International Applied Mechanics, Vol. 48, No. 5, September, 2012

INFLUENCE OF BOUNDARY CONDITIONS ON THE NATURAL FREQUENCIES OF NONAXISYMMETRIC ELECTROELASTIC VIBRATIONS OF PIEZOCERAMIC PLATES

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The general solution to the problem of the nonaxisymmetric electromechanical vibrations of a piezoceramic ring plate is obtained. For plates with radially cut electrode coating and different boundary conditions (clamped edge–free edge, free edge–clamped edge, free edge–free edge), the natural frequency spectra are determined numerically and analyzed for the first circumferential harmonics

Keywords: piezoceramic ring plate, radially cut electrode coating, nonaxisymmetric electromechanical vibrations, natural frequency spectra

Introduction. Thin piezoelectric planar transducers polarized across the thickness are used in devices of various functionality [1, 3, 5, 8, 10, etc.]. Disk- and ring-shaped vibrators with solid electrodes on the faces undergo axisymmetric vibrations [2, 6, 8]. The vibrations will be nonaxisymmetric with respect to the circumferential coordinate if the electroelastic sectors of a ring plate with radially cut electrodes are excited in antiphase. The circumferential vibration modes are a priori determined by the number of radial cuts in the electrodes [7–9, 12]. The present paper compares the frequency spectra of a plate with three types of boundary conditions.

1. Problem Formulation. General Solution. Consider a thin piezoceramic plate of thickness h. To describe the plate, we will use a cylindrical coordinate system r, θ, z with the plane $z = 0$ coinciding with the midsurface of the plate. If a thin piezoceramic plate with electroded faces $z = \pm h/2$ polarized across the thickness is in plane stress state ($u_r(r,\theta,t)$, $u_\theta(r,\theta,t)$, $\sigma_{zz} = \sigma_{z\theta} = \sigma_{zr} = 0$, $E_r = E_\theta = 0$, $E_z(r,\theta,t)$), then the formulas below follow from the general constitutive equations [2, 6, 8]:

$$\sigma_{rr} = \frac{1}{(1-\nu_E^2)s_{11}^E}\left[\frac{\partial u_r}{\partial r} + \nu_E\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right) - (1+\nu_E)d_{31}E_z\right],$$

$$\sigma_{\theta\theta} = \frac{1}{(1-\nu_E^2)s_{11}^E}\left(\nu_E\frac{\partial u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} - (1+\nu_E)d_{31}E_z\right),$$

$$\sigma_{r\theta} = \frac{1}{2(1+\nu_E)s_{11}^E}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right), \tag{1}$$

which include the formulas for strains and $s_{66}^E = 2(s_{11}^E - s_{12}^E)$, $\nu_E = -s_{12}^E / s_{11}^E$. If the thickness accelerations are neglected, two out of the three equations of mechanical vibration remain:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} = \rho\frac{\partial^2 u_r}{\partial t^2}, \qquad \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} = \rho\frac{\partial^2 u_\theta}{\partial t^2}. \tag{2}$$

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 1063-7095/12/4805-0592

After simple transformations in (1) and (2), we arrive at the vibration equations for displacements:

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}+\frac{1-\nu_E}{2}\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{1+\nu_E}{2}\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta}$$

$$+\frac{3-\nu_E}{2}\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}-(1+\nu_E)d_{31}\frac{\partial E_z}{\partial r}=(1-\nu_E^2)s_{11}^E\rho\frac{\partial^2 u_r}{\partial t^2},$$

$$\frac{1+\nu_E}{2}\frac{1}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{3-\nu_E}{2}\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1-\nu_E}{2}\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\right)$$

$$+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}-(1+\nu_E)d_{31}\frac{1}{r}\frac{\partial E_z}{\partial\theta}=(1-\nu_E^2)s_{11}^E\rho\frac{\partial^2 u_\theta}{\partial t^2}. \qquad (3)$$

The solution of the system of equations (3) can be represented [7] in the form

$$u_r=\frac{\partial\Phi}{\partial r}+\frac{1}{r}\frac{\partial\Psi}{\partial\theta},\qquad u_\theta=\frac{1}{r}\frac{\partial\Phi}{\partial\theta}-\frac{\partial\Psi}{\partial r}. \qquad (4)$$

The functions $\Phi(r,\theta,t)$ and $\Psi(r,\theta,t)$ determined from the following wave equations satisfy Eqs. (3):

$$\Delta\Phi-(1+\nu_E)d_{31}E_z=(1-\nu_E^2)s_{11}^E\rho\frac{\partial^2\Phi}{\partial t^2},$$

$$\Delta\Psi=2(1+\nu_E)s_{11}^E\rho\frac{\partial^2\Psi}{\partial t^2}. \qquad (5)$$

The electric potential for a plate with solid electrodes on the faces $z=\pm h/2$ is given by $\varphi=h^{-1}zV_0(t)$, the edge effect being neglected. This potential corresponds, according to [2, 6, 8], to an electric field with $E_r=E_\theta=0$, $E_z=h^{-1}V_0(t)$; hence, the term $(1+\nu_E)d_{31}E_z$ in Eqs. (5) should be omitted, considering (3).

The following expressions for stresses in terms of the potentials Φ and Ψ can be derived from (1), (4):

$$\sigma_{rr}=\frac{1}{(1-\nu_E^2)s_{11}^E}\left[\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)\right)+\nu_E\left(\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)\right)-(1+\nu_E)d_{13}E_z\right],$$

$$\sigma_{\theta\theta}=\frac{1}{(1-\nu_E^2)s_{11}^E}\left[\nu_E\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)\right)+\left(\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)\right)-(1+\nu_E)d_{13}E_z\right],$$

$$\sigma_{r\theta}=\frac{1}{2(1+\nu_E)s_{11}^E}\left(\frac{2}{r}\frac{\partial^2\Phi}{\partial r\partial\theta}-\frac{2}{r^2}\frac{\partial\Phi}{\partial\theta}-\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right). \qquad (6)$$

The homogeneous boundary conditions for displacements and stresses (at $r=r_0$ and $r=r_1$) in a circular piezoceramic plate of radius r_1 with a hole of radius r_0 are taken one from each of the following two pairs ($j=0,1$):

$$u_r(r_j,\theta,t)=0\wedge\sigma_{rr}(r_j,\theta,t)=0,$$

$$u_\theta(r_j,\theta,t)=0\wedge\sigma_{r\theta}(r_j,\theta,t)=0. \qquad (7)$$

The initial conditions for steady-state harmonic vibrations are not formulated.

Consider a circular piezoceramic plate $r_0<r<r_1$. The electrode coating on its faces $z=\pm h/2$ is cut into $2N$ sectors. Adjacent sectors are connected in antiphase so that $E_z^{\mathrm{a}}=(-1)^{n-1}V_0/h$, $n=1,\ldots,2N$. If vibrations are harmonic,

$f(r,\theta,t)=\mathrm{Re}\, f^{a}(r,\theta)\exp i\omega t$, where ω is the angular frequency, then a candidate solution to Eqs. (5) (the term $(1+\nu)d_{31}E_z$ in the first equation should be equated to zero [7]) in polar coordinates r,θ can be chosen in the form of series:

$$\Phi(r,\theta,t)=R^2\ \mathrm{Re}\sum_m \{A_{m,1}J_m(k_1 r)+A_{m,2}Y_m(k_1 r)\}\sin m\theta \exp i\omega t,$$

$$\Psi(r,\theta,t)=R^2\ \mathrm{Re}\sum_m \{A_{m,3}J_m(k_2 r)+A_{m,4}Y_m(k_2 r)\}\cos m\theta \exp i\omega t, \tag{8}$$

where $J_m(k_j r)$ and $Y_m(k_j r)$ are mth-order cylindrical functions of the first and second kinds [4]; $k_1^2=(1-\nu_E^2)s_{11}^E\rho\omega^2$, $k_2^2=2(1+\nu_E)s_{11}^E\rho\omega^2$; $A_{m,i}$ are dimensionless constants; R is a parameter with units of meters.

From (4), (6), and (8), we can find the displacements [9, 12]:

$$u_r=R\ \mathrm{Re}\sum_m [u_{m1}(k_1 r)A_{m,1}+u_{m2}(k_1 r)A_{m,2}-u_{m3}(k_2 r)A_{m,3}-u_{m4}(k_2 r)A_{m,4}]\sin m\theta \exp i\omega t,$$
$$u_\theta=R\ \mathrm{Re}\sum_m [l_{m3}(k_1 r)A_{m,1}+l_{m4}(k_1 r)A_{m,2}+l_{m1}(k_2 r)A_{m,3}+l_{m2}(k_2 r)A_{m,4}]\cos m\theta \exp i\omega t \tag{9}$$

and the stresses:

$$\sigma_{rr}(r,\theta,t)=-\mathrm{Re}\frac{1}{(1-\nu_E^2)s_{11}^E}\left\{\sum_m\Big(a_{m1}(k_1 r)A_{m,1}+a_{m2}(k_1 r)A_{m,2}+a_{m3}(k_2 r)A_{m,3}+a_{m4}\big(k_2 r\big)A_{m,4}\Big)\sin m\theta\right.$$
$$\left.+\frac{4}{\pi}V_0(1+\nu_E)d_{13}\sum_{n=1}^{\infty}\frac{\sin N(2n-1)\theta}{2n-1}\right\}\exp i\omega t,$$

$$\sigma_{\theta\theta}(r,\theta,t)=-\mathrm{Re}\frac{1}{(1-\nu_E^2)s_{11}^E}\left\{\sum_m\Big(b_{m1}(k_1 r)A_{m,1}+b_{m2}(k_1 r)A_{m,2}+b_{m3}(k_2 r)A_{m,3}+b_{m4}\big(k_2 r\big)A_{m,4}\Big)\sin m\theta\right.$$
$$\left.+\frac{4}{\pi}V_0(1+\nu_E)d_{13}\sum_{n=1}^{\infty}\frac{\sin N(2n-1)\theta}{2n-1}\right\}\exp i\omega t,$$

$$\sigma_{r\theta}(r,\theta,t)=\mathrm{Re}\frac{1}{(1+\nu_E)s_{11}^E}\sum_m\Big(c_{m1}(k_1 r)A_{m,1}+c_{m2}(k_1 r)A_{m,2}+c_{m3}(k_2 r)A_{m,3}$$
$$+c_{m4}(k_2 r)A_{m,4}\Big)\cos m\theta \exp i\omega t, \tag{10}$$

where

$$a_{m1}(k_1 r)=\Big[(1-\nu_E)k_1 rJ_{m-1}(k_1 r)+\Big(k_1^2r^2-(1-\nu_E)m(m+1)\Big)J_m(k_1 r)\Big]R^2/r^2,$$

$$a_{m2}(k_1 r)=\Big[(1-\nu_E)k_1 rY_{m-1}(k_1 r)+\Big(k_1^2r^2-(1-\nu_E)m(m+1)\Big)Y_m(k_1 r)\Big]R^2/r^2,$$

$$a_{m3}(k_2 r)=(1-\nu_E)m\big[k_2 rJ_{m-1}(k_2 r)-(m+1)J_m(k_2 r)\big]R^2/r^2,$$

$$a_{m4}(k_2 r)=(1-\nu_E)m\big[k_2 rY_m(k_2 r)-(m+1)Y_m(k_2 r)\big]R^2/r^2,$$

$$b_{m1}(k_1 r)=\Big[-(1-\nu_E)k_1 rJ_{m-1}(k_1 r)+\Big(\nu_E k_1^2r^2+(1-\nu_E)m(m+1)\Big)J_m(k_1 r)\Big]R^2/r^2,$$

$$b_{m2}(k_1 r)=\Big[-(1-\nu_E)k_1 rY_{m-1}(k_1 r)+\Big(\nu_E k_1^2r^2+(1-\nu_E)m(m+1)\Big)Y_m(k_1 r)\Big]R^2/r^2,$$

$$b_{m3}(k_2 r) = -(1-\nu_E)m\left[k_2 r J_{m-1}(k_2 r) - (m+1)J_m(k_2 r)\right]R^2/r^2,$$

$$b_{m4}(k_2 r) = -(1-\nu_E)m\left[k_2 r Y_{m-1}(k_2 r) - (m+1)Y_m(k_2 r)\right]R^2/r^2,$$

$$c_{m1}(k_1 r) = m\left[k_1 r J_{m-1}(k_1 r) - (m+1)J_m(k_1 r)\right]R^2/r^2,$$

$$c_{m2}(k_1 r) = m\left[k_1 r Y_{m-1}(k_1 r) - (m+1)Y_m(k_1 r)\right]R^2/r^2,$$

$$c_{m3}(k_2 r) = \left[k_2 r J_{m-1}(k_2 r) + \left(\frac{1}{2}k_2^2 r^2 - m(m+1)\right)J_m(k_2 r)\right]R^2/r^2,$$

$$c_{m4}(k_2 r) = \left[k_2 r Y_{m-1}(k_2 r) + \left(\frac{1}{2}k_2^2 r^2 - m(m+1)\right)Y_m(k_2 r)\right]R^2/r^2,$$

$$u_{m1}(k_1 r) = -m\frac{R}{r}J_m(k_1 r) + k_1 R J_{m-1}(k_1 r), \quad u_{m2}(k_1 r) = -m\frac{R}{r}Y_m(k_1 r) + k_1 R Y_{m-1}(k_1 r),$$

$$u_{m3}(k_2 r) = m\frac{R}{r}J_m(k_2 r), \quad u_{m4}(k_2 r) = m\frac{R}{r}Y_m(k_2 r),$$

$$l_{m1}(k_1 r) = u_{m3}(k_1 r), \quad l_{m2}(k_1 r) = u_{m2}(k_1 r), \quad l_{m3}(k_2 r) = -u_{m3}(k_2 r), \quad l_{m4}(k_2 r) = -u_{m2}(k_2 r). \tag{11}$$

Since $E_z^{\mathrm{a}} = (-1)^{n-1}V_0 h^{-1}$, $n = 1, 2, \ldots, 2N$ the electric-field strength $E_z = \operatorname{Re} E_z^{\mathrm{a}} \exp i\omega t$ can be expanded into a Fourier series with respect to the angular coordinate θ:

$$E_z^{\mathrm{a}} = -\frac{2V_0}{\pi h}\sum_{n=1}^{\infty}\frac{\sin N(2n-1)\theta}{2n-1}, \tag{12}$$

we have $m = N(2n-1)$, $n = 1, 2, \ldots$ in (9) and (10).

In the resonance case, the concept of complex moduli [6, 8] has to be used, i.e., the material constants should be considered complex ($\tilde{s}_{ij}^E = s_{ij}^E - is_{ij}^{E\,\mathrm{Im}}$, $\tilde{d}_{ij} = d_{ij} - id_{ij}^{\mathrm{Im}}$, $\tilde{\varepsilon}_{ij}^T = \varepsilon_{ij}^T - i\varepsilon_{ij}^{T\,\mathrm{Im}}$).

To determine the resonance frequencies, it is possible to neglect the loss tangents as small and to use the real values of the material constants.

Consider a ring plate with the following boundary conditions:

inner edge $r = r_0$ clamped and outer edge $r = r_1$ free,

$$u_r(r_0, \theta, t) = 0, \quad u_\theta(r_0, \theta, t) = 0, \quad \sigma_{rr}(r_1, \theta, t) = 0, \quad \sigma_{r\theta}(r_1, \theta, t) = 0, \tag{13}$$

inner edge $r = r_0$ free and outer edge $r = r_1$ clamped:

$$\sigma_{rr}(r_0, \theta, t) = 0, \quad \sigma_{r\theta}(r_0, \theta, t) = 0, \quad u_r(r_1, \theta, t) = 0, \quad u_\theta(r_1, \theta, t) = 0, \tag{14}$$

inner edge $r = r_0$ free and outer edge $r = r_1$ free:

$$\sigma_{rr}(r_0, \theta, t) = 0, \quad \sigma_{r\theta}(r_0, \theta, t) = 0, \quad \sigma_{rr}(r_1, \theta, t) = 0, \quad \sigma_{r\theta}(r_1, \theta, t) = 0. \tag{15}$$

Using expressions (9), (10) and boundary conditions (13), we obtain block systems of algebraic equations for the dimensionless constants $A_{N(2n-1),i}$ $(n = 1, 2, \ldots)$:

$$u_{N(2n-1),1}(k_1 r_0)A_{N(2n-1),1} + u_{N(2n-1),2}(k_1 r_0)A_{N(2n-1),2}$$

$$+ u_{N(2n-1),3}(k_2 r_0)A_{N(2n-1),3} + u_{N(2n-1),4}(k_2 r_0)A_{N(2n-1),4} = 0,$$

$$l_{N(2n-1),1}(k_1 r_0)A_{N(2n-1),1} + l_{N(2n-1),2}(k_1 r_0)A_{N(2n-1),2}$$

$$+l_{N(2n-1),3}(k_2 r_0)A_{N(2n-1),3} + l_{N(2n-1),4}(k_2 r_0)A_{N(2n-1),4} = 0,$$

$$a_{N(2n-1),1}(k_1 r_1)A_{N(2n-1),1} + a_{N(2n-1),2}(k_1 r_1)A_{N(2n-1),2}$$

$$+a_{N(2n-1),3}(k_2 r_1)A_{N(2n-1),3} + a_{N(2n-1),4}(k_2 r_1)A_{N(2n-1),4} = -\frac{4}{\pi}V_0 \frac{(1+\nu_E)d_{13}}{2n-1},$$

$$c_{N(2n-1),1}(k_1 r_1)A_{N(2n-1),1} + c_{N(2n-1),2}(k_1 r_1)A_{N(2n-1),2}$$

$$+c_{N(2n-1),3}(k_2 r_1)A_{N(2n-1),3} + c_{N(2n-1),4}(k_2 r_1)A_{N(2n-1),4} = 0. \tag{16}$$

The resonance frequencies can be determined by equating the fourth-order determinants of the homogeneous (at $V_0 = 0$) systems of algebraic equations (16) to zero:

$$\begin{vmatrix} u_{m1}(k_1 r_0) & u_{m2}(k_1 r_0) & u_{m3}(k_2 r_0) & u_{m4}(k_2 r_0) \\ l_{m1}(k_1 r_0) & l_{m2}(k_1 r_0) & l_{m3}(k_2 r_0) & l_{m4}(k_2 r_0) \\ a_{m1}(k_1 r_1) & a_{m2}(k_1 r_1) & a_{m3}(k_2 r_1) & a_{m4}(k_2 r_1) \\ c_{m1}(k_1 r_1) & c_{m2}(k_1 r_1) & c_{m3}(k_2 r_1) & c_{m4}(k_2 r_1) \end{vmatrix} = 0. \tag{17}$$

Using expressions (9), (10) and boundary conditions (14), we obtain block systems of algebraic equations for the dimensionless constants $A_{N(2n-1),i}$ $(n = 1, 2, \dots)$:

$$a_{N(2n-1),1}(k_1 r_0)A_{N(2n-1),1} + a_{N(2n-1),2}(k_1 r_0)A_{N(2n-1),2}$$

$$+a_{N(2n-1),3}(k_2 r_0)A_{N(2n-1),3} + a_{N(2n-1),4}(k_2 r_0)A_{N(2n-1),4} = -\frac{4}{\pi}V_0 \frac{(1+\nu_E)d_{13}}{2n-1},$$

$$c_{N(2n-1),1}(k_1 r_0)A_{N(2n-1),1} + c_{N(2n-1),2}(k_1 r_0)A_{N(2n-1),2}$$

$$+c_{N(2n-1),3}(k_2 r_0)A_{N(2n-1),3} + c_{N(2n-1),4}(k_2 r_0)A_{N(2n-1),4} = 0,$$

$$u_{N(2n-1),1}(k_1 r_1)A_{N(2n-1),1} + u_{N(2n-1),2}(k_1 r_1)A_{N(2n-1),2}$$

$$+u_{N(2n-1),3}(k_2 r_1)A_{N(2n-1),3} + u_{N(2n-1),4}(k_2 r_1)A_{N(2n-1),4} = 0,$$

$$l_{N(2n-1),3}(k_1 r_1)A_{N(2n-1),1} + l_{N(2n-1),4}(k_1 r_1)A_{N(2n-1),2}$$

$$+l_{N(2n-1),3}(k_2 r_1)A_{N(2n-1),3} + l_{2N(2n-1),4}(k_2 r_1)A_{N(2n-1),4} = 0. \tag{18}$$

The equations for the resonance frequencies of a plate with the boundary conditions (14) can be derived from the existence condition for nontrivial solutions of the homogeneous (at $V_0 = 0$) systems of equations (18):

$$\begin{vmatrix} a_{m1}(k_1 r_0) & a_{m2}(k_1 r_0) & a_{m3}(k_2 r_0) & a_{m4}(k_2 r_0) \\ c_{m3}(k_1 r_0) & c_{m3}(k_1 r_0) & c_{m1}(k_2 r_0) & c_{m2}(k_2 r_0) \\ u_{m1}(k_1 r_1) & u_{m2}(k_1 r_1) & u_{m3}(k_2 r_1) & u_{m4}(k_2 r_1) \\ l_{m1}(k_1 r_1) & l_{m2}(k_1 r_1) & l_{m3}(k_2 r_1) & l_{m4}(k_2 r_1) \end{vmatrix} = 0. \tag{19}$$

Using expressions (9), (10) and boundary conditions (15), we obtain block systems of algebraic equations for the dimensionless constants $A_{N(2n-1),i}$ $(n = 1, 2, \dots)$:

$$a_{N(2n-1),1}(k_1 r_0)A_{N(2n-1),1} + a_{N(2n-1),2}(k_1 r_0)A_{N(2n-1),2}$$

$$+a_{N(2n-1),3}(k_2 r_0)A_{N(2n-1),3} + a_{N(2n-1),4}(k_2 r_0)A_{N(2n-1),4} = -\frac{4}{\pi}V_0 \frac{(1+\nu_E)d_{13}}{2n-1},$$

$$c_{N(2n-1),1}(k_1 r_0)A_{N(2n-1),1} + c_{N(2n-1),2}(k_1 r_0)A_{N(2n-1),2}$$

$$+c_{N(2n-1),3}(k_2 r_0)A_{N(2n-1),3} + c_{N(2n-1),4}(k_2 r_0)A_{N(2n-1),4} = 0,$$

$$a_{N(2n-1),1}(k_1 r_1)A_{N(2n-1),1} + a_{N(2n-1),2}(k_1 r_1)A_{N(2n-1),2}$$

$$+a_{N(2n-1),3}(k_2 r_1)A_{N(2n-1),3} + a_{N(2n-1),4}(k_2 r_1)A_{N(2n-1),4} = -\frac{4}{\pi}V_0 \frac{(1+\nu_E)d_{13}}{2n-1},$$

$$c_{N(2n-1),1}(k_1 r_1)A_{N(2n-1),1} + c_{N(2n-1),2}(k_1 r_1)A_{N(2n-1),2}$$

$$+c_{N(2n-1),3}(k_2 r_1)A_{N(2n-1),3} + c_{N(2n-1),4}(k_2 r_1)A_{N(2n-1),4} = 0. \tag{20}$$

The resonance frequencies can be determined by equating the fourth-order determinants of the homogeneous (at $V_0 = 0$) systems of algebraic equations (20) to zero:

$$\begin{vmatrix} a_{m1}(k_1 r_0) & a_{m2}(k_1 r_0) & a_{m3}(k_2 r_0) & a_{m4}(k_2 r_0) \\ c_{m1}(k_1 r_0) & c_{m2}(k_1 r_0) & c_{m3}(k_2 r_0) & c_{m4}(k_2 r_0) \\ a_{m1}(k_1 r_1) & a_{m2}(k_1 r_1) & a_{m3}(k_2 r_1) & a_{m4}(k_2 r_1) \\ c_{m1}(k_1 r_1) & c_{m2}(k_1 r_1) & c_{m3}(k_2 r_1) & c_{m4}(k_2 r_1) \end{vmatrix} = 0. \tag{21}$$

In (16)–(21), $m = N(2n-1)$, $n = 1, 2, \ldots$; N is the number of radial cuts of the electrode coating.

The following general properties of the theoretical frequency spectrum of a plate with different number (N) of radial cuts can be found from the boundary conditions (13)–(15), formulas (9), (10), and frequency equations (17), (19), and (21):

if $N = 1$ (two electrodes), then $f_{1,k}, f_{3,k}, f_{5,k}, \ldots$;
if $N = 2$ (four electrodes), then $f_{2,k}, f_{6,k}, f_{10,k}, \ldots$;
if $N = 3$ (six electrodes), then $f_{3,k}, f_{9,k}, f_{15,k}, \ldots$;
if $N = 4$ (eight electrodes), then $f_{4,k}, f_{12,k}, f_{20,k}, \ldots$;
if $N = 5$ (10 electrodes), then $f_{5,k}, f_{15,k}, f_{25,k}, \ldots$;
if $N = 6$ (12 electrodes), then $f_{6,k}, f_{18,k}, f_{30,k}, \ldots$;
if $N = 7$ (14 electrodes), then $f_{7,k}, f_{21,k}, f_{35,k}, \ldots$;
if $N = 8$ (16 electrodes), then $f_{8,k}, f_{24,k}, f_{40,k}, \ldots$.

In the notation of frequencies $f_{m,k}$ the subscript "m" is the harmonic number with respect to the azimuth θ (circumferential mode number) and the subscript "k" is the root sequence number of the respective frequency equation.

2. Analysis of the Results. The results of analysis of the frequency equations (17), (19), and (21) are summarized in Tables 1, 2, and 3, respectively. They present the values of dimensionless resonant frequencies $\overline{\omega} = \sqrt{(1-\nu_E^2)s_{11}^E \rho}\,\omega r_1$ determined from (17), (19), and (21), respectively, for different number N of cuts in the electrode coating for lower harmonics $m = N$, and for $r_0 / r_1 = 0.4$ and $\rho = 7740$ kg/m^3, $s_{11}^E = 15.2 \cdot 10^{-12}$ m^2/N, $s_{12}^E = -5.8 \cdot 10^{-12}$ m^2/N, $d_{31} = -125 \cdot 10^{-12}$ C/N, which corresponds to TsTS-19 piezoceramics [2].

When $N = 0$, the plate undergoes radial electroelastic vibrations,

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 u_r}{\partial t^2}, \tag{22}$$

$$\sigma_{rr} = \frac{1}{(1-\nu_E^2)s_{11}^E}\left(\frac{\partial u_r}{\partial r} + \nu_E \frac{u_r}{r} - (1+\nu_E)d_{31}E_z\right)$$

and azimuthal vibrations,

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = \rho \frac{\partial^2 u_\theta}{\partial t^2},$$

TABLE 1

k	$N=0$ $\overline{\omega}_{0,k}$	$N=1$ $\overline{\omega}_{1,k}$	$N=2$ $\overline{\omega}_{2,k}$	$N=3$ $\overline{\omega}_{3,k}$
1	0.77405	1.2108	1.85491	2.31119
2	2.7658	2.70754	2.87211	3.50261
3	4.24997	4.61023	5.4193	6.40551
4	7.211044	7.05481	6.88761	6.87753
5	7.93913	8.24799	8.85883	9.53872
6	10.14251	10.15934	10.23068	10.43583
7	13.06489	13.04537	13.09362	13.22587

TABLE 2

k	$N=0$ $\overline{\omega}_{0,k}$	$N=1$ $\overline{\omega}_{1,k}$	$N=2$ $\overline{\omega}_{2,k}$	$N=3$ $\overline{\omega}_{3,k}$
1	2.31578	2.46586	2.70391	3.14207
2	3.20884	3.3252	3.93292	4.80765
3	4.81907	5.20747	6.05161	6.743159
4	7.56991	7.377518	7.278965	7.747007
5	8.04387	8.391521	8.976372	9.558131
6	10.40329	10.43448	10.56156	10.88514
7	13.20124	13.1704	13.22125	13.38097

TABLE 3

k	$N=0$ $\overline{\omega}_{0,k}$	$N=1$ $\overline{\omega}_{1,k}$	$N=2$ $\overline{\omega}_{2,k}$	$N=3$ $\overline{\omega}_{3,k}$
1	1.42334	1.6265	0.69281	1.54389
2	3.31746	3.85103	2.34721	3.18735
3	5.49151	5.20302	4.85053	4.97488
4	6.05803	6.53165	5.05288	6.17078
5	8.896337	8.88278	7.294018	7.983204
6	10.59329	10.72654	8.93175	9.28126
7	11.76887	11.81621	11.049	11.42551

TABLE 4

r_0 / r_1	$N = 1$ $\bar{\omega}_{1,1}$	$N = 2$ $\bar{\omega}_{2,1}$	$N = 3$ $\bar{\omega}_{3,1}$	$N = 4$ $\bar{\omega}_{4,1}$
0.1	0.71165	1.34364	2.00921	2.61817
0.2	0.87091	1.45357	2.03302	2.62141
0.3	1.02743	1.62725	2.1194	2.6486
0.4	1.21076	1.85491	2.31119	2.75133
0.5	1.45919	2.12615	2.64278	3.01045
0.6	1.84263	2.46673	3.10797	3.52854
0.7	2.5143	3.02277	3.6822	4.32559
0.8	3.91731	4.2612	4.77424	5.39814
0.9	8.23944	8.40371	8.67026	9.02956

TABLE 5

r_0 / r_1	$N = 1$ $\bar{\omega}_{1,1}$	$N = 2$ $\bar{\omega}_{2,1}$	$N = 3$ $\bar{\omega}_{3,1}$	$N = 4$ $\bar{\omega}_{4,1}$
0.1	1.95992	2.75578	3.79396	4.56191
0.2	2.06778	2.50631	3.55341	4.4994
0.3	2.23762	2.50406	3.22194	4.15698
0.4	2.46665	2.70391	3.14207	3.81339
0.5	2.7481	3.08122	3.36569	3.78359
0.6	3.11977	3.60247	3.93426	4.18378
0.7	3.74549	4.21597	4.81566	5.20081
0.8	5.0915	5.42034	5.92438	6.5546
0.9	9.3562	9.51646	9.77759	10.13163

$$\sigma_{r\theta} = \frac{1}{2(1+\nu_E)s_{11}^E}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right). \tag{23}$$

The natural frequencies of electroelastic radial vibrations (22) were analyzed in [11]. The circumferential vibrations (23) cannot be excited electrically.

It follows from Tables 1–3 that when $N = 0$ (axisymmetric vibrations) and one (inner or outer) edge is clamped (boundary conditions (13) and (14)), the second, fifth, and seventh frequencies represent the radial vibrations (22), while the

TABLE 6

r_0 / r_1	$N = 1$ $\bar{\omega}_{1,1}$	$N = 2$ $\bar{\omega}_{2,1}$	$N = 3$ $\bar{\omega}_{3,1}$	$N = 4$ $\bar{\omega}_{4,1}$
0.1	1.56127	1.23169	2.00392	2.618
0.2	1.58725	1.05552	1.95461	2.61131
0.3	1.61479	0.86562	1.79267	2.55171
0.4	1.62666	0.69304	1.54389	2.35656
0.5	1.61051	0.54047	1.27326	2.04243
0.6	1.56808	0.4038	1.00296	1.67866
0.7	1.50873	0.28052	0.73471	1.28545
0.8	1.44201	0.17266	0.4703	0.8614
0.9	1.37384	0.07913	0.22072	0.41876

first, third, fourth, and sixth frequencies represent the circumferential vibrations (23). When the edges are free (boundary conditions (15)), the first, third, and sixth frequencies represent radial vibrations and the second, fourth, fifth, and seventh frequencies represent circumferential vibrations. For $N > 0$, the frequencies of radial (problem (22)) and circumferential (problem (23)) vibrations will be called quasiradial and quasicircumferential, respectively. When one edge is clamped (boundary conditions (13) and (14)), the natural frequencies equalize with increasing frequency number at $N = 0, 1, 2, 3$. This is also true for the boundary conditions (15) (free edges), but the associated frequencies are lower.

Depending on the boundary conditions and the number N of cuts in the electrode coating, the first natural frequencies, both quasiradial and quasicircumferential, for conditions (13) and (14) are considerably different (sometimes two- or three-fold) from those for condition (15). As the frequency number increases, the difference decreases to approximately 10% for the seventh frequency.

With increase in the number N of cuts in the electrode coating of the plate (with at least one edge clamped), the frequency corresponding to small k becomes higher and the frequency spectrum becomes more crowded in the high-frequency range.

It is of interest to analyze the dependence of the frequency spectrum on the geometry of the ring. Tables 4, 5, and 6 give the values of the first frequency as a function of the ratio r_0 / r_1 for different values of N and the following boundary conditions: clamped edge–free edge, free edge–clamped edge, and free edge–free edge, respectively.

The tables indicate that the way the principal frequency depends on the ring geometry is strongly dependent on the number of cuts. This frequency peaks for $N = 1$ and decreases with increasing radius of the hole for $N = 2, 3, 4$. For $N = 2, 3, 4$, the more there are the cuts, the higher the frequency.

Conclusions. Nonaxisymmetric planar electroelastic vibrations can be excited in thin piezoceramic ring plates with radially cut electrode coating on their faces. The general solution of the relevant problem has been obtained. The natural frequency spectra for lower circumferential harmonics have been numerically analyzed for three types of boundary conditions, different number of radial cuts in the electrode coating, and different ratios of inner and outer radii of the plate. The dependence of the quasiradial and quasiazimuthal natural frequencies on the frequency number and the number of cuts in the electrode coating has been examined. It has been established that the natural frequencies of the plate with one edge clamped are higher than those of the plate with all edges free. For all types of boundary conditions, the natural frequencies of the plate can be changed considerably by changing its geometry.

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