cylindrical shell has been obtained by introducing an additional load to satisfy the kinematic conditions at the edges of the shell. Mathematically, the problem has been reduced to a system of integral equations over time derived from the boundary conditions. The calculations have demonstrated the efficiency of the developed numerical analytic method for the determination of the mechanical and electric characteristics of the transient under consideration. The results have been validated by solving a test problem for a bimorph asymmetric strip and comparing with finite-element solutions.

Note that the formulas presented here can be used to analyze other types of boundary conditions of the bimorph shell and the geometries of the electrodes on its electroelastic layer. The obtained results can be used to solve applied problems of the active control of the nonstationary vibrations of structural elements in the form of nonclosed cylindrical shells.

REFERENCES

1. A. E. Babaev, V. G. Savin, and Yu. V. Kozhemyaka, "Radiation of acoustic pulses of set profile by an electroelastic cylindrical shell," *Teor. Prikl. Mekh*., **33**, 186–191 (2001).
2. A. E. Babaev, A. A. Babaev, and I. V. Yanchevskii, "Active damping of the nonstationary flexural vibrations of a bimorph beam," *Int. Appl. Mech*., **46**, No. 7, 806–811 (2010).
3. E. G. Goloskokov and A. P. Filippov, *Nonstationary Vibrations of Deformable Systems* [in Russian], Naukova Dumka, Kyiv (1977).
4. A. N. Guz, V. D. Kubenko, and A. E. Babev, "Dynamics of systems of shells interacting with a fluid," Vol. 3 of the six-volume series *Advances in Mechanics* [in Russian], A.S.K., Kyiv (2007), pp. 185–245.
5. V. L. Karlash, "Resonant electromechanical vibrations of piezoelectric shells of revolution (review)," *Int. Appl. Mech*., **44**, No. 4, 361–387 (2008).
6. T. V. Karnaukhova, "Active damping of the forced resonant vibrations of a shallow viscoelastic composite cylindrical panel under unknown mechanical load," *Akust. Visn*., **11**, No. 4, 24–30 (2008).
7. I. F. Kirichok, T. V. Karnaukhova, and N. V. Peresun'ko, "Resonant axisymmetric vibrations and self-heating of cylindrical shells controlled with piezoelectric actuators," *Teor. Prikl. Mekh*., **46**, 132–140 (2009).
8. V. D. Kubenko and I. V. Yanchevskii, "Nonstationary nonaxisymmetric vibrations of a cylindrical metal–piezoceramic shell," in: *Methods for Solving Applied Problems in Solid Mechanics* [in Russian and Ukrainian], issue 12, DNU, Dnipropetrovsk (2011), pp. 188–196.
9. S. I. Rudnitskii, V. M. Sharapov, and N. A. Shul'ga, "Vibrations of a bimorphic disk transducer of the metal-piezoceramic type," *Int. Appl. Mech*., **26**, No. 10, 973–980 (1990).
10. E. G. Yanyutin and I. V. Yanchevskii, *Impulsive Loads on Elastically Deformed Structural Members* [in Russian], Izd. KhADI, Kharkov (2001).
11. H. J. Ding, R. Q. Xu, and W.Q. Chen, "Free vibration of transversely isotropic piezoelectric circular cylindrical panels," *Int. J. Mech. Sci*., **44**, No. 1, 191–206 (2002).
12. K. M. Gupta, "An orthotropic adaptive shallow cylindrical shell on elastic foundation," *Int. J. Res. Rev. Appl. Sci*., **2**, No. 1, 67–87 (2010).
13. A. Nosier and M. Ruhi, "Three dimensional analysis of laminated cylindrical panels with piezoelectric layers," *IJE. Trans. B: Applications*, **19**, No. 1, 61–72 (2006).
14. Y. Ootao and Y. Tanigawa, "Control of transient thermoelastic displacement of an angle-ply laminated cylindrical panel bonded to a piezoelectric layer,"*Appl. Math. Comp*., **148**, No. 1, 263–286 (2004).

15. J. Qiu and H. Ji, "The application of piezoelectric materials in smart structures," *Int. J. Aeron. Space Sci*., **11**, No. 4, 266–284 (2010).
16. G. G. Sheng and X. Wang, "Active control of functionally graded laminated cylindrical shells," *Compos. Struct*., **90**, No. 4, 448–457 (2009).
17. H. M. Wang, H. J. Ding, and Y. M. Chen, "Dynamic solution of a multilayered orthotropic piezoelectric hollow cylinder for axisymmetric plane strain problems," *Int. J. Solid. Struct.*, **42**, No. 1, 85–102 (2005).
18. J. S. Yang, "Piezoelectric transformer structural modeling — A review," *IEEE Trans. Ultrason. Ferroelectr. Freq. Control*, **54**, No. 6, 1154–1170 (2007).