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MIXED SYSTEM OF EQUATIONS IN KIRCHHOFF'S THEORY OF THE TRANSVERSE VIBRATIONS OF PLATES

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A general analysis of the mixed systems of four equations in Kirchhoff's theory of the vibrations of plates in rectangular and polar coordinates is carried out. It is shown that these systems can be represented in Hamiltonian (canonical) operator form in space coordinate after the selection of the appropriate "canonical" variables and operator Hamiltonian. Functionals for canonical systems are formulated

Keywords: mixed equations of vibrations of plates in rectangular and polar coordinates, Hamiltonian system in space coordinate, functional for canonical system

Introduction. The monograph [3] was the first in the scientific literature to represent the mixed system of equations describing the vibrations of an elastic plate in Hamiltonian operator form in space coordinate. In many subsequent studies reviewed in [2, 6–8, etc.], the Hamiltonian formalism in the form developed in [3] was extended to the equations and problems of elasticity, electroelasticity, and magnetoelasticity. These results made it possible to determine the properties of the characteristic equations and the general structure of solutions of wave problems for periodic media. Canonical equations in a space coordinate in problems of the harmonic flexural vibrations of plates with parameters periodic in one coordinate were derived in [4]. Similar studies were conducted in [6, etc.] and in the theory of vibrations of beams with periodic parameters.

Here the Hamiltonian formalism is applied to Kirchhoff's theory of the bending of plates. It will also be shown that canonical operator equations in a space coordinate can be derived from the variational principle.

1. Problem Formulation. In Kirchhoff's (classical) theory of the transverse vibrations of a thin plate in orthogonal curvilinear coordinates α_1, α_2, the bending moments, M_{11} and M_{22}, twisting moment, $M_{12} = M_{21}$, transverse forces, Q_1, Q_2, and deflection, w, in the midplane are related by the vibration equations

$$\frac{1}{A_1 A_2}\left\{\frac{\partial A_2 M_{11}}{\partial\alpha_1} - \frac{\partial A_2}{\partial\alpha_1} M_{22} + \frac{1}{A_1}\frac{\partial A_1^2 M_{12}}{\partial\alpha_2}\right\} - Q_1 = 0,$$

$$\frac{1}{A_1 A_2}\left\{\frac{\partial A_1 M_{22}}{\partial\alpha_2} - \frac{\partial A_1}{\partial\alpha_2} M_{11} + \frac{1}{A_2}\frac{\partial A_2^2 M_{21}}{\partial\alpha_1}\right\} - Q_2 = 0,$$

$$\frac{1}{A_1 A_2}\left\{\frac{\partial A_2 Q_1}{\partial\alpha_1} + \frac{\partial A_1 Q_2}{\partial\alpha_2}\right\} = \rho h \frac{\partial^2 w}{\partial t^2} \tag{1.1}$$

and the constitutive equations

$$M_{11} = -D\left[\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right) + \frac{1}{A_1 A_2^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_2} + \nu\left(\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\right) + \frac{1}{A_1^2 A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial w}{\partial\alpha_1}\right)\right],$$

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$$M_{22}=-D\left[\nu\left(\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right)+\frac{1}{A_1A_2^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_2}\right)+\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\right)+\frac{1}{A_1^2A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial w}{\partial\alpha_1}\right],$$

$$M_{12}=M_{21}=-\frac{1-\nu}{2}D\left[\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_2^2}\frac{\partial w}{\partial\alpha_2}\right)+\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_1^2}\frac{\partial w}{\partial\alpha_1}\right)\right], \tag{1.2}$$

which include the formulas for strains. In (1.1) and (1.2): ρ, E, ν are density, Young's modulus, and Poisson's ratio; $D=I_1E/(1-\nu^2)$ is flexural stiffness; h is the thickness of the plate; $I_1=h^3/12$ is the area moment of inertia per unit length; A_1 and A_2 are the Lame parameters.

The theory also employs the formulas for the angles between the normal and the bent surface

$$\varphi_1=\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1},\quad \varphi_2=\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} \tag{1.3}$$

and for the generalized transverse forces

$$Q_1^*=Q_1+\frac{1}{A_2}\frac{\partial M_{12}}{\partial\alpha_2},\quad Q_2^*=Q_2+\frac{1}{A_1}\frac{\partial M_{21}}{\partial\alpha_1}. \tag{1.4}$$

The six equations (1.1), (1.2) for six unknown functions $M_{11}, M_{22}, M_{12}=M_{21}, Q_1, Q_2, w$ of the coordinates α_1, α_2 and time t are commonly reduced to one equation for the deflection $w(\alpha_1,\alpha_2,t)$.

For numerical purposes, it is reasonable to use a mixed system of four operator equations in Cauchy normal form. Here we perform a general analysis of Eqs. (1.1) and (1.2) in rectangular (x_1,x_2) and polar (r,θ) coordinates.

2. Rectangular Coordinates. When $A_1=A_2=1$, the system of equations (1.1) in rectangular coordinates ($x_1=\alpha_1$, $x_2=\alpha_2$) can be reduced to

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{21}}{\partial x_2}-Q_1=0,$$

$$\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-Q_2=0,$$

$$\frac{\partial Q_1}{\partial x_1}+\frac{\partial Q_2}{\partial x_2}=\rho h\frac{\partial^2 w}{\partial t^2} \tag{2.1}$$

and the constitutive relations (1.2) can be written as

$$M_{11}=-D\left(\frac{\partial^2 w}{\partial x_1^2}+\nu\frac{\partial^2 w}{\partial x_2^2}\right),$$

$$M_{22}=-D\left(\nu\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}\right),$$

$$M_{12}=-(1-\nu)D\frac{\partial^2 w}{\partial x_1\partial x_2}. \tag{2.2}$$

The formulas for the angles between the normal and the midsurface become

$$\varphi_1=\frac{\partial w}{\partial x_1},\quad \varphi_2=\frac{\partial w}{\partial x_2}. \tag{2.3}$$

The formulas for the generalized transverse forces take the form

$$Q_1^* = Q_1 + \frac{\partial M_{12}}{\partial x_2}, \quad Q_2^* = Q_2 + \frac{\partial M_{21}}{\partial x_1}. \tag{2.4}$$

Let us write the system of equations (2.1)–(2.3) in mixed form:

$$\frac{\partial M_{11}}{\partial x_1} = Q_1 - \frac{\partial M_{21}}{\partial x_2}, \quad \frac{\partial w}{\partial x_1} = \varphi_1,$$

$$\frac{\partial \varphi_1}{\partial x_1} = -\frac{M_{11}}{D} - \nu \frac{\partial^2 w}{\partial x_2^2}, \quad \frac{\partial Q_1}{\partial x_1} = \rho h \frac{\partial^2 w}{\partial t^2} - \frac{\partial Q_2}{\partial x_2}. \tag{2.5}$$

At perfect mechanical contact, the bending moment M_{11}, deflection w, angle φ_1, and generalized transverse force Q_1^* remain continuous in the sections $x_1 = \text{const}$ where the mechanical characteristics of the plate discontinue. These functions should be considered unknown, and system (2.5) should be transformed appropriately.

To this end, it is necessary to replace Q_1 in (2.5) by Q_1^* and to eliminate M_{12}, M_{22}, and Q_2.

Let us express M_{22}, M_{12}, and Q_2 in terms of the unknown functions M_{11}, w, φ_1:

$$M_{22} = \nu M_{11} - (1-\nu^2) D \frac{\partial^2 w}{\partial x_2^2},$$

$$M_{12} = -(1-\nu) D \frac{\partial \varphi_1}{\partial x_2},$$

$$Q_2 = \frac{\partial M_{11}}{\partial x_2} - (1-\nu) D \frac{\partial^3 w}{\partial x_2^3}, \tag{2.6}$$

and replace Q_1 by Q_1^* according to (2.4).

After appropriate transformations, we obtain

$$\frac{\partial M_{11}}{\partial x_1} = 2(1-\nu) D \frac{\partial^2 \varphi_1}{\partial x_2^2} + Q_1^*, \quad \frac{\partial w}{\partial x_1} = \varphi_1, \quad \frac{\partial \varphi_1}{\partial x_1} = -\frac{M_{11}}{D} - \nu \frac{\partial^2 w}{\partial x_2^2},$$

$$\frac{\partial Q_1^*}{\partial x_1} = -\nu \frac{\partial^2 M_{11}}{\partial x_2^2} + (1-\nu^2) D \frac{\partial^4 w}{\partial x_2^4} + \rho h \frac{\partial^2 w}{\partial t^2}. \tag{2.7}$$

The functions M_{22}, M_{12}, Q_2, absent in (2.7), are determined in terms of M_{11}, w, φ_1 by formulas (2.6), and

$$Q_1 = Q_1^* + (1-\nu) D \frac{\partial^2 \varphi_1}{\partial x_2^2}.$$

The coefficients of system (2.5) and, hence, of Eqs. (2.1) and (2.2) may be arbitrary functions of the coordinate x_1 with discontinuities of the first kind.

System (2.7) is an operator system in Cauchy normal form in coordinate x_1. Let us show that this system is a Hamiltonian operator system [1] in space coordinate x_1:

$$\frac{\partial q_i}{\partial x_1} = \frac{\partial \hat{H}}{\partial p_i}, \quad \frac{\partial p_i}{\partial x_1} = -\frac{\partial \hat{H}}{\partial q_i}, \quad i = 1, 2. \tag{2.8}$$

To this end, it is necessary to appropriately select canonical variables $\{q_1, q_2\} = \{M_{11}, w\}$, $\{p_1, p_2\} = \{\varphi_1, Q_1^*\}$ and operator Hamiltonian

$$\hat{H} = \frac{1}{2}\hat{P}_{ij} q_i q_j + \frac{1}{2}\hat{Q}_{ij} p_i p_j \tag{2.9}$$

with the following values of operator elements $\hat{P}_{ij}$ and $\hat{Q}_{ij}$ of the symmetric operator matrices $\hat{P}$ and $\hat{Q}$:

$$-\hat{P}_{11} = -\frac{1}{D}, \quad -\hat{P}_{12} = -\hat{P}_{21} = -\nu\frac{\partial^2}{\partial x_2^2}, \quad -\hat{P}_{22} = (1-\nu^2)D\frac{\partial^4}{\partial x_2^4} + \rho h\frac{\partial^2}{\partial t^2},$$

$$\hat{Q}_{11} = 2(1-\nu)D\frac{\partial^2}{\partial x_2^2}, \quad \hat{Q}_{12} = \hat{Q}_{21} = 1, \quad \hat{Q}_{22} = 0. \tag{2.10}$$

In (2.8) and (2.9), the operators $\hat{P}_{ij}$ and $\hat{Q}_{ij}$ should be considered constant. This reduces (2.8) to (2.7).

The Hamiltonian operator system in a space coordinate x_1 can be derived from the “isochronous” variation of the functional

$$I(M_{11}, w, \varphi_1, Q_1^*) = \int_a^b \left\{ \varphi_1 \frac{\partial M_{11}}{\partial x_1} + Q_1^* \frac{\partial w}{\partial x_1} - \left[\frac{1}{2} D^{-1} M_{11}^2 + \nu\partial_2^2 M_{11} w \right.\right.$$

$$\left.\left. + \frac{1}{2}\left(-(1-\nu^2)D\partial_2^4 - \rho h \partial_t^2 \right) w^2 + (1-\nu)D\partial_2^2\varphi_1^2 + \varphi_1 Q_1^* \right] \right\} dx_1. \tag{2.11}$$

In varying it, the following rule should be used:

$$\delta(P_{ij}, Q_{ij})a_m b_n = (P_{ij}, Q_{ij})(a_m \delta b_n + b_n \delta a_m) = \delta b_n (P_{ij}, Q_{ij})a_m + \delta a_m (P_{ij}, Q_{ij})b_n. \tag{2.12}$$

The Hamiltonian operator system in space coordinate x_2 can be obtained in a similar way if M_{22}, w, φ_2, Q_2^* are used as unknown functions. Let us write the system of equations (2.1)–(2.3) in mixed form:

$$\frac{\partial M_{22}}{\partial x_2} = Q_1 - \frac{\partial M_{12}}{\partial x_1}, \quad \frac{\partial w}{\partial x_2} = \varphi_2,$$

$$\frac{\partial \varphi_2}{\partial x_2} = -\frac{M_{22}}{D} - \nu\frac{\partial^2 w}{\partial x_1^2}, \quad \frac{\partial Q_2}{\partial x_2} = \rho h\frac{\partial^2 w}{\partial t^2} - \frac{\partial Q_1}{\partial x_1}. \tag{2.13}$$

The functions M_{11}, M_{12}, and Q_1, absent in (2.13), are determined in terms of M_{22}, w, φ_2 by the formulas

$$M_{11} = \nu M_{22} - (1-\nu^2)D\frac{\partial^2 w}{\partial x_1^2}, \quad M_{12} = -(1-\nu)D\frac{\partial \varphi_2}{\partial x_1},$$

$$Q_1 = \frac{\partial M_{22}}{\partial x_1} - (1-\nu)D\frac{\partial^3 w}{\partial x_1^3} \quad \text{or} \quad Q_2 = Q_2^* + (1-\nu)D\frac{\partial^2 \varphi_2}{\partial x_1^2}. \tag{2.14}$$

System (2.13) can be reduced to the form

$$\frac{\partial M_{22}}{\partial x_2} = 2(1-\nu)D\frac{\partial \varphi_2}{\partial x_1^2} + Q_2^*, \quad \frac{\partial w}{\partial x_2} = \varphi_2, \quad \frac{\partial \varphi_2}{\partial x_2} = -\frac{M_{22}}{D} - \nu\frac{\partial^2 w}{\partial x_1^2},$$

$$\frac{\partial Q_2^*}{\partial x_2} = -\nu \frac{\partial^2 M_{22}}{\partial x_1^2} + (1-\nu^2) D \frac{\partial^4 w}{\partial x_1^4} + \rho h \frac{\partial^2 w}{\partial t^2}. \tag{2.15}$$

With column vectors $\{q_1, q_2\} = \{M_{22}, w\}$, $\{p_1, p_2\} = \{\varphi_2, Q_2^*\}$ and operator matrices $\hat{Q}$, $\hat{P}$ with operator elements

$$-\hat{P}_{11} = -\frac{1}{D}, \quad -\hat{P}_{12} = -\hat{P}_{21} = -\nu \frac{\partial^2}{\partial x_1^2}, \quad -\hat{P}_{22} = (1-\nu^2) D \frac{\partial^4}{\partial x_1^4} + \rho h \frac{\partial^2}{\partial t^2},$$

$$\hat{Q}_{11} = 2(1-\nu) D \frac{\partial^2}{\partial x_1^2}, \quad \hat{Q}_{12} = \hat{Q}_{21} = 1, \quad \hat{Q}_{22} = 0 \tag{2.16}$$

system (2.15) becomes a Hamiltonian operator system in space coordinate x_2:

$$\frac{\partial q_i}{\partial x_2} = \hat{Q}_{ij} p_j, \quad \frac{\partial p_i}{\partial x_2} = -\hat{P}_{ij} q_j, \quad i = 1, 2. \tag{2.17}$$

System (2.15) becomes equivalent to (2.7) if the indices 1 and 2 are interchanged.

3. Polar Coordinates. When $A_1 = 1$ and $A_2 = r$, the system of equations (1.1) in polar coordinates $r = \alpha_1$, $\theta = \alpha_2$ takes a simpler form:

$$\frac{1}{r}\frac{\partial r M_{rr}}{\partial r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} - \frac{M_{\theta\theta}}{r} - Q_r = 0,$$

$$\frac{1}{r^2}\frac{\partial r^2 M_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial \theta} - Q_\theta = 0,$$

$$\frac{1}{r}\frac{\partial r Q_r}{\partial r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} = \rho h \frac{\partial^2 w}{\partial t^2}. \tag{3.1}$$

The constitutive equations (1.2) become:

$$M_{rr} = -D\left(\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right),$$

$$M_{\theta\theta} = -D\left(\nu \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right),$$

$$M_{r\theta} = -(1-\nu) D \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right). \tag{3.2}$$

The formulas for the angles between the normal and the bent midsurface take the form

$$\varphi_r = \frac{\partial w}{\partial r}, \quad \varphi_\theta = \frac{1}{r}\frac{\partial w}{\partial \theta} \tag{3.3}$$

and the formulas for the generalized transverse forces are

$$Q_r^* = Q_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta}, \quad Q_\theta^* = Q_\theta + \frac{1}{r}\frac{\partial M_{\theta r}}{\partial r}. \tag{3.4}$$

Let us write the system of equations (3.1)–(3.4) in the form

$$\frac{\partial rM_{rr}}{\partial r} = M_{\theta\theta} + Q_r - \frac{\partial M_{r\theta}}{\partial r},$$

$$\frac{\partial w}{\partial r} = \varphi_r,$$

$$\frac{\partial \varphi_r}{\partial r} = -\frac{M_{rr}}{D} - \nu\left(\frac{\varphi_r}{r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right),$$

$$\frac{\partial rQ_r}{\partial r} = r\rho h\frac{\partial^2 w}{\partial t^2} - \frac{\partial Q_\theta}{\partial \theta}. \tag{3.5}$$

System (3.5) is an operator system in Cauchy normal form in radial coordinate r. At perfect mechanical contact, the bending moment M_{rr}, deflection w, angle φ_r, and generalized transverse force Q_r^* remain continuous in the sections $r = \text{const}$ where the mechanical characteristics of the plate discontinue. Choosing rM_{rr}, w, φ_r, rQ_r^* [5] as unknown functions, we will transform the system.

To this end, Q_r in (3.5) is replaced by Q_r^* by formula (3.4), and $M_{\theta\theta}$, $M_{r\theta}$, and Q_θ are eliminated from Eqs. (3.5). Using Eqs. (3.2) and (3.1), we express $M_{\theta\theta}$, $M_{r\theta}$, and Q_θ in terms of the unknown functions rM_{rr}, w, φ_r:

$$M_{\theta\theta} = \nu M_{rr} - (1-\nu^2)D\left(\frac{1}{r}\varphi_r + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right),$$

$$M_{r\theta} = -(1-\nu^2)D\left(\frac{1}{r}\frac{\partial \varphi_r}{\partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right),$$

$$Q_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}\left[\nu M_{rr} - (1-\nu^2)D\left(\frac{1}{r}\varphi_r + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] - (1-\nu)D\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi_r}{\partial \theta}\right) - \frac{1}{r^2}\frac{\partial^2 w}{\partial r \partial \theta}\right]. \tag{3.6}$$

Transforming Eqs. (3.5), we obtain the following system of mixed equations in polar coordinates describing the transverse vibrations of a plate according to Kirchhoff theory:

$$\frac{\partial rM_{rr}}{\partial r} = \frac{\nu}{r}rM_{rr} - (1-\nu)(3+\nu)D\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - (1-\nu^2)D\frac{\varphi_r}{r} + 2(1-\nu)D\frac{1}{r}\frac{\partial^2 \varphi_r}{\partial \theta^2} + rQ_r^*,$$

$$\frac{\partial w}{\partial r} = \varphi_r,$$

$$\frac{\partial \varphi_r}{\partial r} = -\frac{1}{Dr}rM_{rr} - \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{\nu}{r}\varphi_r,$$

$$\frac{\partial rQ_r^*}{\partial r} = -\frac{\nu}{r^2}\frac{\partial^2 rM_{rr}}{\partial \theta^2} + (1-\nu^2)D\frac{1}{r^3}\frac{\partial^4 w}{\partial \theta^4} - 2(1-\nu)D\frac{1}{r^3}\frac{\partial^2 w}{\partial \theta^2} + r\rho h\frac{\partial^2 w}{\partial t^2} + (1-\nu)(3+\nu)D\frac{1}{r^2}\frac{\partial^2 \varphi_r}{\partial \theta^2}. \tag{3.7}$$

Using the general idea of [3] and following [5], we will show that system (3.7) is a Hamiltonian operator system [1] in space coordinate r:

$$\frac{\partial q_i}{\partial r} = \frac{\partial \hat{H}}{\partial p_i}, \qquad \frac{\partial p_i}{\partial r} = -\frac{\partial \hat{H}}{\partial q_i}, \qquad i = 1, 2. \tag{3.8}$$

To this end, the "canonical" variables q_i, p_i and the operator Hamiltonian are represented as

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} rM_{rr} \\ w \end{bmatrix}, \qquad \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \varphi_r \\ rQ_r^* \end{bmatrix},$$

$$\hat{H} = \frac{1}{2}\hat{P}_{ij} q_i q_j + \hat{R}_{ij} q_i p_j + \frac{1}{2}\hat{Q}_{ij} p_i p_j, \tag{3.9}$$

where the elements of the symmetric operator matrices $\hat{P}_{ij}, \hat{Q}_{ij}$ and the nonzero elements of the operator matrix $\hat{R}_{ij}$ are given by

$$-\hat{P}_{11} = -\frac{1}{Dr}, \qquad -\hat{P}_{12} = -\hat{P}_{21} = -\frac{\nu}{r^2}\frac{\partial^2}{\partial\theta^2},$$

$$-\hat{P}_{22} = (1-\nu^2)D\frac{1}{r^3}\frac{\partial^4}{\partial\theta^4} - 2(1-\nu)D\frac{1}{r^3}\frac{\partial^2}{\partial\theta^2} + r\rho h\frac{\partial^2}{\partial t^2},$$

$$\hat{Q}_{11} = -(1-\nu^2)D\frac{1}{r} + 2(1-\nu)D\frac{1}{r}\frac{\partial^2}{\partial\theta^2}, \qquad \hat{Q}_{12} = \hat{Q}_{21} = 1, \qquad \hat{Q}_{22} = 0,$$

$$\hat{R}_{11} = \frac{\nu}{r}, \qquad \hat{R}_{12} = -(1-\nu)(3+\nu)D\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \qquad \hat{R}_{21} = \hat{R}_{22} = 0. \tag{3.10}$$

In (3.8) and (3.9), the operators $\hat{P}_{ij}, \hat{Q}_{ij}, \hat{R}_{ij}$ in (3.10) are assumed constant. This reduces (3.8) to (3.7). Thus, it has been proved that (3.7) is a Hamiltonian operator system in space coordinate r.

The coefficients of system (3.7) and, hence, of Eqs. (3.1) and (3.2) may be arbitrary functions of the coordinate r with discontinuities of the first kind.

The functions $M_{\theta\theta}, M_{r\theta}, Q_r, Q_\theta$ are determined in terms of the unknown functions by formulas (3.4) and (3.6).

The Hamiltonian operator system in space coordinate r can be derived from the “isochronous” variation of the functional

$$I(rM_{rr}, w, \varphi_r, rQ_r^*) = \int_a^b \Bigg\{ \varphi_r \frac{\partial(rM_{rr})}{\partial r} + (rQ_r^*)\frac{\partial w}{\partial r} - \Bigg\{ \frac{1}{2}\frac{1}{Dr}(rM_{rr})^2 + \frac{\nu}{r^2}\partial_\theta^2(rM_{rr})w$$

$$+\frac{1}{2}\left[2(1-\nu)D\frac{1}{r^3}\partial_\theta^2 - (1-\nu^2)D\frac{1}{r^3}\partial_\theta^4 - r\rho h\partial_t^2\right]w^2 + \frac{\nu}{r}(rM_{rr})\varphi_r - (1-\nu)(3+\nu)D\frac{1}{r^2}\partial_\theta^2 w\varphi_r$$

$$+\frac{1}{2}\left[-(1-\nu^2)D\frac{1}{r} + 2(1-\nu)D\frac{1}{r}\partial_\theta^2\right]\varphi_r^2 + \varphi_r(Q_r^*)\Bigg\}\Bigg\} dr. \tag{3.11}$$

In varying the functional, the operators $\hat{P}_{ij} = \hat{P}_{ji}$, $\hat{Q}_{ij} = \hat{Q}_{ji}$, and $\hat{R}_{ij}$ should be considered “frozen” (constant) and permutable with variations

$$\delta(\hat{P}_{ij}, \hat{Q}_{ij}, \hat{R}_{ij})a_m b_n = (\hat{P}_{ij}, \hat{Q}_{ij}, \hat{R}_{ij})(a_m \delta b_n + b_n \delta a_m)$$

$$= \delta b_n(\hat{P}_{ij}, \hat{Q}_{ij}, \hat{R}_{ij})a_m + \delta a_m(\hat{P}_{ij}, \hat{Q}_{ij}, \hat{R}_{ij})b_n. \tag{3.12}$$

Conclusions. A general analysis of the mixed systems of four equations in Kirchhoff’s theory of the vibrations of plates in rectangular and polar coordinates has been performed. It has been shown that these systems can be represented in Hamiltonian operator form in space coordinate after selection of the appropriate “canonical” variables and operator Hamiltonian. Functionals for canonical systems have been formulated.

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