International Applied Mechanics, Vol. 48, No. 4, July, 2012

STRESS STATE OF A FINITE ELASTIC CYLINDER WITH A CIRCULAR CRACK UNDERGOING TORSIONAL VIBRATIONS

V. G. Popov

The stress intensity factors (SIF) for a plane circular crack in a finite cylinder undergoing torsional vibrations are determined. The vibrations are generated by a rigid circular plate attached to one end of the cylinder and subjected to a harmonic moment. The boundary-value problem is reduced to the Fredholm equation of the second kind. This equation is solved numerically, and the solution is used to derive a highly accurate approximate formula to calculate the SIFs. The calculated results are plotted and analyzed

Keywords: finite cylinder, plane circular crack, stress intensity factor, torsional vibrations

Introduction. Elements of machines and structures often have cylindrical shape. Cracks considerably reduce their performance and may lead to fracture, especially under dynamic loading. Therefore, it is important to analyze the stress distribution in cylindrical bodies with cracks under dynamic loading.

A review of the modern scientific literature suggests that the stress state of finite and infinite cylindrical bodies with cracks under static loading has been studied adequately. Examples of solving similar problems by various methods can be found in [10, 11, 16–18, 22]. Dynamic problems have been mainly solved for unbounded bodies with cracks, mainly circular even in the case of harmonic vibrations. The relevant results are detailed in [4, 19].

Recently, a new research area has been developed in dynamic fracture mechanics and presented in [6, 13–15]. These papers propose a method for numerical solution of spatial problems for cracked bodies under harmonic loading that takes into account the normal contact interaction of and the friction between the crack faces. The effect of these factors on the distribution of stress intensity factors (SIFs) is assessed by comparing with the results obtained regardless of the interaction of the crack faces. As regards the harmonic vibrations of cylindrical bodies with cracks, there are publications such as [20, 21] that address circular cracks in plates and infinitely long cylinders.

Thus, the stress concentration around cracks in finite cylinders under dynamic loading has been studied inadequately. Here we will determine the stress intensity factor (SIF) near a plane circular crack in a cylinder undergoing torsional vibrations.

1. Problem Formulation. Consider an elastic cylinder of finite length a and radius r_0 made of an isotropic material. To describe the cylinder, we will use a cylindrical coordinate system with origin at the center of the lower end (Fig. 1). The lower end is fixed, and the upper end is covered by a rigid plate of the same radius to which a harmonic torque $Me^{-i\omega t}$ is applied (hereafter the factor $e^{-i\omega t}$ indicating dependence on time is omitted). The cylinder has a circular crack of radius $b < r_0$ with center $z = c$ $(0 < c < a)$ on the cylinder axis. The cylinder undergoes axisymmetric torsional deformation. Only the angular displacement $w(r, z)$ is nonzero which can be found from the equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2} + \frac{\partial^2 w}{\partial^2 z} + \kappa_2^2 w = 0,$$

Odessa National Maritime Academy, 8 Didrikhsona St., Odessa, Ukraine 65029, e-mail: dr.vg.popov@gmail.com.
Translated from Prikladnaya Mekhanika, Vol. 48, No. 4, pp. 86–93, July–August 2012. Original article submitted October 18, 2010.

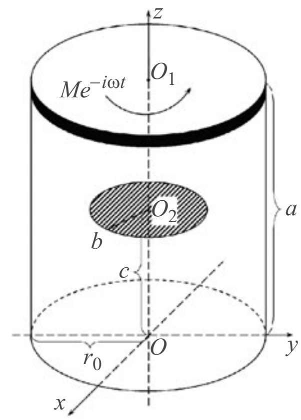

Fig. 1

$$0 < r < r_0, \qquad 0 < z < a, \qquad \kappa_2^2 = \frac{\omega^2}{c_2^2}, \qquad c_2^2 = \frac{\rho}{G}, \tag{1.1}$$

where ρ is density; G is the shear modulus of the material of the cylinder.

The following equalities hold at the ends of the cylinder:

$$w(r,0) = 0, \qquad w(r,a) = \alpha r, \qquad 0 \le r \le r_0, \tag{1.2}$$

where α is the unknown angle of rotation of the cover plate.

To determine it, it is necessary to use the equation of the torsional vibrations of the cover plate:

$$-\omega^2 j_0 \alpha = M - M_R, \tag{1.3}$$

where j_0 is the moment of inertia of the plate about the axis; M_R is the moment of reaction forces exerted by the cylinder on the plate.

These moments are defined by

$$j_0 = \frac{r_0^2 m_0}{2}, \qquad M_R = 2\pi \int_0^{r_0} r^2 \tau_{\varphi z}(r,a)\,dr, \tag{1.4}$$

where m_0 is the mass of the plate; $\tau_{\varphi z}(r,a)$ is the shear stress under the plate.

The lateral surface of the cylinder is free from stresses:

$$\tau_{\varphi r}(r_0, z) = Gr \frac{\partial}{\partial r}\left(\frac{w}{r}\right)_{r=r_0} = 0, \qquad 0 \le z \le a. \tag{1.5}$$

The crack surface is free from stresses as well:

$$\tau_{\varphi z}(r,c) = G\left(\frac{\partial w}{\partial z}\right)_{z=c} = 0, \qquad 0 \le r \le b. \tag{1.6}$$

The displacements discontinue on the crack surface:

$$w(r, c+0) - w(r, c-0) = \chi(r), \qquad 0 \le r \le b. \tag{1.7}$$

2. Solution of the Dynamic Problem. To solve the boundary-value problem, we will represent the angular displacement of the cylinder as two terms:

$$w(r,z)=w_0(r,z)+w_1(r,z). \tag{2.1}$$

The first term is the displacement of the cylinder without the crack satisfying conditions (1.2), (1.5) on its surface. It is defined by the formula

$$w_0(r,z)=\alpha r\frac{\sin \kappa_2 z}{\sin \kappa_2 a}. \tag{2.2}$$

Then the second term in (2.1) is the solution of Eq. (1.1) for which the zero conditions at the ends are satisfied:

$$w_1(r,0)=0, \qquad w_1(r,a)=0 \qquad (0\le r\le r_0), \tag{2.3}$$

and so is condition (1.5) on the lateral surface of the cylinder. Moreover, this solution undergoes discontinuity (1.7) on the crack surface and satisfies the condition

$$\tau^1_{\varphi z}(r,c)=-\tau^0_{\varphi z}(r,c), \qquad 0\le r\le c \tag{2.4}$$

$$\left(\tau^1_{\varphi z}(r,c)=G\frac{\partial w_1}{\partial z}, \qquad \tau^0_{\varphi z}(r,c)=Gr\kappa_2\frac{\cos \kappa_2 z}{\sin \kappa_2 a}\right).$$

The solution of Eq. (1.1) for which conditions (1.5), (1.7), and (2.4) are satisfied is found by the method of integral transforms generalized to discontinuous problems [7]. It is necessary to use the finite Fourier sine transform with respect to the variable z [9]:

$$w_{1k}(r)=\int_0^a w_1(r,z)\sin \lambda_k z dz,$$

$$w_1(r,z)=\frac{2}{a}\sum_{k=1}^{\infty} w_{1k}(r)\sin \lambda_k z \qquad \left(\lambda_k=\frac{\pi k}{a}\right). \tag{2.5}$$

Applying this integral transform yields the following one-dimensional problem:

$$w''_{1k}(r)+\frac{w'_{1k}(r)}{r}-\frac{w_{1k}(r)}{r^2}-q_k^2 w_{1k}(r)=-\lambda_k \chi(r)\cos \lambda_k c \qquad (q_k=\sqrt{\lambda_k^2-\kappa_2^2}),$$

$$Gr(r^{-1}w_k(r))'_{r=r_0}=0 \qquad (k=1,2,\ldots,\infty). \tag{2.6}$$

A partial solution of the inhomogeneous equation (2.5) is given by the following formula [8]:

$$\overline{w}_{1k}(r)=-\lambda_k \cos \lambda_k c\int_0^c \eta\chi(\eta)g_k(\eta,r)\, d\eta, \tag{2.7}$$

where $g_k(\eta,r)$ is the fundamental function of this equation,

$$g_k(\eta,r)=\int_0^{\infty}\frac{\beta}{\beta^2+q_k^2}J_1(\beta r)J_1(\beta\eta)d\beta=\begin{cases} I_1(q_k\eta)K_1(q_k r), \eta<r, \\ K_1(q_k\eta)I_1(q_k r), \eta>r. \end{cases}$$

A solution bounded as $r\to 0$ of the boundary-value problem (2.6) is given by

$$w_{1k}(r)=C_k I_1(q_k r)+\overline{w}_{1k}(r), \tag{2.8}$$

where C_k is an arbitrary constant determined from the boundary condition (1.5):

$$C_k = \lambda_k \cos \lambda_k c \int_0^b \eta\chi(\eta) \frac{I_1(q_k\eta) K_2(q_k r_0)}{I_2(q_k r_0)} d\eta. \tag{2.9}$$

Substituting (2.9) into (2.8) and applying the inverse Fourier sine transform (2.5), we find

$$w_1(r,z) = \int_0^b \eta\chi(\eta)[S(\eta,r,z) + D(\eta,r,z)]\, d\eta, \tag{2.10}$$

where

$$S = \frac{S^+ + S^-}{2}, \qquad S^{\pm}(\eta,r,z) = \frac{a}{2}\int_0^\infty \beta J_1(\beta r) J_1(\beta\eta) F(\beta, z \pm c)\, d\beta,$$

$$F(\beta,z) = \operatorname{sign}(z) \frac{\sinh(d_2(\beta)(a - |z|))}{\sinh(a d_2(\beta))}, \qquad d_2(\beta) = \sqrt{\beta^2 - \kappa_2^2},$$

$$D(r,\eta,z) = \frac{2}{a}\sum_{k=1}^\infty \lambda_k \cos(\lambda_k c)\sin(\lambda_k z) \frac{I_1(q_k r) I_1(q_k \eta) K_2(q_k r_0)}{I_2(q_k r_0)}. \tag{2.11}$$

Formulas (2.11) are derived using formula 1.445 (1) in [3] for the summation of trigonometric series.

To determine the angular displacement in the cylinder, it is necessary to find the unknown discontinuity $\chi(\eta)$. To this end, we substitute (2.11) into Eq. (2.4) to obtain the integral equation

$$\int_0^b \eta\chi(\eta)[F_0(\eta,r) + D_0(\eta,r)]d\eta = -\alpha\kappa_2 r \frac{\cos \kappa_2 c}{\sin \kappa_2 a} \qquad (0 \le r \le b), \tag{2.12}$$

where, according to (2.11), the following formulas are used:

$$F_0(\eta,r) = -\int_0^\infty \frac{\beta d_2(\beta)}{\sinh(a d_2(\beta))} \cosh(d_2(\beta)(a-c)) \cosh(a d_2(\beta)) J_1(\beta r) J_1(\beta\eta)\, d\beta,$$

$$D_0(\eta,r) = \frac{2}{a}\sum_{k=1}^\infty \frac{\lambda_k^2 \cos^2 \lambda_k c}{I_2(q_k r_0)} K_2(q_k r_0) I_1(q_k r) I_1(q_k \eta).$$

The integral equation (2.12) is reduced to the Fredholm equation of the second kind by introducing new unknown functions and transforms similar to those detailed in [1, 2]. First, after integration by parts in (2.12) we obtain

$$\int_0^b \psi(\eta)[F_1(\eta,r) + D_1(\eta,r)]d\eta = -\alpha\kappa_2 r \frac{\cos \kappa_2 c}{\sin \kappa_2 a} \qquad (0 \le r \le b),$$

$$F_1(\eta,r) = -\int_0^\infty \frac{d_2(\beta)}{\sinh(a d_2(\beta))} \cosh(d_2(\beta)(a-c)) \cosh(a d_2(\beta)) J_1(\beta r) J_1(\beta\eta)\, d\beta,$$

$$D_1(\eta,r) = \frac{2}{a}\sum_{k=1}^\infty \frac{\lambda_k^2 \cos^2 \lambda_k c}{q_k I_2(q_k r_0)} K_2(q_k r_0) I_1(q_k r) I_1(q_k \eta). \tag{2.13}$$

Next we introduce an unknown function:

$$\psi(\eta) = -\frac{2}{\pi}\int_{\eta}^{b} \frac{\tau\varphi(\tau)d\tau}{\sqrt{\tau^2 - \eta^2}} \tag{2.14}$$

and apply the following operator to both sides of Eq. (2.13):

$$D_2[f] = \frac{d}{dx}\int_0^x \frac{ydy}{\sqrt{x^2 - y^2}}\int_0^y f(r)dr.$$

The following formulas should be used:

$$\int_0^b \psi(\eta)J_0(\beta\eta)d\eta = \frac{2}{\pi}\int_0^b \varphi(\tau)\cos(\beta\tau)d\tau, \qquad \int_0^b \psi(\eta)I_0(q_k\eta)d\eta = \frac{2}{\pi}\int_0^b \varphi(\tau)\cosh(q_k\tau)d\tau,$$

$$D_2[J_1(\beta r)] = \frac{1-\cos(\beta x)}{x}, \qquad D_2[I_1(q_k r)] = \frac{\cosh(q_k x) - 1}{q_k}, \qquad D_2[r] = x^2.$$

After these transformations, Eq. (2.13) becomes

$$\frac{1}{\pi}\int_{-1}^{1} g(y)F(y-\zeta)dy - \frac{1}{\pi}\int_{-1}^{1} g(y)[R(y,\zeta) + G(y,\zeta)dy$$

$$-\frac{1}{\pi}\int_{-1}^{1} g(y)F(y)dy = -\alpha\kappa_0\beta_0\frac{\cos(\kappa_0\gamma d_0)}{\sin(\kappa_0\gamma)}\zeta^2 \qquad (-1 \le \zeta \le 1), \tag{2.15}$$

where $y = \tau/b$, $\zeta = x/b$, $g(y)$ is an even continuation of the function $b^{-1}\varphi(by)$ to the interval [−1, 1];

$$F(y) = \int_1^{\infty} D(u)\cos(u\kappa_0\beta y)du,$$

$$D(u) = \kappa_0\beta_0\frac{p_2\cosh(\gamma\kappa_0(1-d_0)p_2)\cosh(\gamma\kappa_0 d_0 p_2)}{u\sinh(\gamma\kappa_0 p_2)},$$

$$\frac{R(y,\zeta)}{2\kappa_0\beta_0} = \int_0^1 \frac{\cos(\gamma\kappa_0 d_0 h_2)\cos(\kappa_0\gamma(1-d_0)h_2)}{h_2^{-1}u\sin(\kappa_0\gamma h_2)}\cos(u\kappa_0\beta_0 y)\sin^2\left(\frac{u\kappa_0\beta_0 y}{2}\right)du,$$

$$G(y,\zeta) = \frac{4\beta_0}{\gamma}\sum_{k=1}^{\infty} A_k\cosh\left(\frac{\pi\beta_0\sigma_k}{\gamma}y\right)\sinh^2\left(\frac{\pi\beta_0\sigma_k}{2\gamma}\zeta\right), \qquad A_k = \frac{k^2\cos^2(\pi d_0 k)K_2\left(\frac{\pi\sigma_k}{\gamma}\right)}{\sigma_k^2 I_2\left(\frac{\pi\sigma_k}{\gamma}\right)},$$

$$p_2 = \sqrt{u^2 - 1}, \qquad h_2 = \sqrt{u^2 - 1}, \qquad \sigma_k = \sqrt{k^2 - q_0^2},$$

$$q_0 = \frac{\kappa_0\gamma}{\pi}, \qquad \gamma = \frac{a}{r_0}, \qquad d_0 = \frac{c}{a}, \qquad \beta_0 = \frac{b}{r_0}, \qquad \kappa_0 = \kappa_2 r_0.$$

The above formulas indicate that the function $F(y)$ is represented by an improper integral that should be additionally analyzed. The following asymptotic formula holds for the subintegral function:

$$D(u)=\kappa_0\beta_0(0.5+O(u^{-2}))\quad (u\to\infty).$$

Thus, this integral is divergent and can be evaluated using the theory of distributions [5] as follows:

$$F(y)=\frac{\pi}{2}\delta(y)+Q(y), \tag{2.16}$$

where $\delta(y)$ is the Dirac delta function, and $Q(y)$ is defined by uniformly converging integrals.

Substituting (2.16) into (2.15), we arrive at the Fredholm equation

$$g(\zeta)+\frac{2}{\pi}\int_{-1}^{1}g(y)Q(y-\zeta)dy-\frac{2}{\pi}\int_{-1}^{1}g(y)B(y-\zeta)dy$$

$$-g(0)-\frac{2}{\pi}\int_{-1}^{1}g(y)Q(y)dy=-2d_0\kappa_0\frac{\cos(\kappa_0\gamma d_0)}{\sin(\kappa_0\gamma)}\alpha\zeta^2\qquad(-1\le\zeta\le1),$$

$$B(y,\zeta)=R(y,\zeta)+G(y,\zeta). \tag{2.17}$$

Note that the right-hand side of Eq. (2.17) includes the unknown angle α of rotation of the cover plate. This angle can be calculated using (1.3) and (1.4). Substituting the expressions of the stresses and performing some transformations, we get

$$\alpha=\frac{\beta_0}{q}\int_{-1}^{1}g(y)(Z_1(y)-Z_2(y))dy-\frac{M_0}{q}, \tag{2.18}$$

where

$$Z_1(y)=2\kappa_0\beta_0\int_0^{\infty}\frac{p_2\cosh(\kappa_0\gamma d_0 p_2)}{u\sinh(\kappa_0\gamma p_2)}J_2(\kappa_0 u)\cos(\kappa_0\beta_0 yu)du,$$

$$Z_2(y)=\frac{4\beta_0}{\gamma}\sum_{k=1}^{\infty}d_k\sinh\left(\frac{\pi\beta_0\sigma_k}{\gamma}y\right),\qquad d_k=\frac{(-1)^k\kappa^2}{\sigma_k^2}\cos(\pi d_0 k)K_2\left(\frac{\pi\sigma_k}{\gamma}\right),$$

$$M_0=\frac{M}{Gr_0^3},\qquad q=\frac{\pi}{2}(\gamma\kappa_0^2 m_0+\kappa_0\cot(\gamma k_0)),$$

m_0 is the ratio of the mass of the cover plate to the mass of the cylinder.

An approximate solution of Eq. (2.18) can be represented by the best interpolation polynomial:

$$g(y)\approx g^n(y),\qquad g^n(y)=\sum_{m=1}^{n}g_m\frac{P_n(y)}{(y-y_m)P_n'(y_m)}\qquad(g_m=g(y_m),\quad m=1,2,\dots,n), \tag{2.19}$$

where $P_n(y)$ is nth-order Legendre polynomial, and y_m are the roots of this polynomial. A system of linear algebraic equations for the values of the unknown function at interpolation nodes (as in [2]) is derived from (2.17) and (2.18):

$$g_j+\frac{2}{\pi}\sum_{m=1}^{n}g_m A_m[Q(y_m-y_j)-B(y_m,y_j)-\frac{\pi}{2}b_m^0]=-2d_0\kappa_0\frac{\cos(\kappa_0\gamma d_0)}{\sin(\kappa_0\gamma)}\alpha y_j^2,\qquad j=1,2,\dots,n,$$

$$A_m=\frac{2}{(1-y_m^2)P_n'(y_m)^2},\qquad b_m^0=\frac{1}{2}\sum_{j=0}^{n-1}(2j+1)P_j(0)P_j(y_m). \tag{2.20}$$

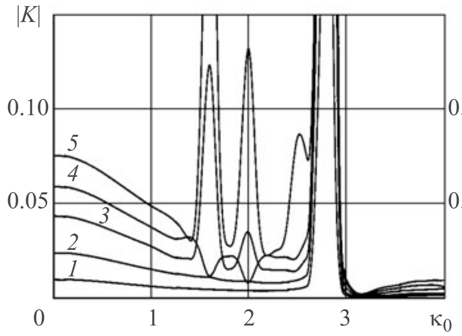

Fig. 2

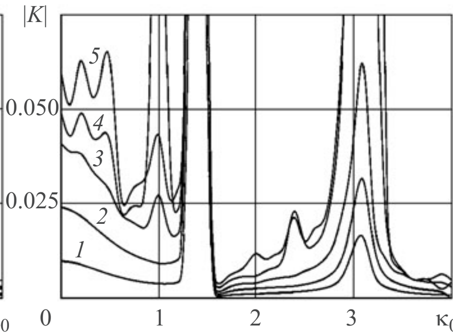

Fig. 3

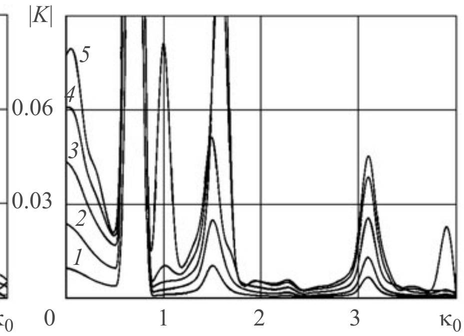

Fig. 4

After solving system (2.20) and finding the unknown discontinuity, formulas (2.10) and (2.11) can be used to determine the displacements and, hence, the stresses at any point of the cylinder. Of great interest for fracture mechanics is the SIF:

$$K = \lim_{r \to b+0} \sqrt{r-b}\,\tau^1_{\varphi z}(r, c). \tag{2.21}$$

The stresses appearing in (2.21) are subject to the asymptotics

$$\tau^1_{\varphi z}(r, c) = -\frac{G_1}{\pi} \frac{b}{r\sqrt{r^2 - b^2}}\, g\left(\frac{r}{b}\right) + O(1), \qquad r \to b + 0.$$

Substituting the last expression into (2.21) and passing to the limit using (2.19), we obtain an approximate expression for the SIF in terms of the solution of system (2.19):

$$K = -G_1 \sqrt{b} k, \qquad k = \frac{1}{\pi\sqrt{2}} g(1) = \frac{1}{\pi\sqrt{2}} \sum_{m=1}^{n} \frac{g_m}{(1 - y_m) P_n'(y_m)}. \tag{2.22}$$

3. Numerical Results. We used formula (2.22) for the numerical analysis of the absolute value of the dimensionless SIF k as a function of the frequency of vibration and the aspect ratio of the cylinder. It is assumed that the cover plate and cylinder have equal masses ($m_0 = 1$), the dimensionless moment applied to the plate $M_0 = 1$, and the crack is located in the midsurface of the cylinder ($\beta_0 = c / a = 1$).

The numerical results (SIF versus wave number $\kappa_0 = \kappa_2 r_0$) are presented in Figs. 2, 3, and 4 for $\gamma = 1$, $\gamma = 2$, and $\gamma = 3$, respectively.

Curves *1–5* correspond to the following values of the crack radius $b_0 = b / r_0$: 0.1, 0.25, 0.5, 0.75, 0.9.

Conclusions. Analyzing the results, we may draw the following conclusions.

1. A crack of even small relative radius ($b_0 = 0.1$) changes the natural frequencies of the cylinder, which are defined by $\kappa_0 \gamma = \pi l$, $l = 1, 2, \ldots$, $\gamma = a / r_0$.
2. As the relative crack radius increases toward unity, the SIF acquires additional resonant peaks.
3. An increase in the relative crack radius leads to an increase in the SIF in the subresonance frequency range.
4. As the relative length of the cylinder increases, the SIF decreases in the subresonant frequency range.

REFERENCES

1. L. V. Vakhonina and V. G. Popov, "Stress concentration around a circular thin perfectly rigid inclusion interacting with a torsional wave," *Izv. RAN*, *Mekh. Tverd. Tela*, No. 4, 70–76 (2004).
2. L. V. Vakhonina and V. G. Popov, "Interaction of elastic waves with a circular thin rigid inclusion in the case of smooth contact," *Teor. Prikl. Mekh.*, **38**, 158–166 (2003).

3. I. S. Gradshteyn and I. M. Ryzhik. *Tables of Integrals*, *Sums*, *Series and Products*, Academic Press, New York (1980).
4. A. N. Guz and V. V. Zozulya, *Brittle Fracture of Materials under Dynamic Loading* [in Russian], Naukova Dumka, Kyiv (1993).
5. W. Kecs and P. P. Teodorescu, *Introduction to the Theory of Distributions in Engineering* [in Romanian], Editura Tehnica, Bucharest (1975).
6. A. V. Men'shikov and I. A. Guz, "Friction in harmonic loading of a circular crack," *Dop. NAN Ukrainy*, No. 1, 77–82 (2007).
7. G. Ya. Popov, "On a method of solving mechanics problems for domain with slits or thin inclusions," *J. Appl. Mat. Mech*., **42**, No. 1, 125–139 (1978).
8. G. Ya. Popov, S. A. Abdymanapov, and V. V. Efimov, *Green Functions and Matrices of One-Dimensional Boundary-Value Problems* [in Russian], Rauan, Almaty (1999).
9. G. Ya. Popov, V. V. Reut, and N. D. Vaisfel'd, *Equations of Mathematical Physics. Integral Transform Method* [in Ukrainian], Astroprint, Odessa (2005).
10. T. Akiyawa, T. Hara, and T. Shibua, "Torsion of an infinite cylinder with multiple parallel circular cracks," *Theor. Appl. Mech*., **50**, 137–143 (2001).
11. D.-S. Lee, "Penny shaped crack in a long circular cylinder subjected to a uniform shearing stress," *Europ. J. Mech*., **20**(2), 227–239 (2001).
12. F. Narita, Y. Shindo, and S. Lin, "Impact response of a piezoelectric ceramic cylinder with a penny-shaped crack," *Theor. Appl. Mech*., **52**, 153–162 (2003).
13. A. N. Guz, V. V. Zozulya, and A. V. Men'shikov, "Three-dimensional dynamic contact problem for an elliptic crack interacting with a normally incident harmonic compression-expansion wave," *Int. Appl. Mech*., **39**, No. 12, 1425–1428 (2003).
14. A. N. Guz, V. V. Zozulya, and A. V. Men'shikov, "General spatial dynamic problem for an elliptic crack under the action of a normal shear wave, with consideration for the contact interaction of the crack faces," *Int. Appl. Mech*., **40**, No. 2, 156–159 (2004).
15. A. N. Guz, V. V. Zozulya, and A. V. Men'shikov, "Surface contact of elliptical crack under normally incident tension–compression wave," *Theor. Appl. Fract. Mech*., **40**, No. 3, 285–291 (2008).
16. G.-Y. Huang, Y.-S. Wang, and S.-W. Yu, "Stress concentration at a penny-shaped crack in a nonhomogeneous medium under torsion," *Acta Mech*., *December*, **180**, No. 1–4, 107–115 (2005).
17. Z. H. Jia, D. I. Shippy, and F. I. Rizzo, "Three-dimensional crack analysis using singular boundary element," *Int. J. Numer. Meth. Eng*., **28**(10), 2257–2273 (2005).
18. M. O. Kaman and R. G. Mehmet, "Cracked semi-infinite cylinder and finite cylinder problems," *Int. J. Eng. Sci*., **44**(20), 1534–1555 (2006).
19. P. A. Martin and G. R. Wickham, "Diffraction of elastic waves by a penny-shaped crack: Analytical and numerical results," *Proc. Royal Soc. London*, *Ser. A*, *Math. Phys. Sci*., **390**, No. 1798, 91–129 (1983).
20. B. M. Singh, J. B. Haddow, J. Vibik, and T. B. Moodie, "Dynamic stress intensity factors for penny-shaped crack in twisted plate," *J. Appl. Mech*., **47**, 963–965 (1980).
21. K. N. Srivastava, R. M. Palaiya, and O. P. Cupta, "Interection of elastic waves with a penny-shaped crack in an infinitely long cylinder," *J. Elast*., **12**, No. 1, 143–152 (1982).
22. W. Aishi, "A note on the crack-plane stress field method for analyzing SIFs and ITS to a concentric penny-shaped crack in a circular cylinder opened up by constant pressure," *Int. J. Fract*., **66**, 73–76 (1994).