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DEFORMATION AND DAMAGE OF COMPOSITE MATERIALS OF STOCHASTIC STRUCTURE: PHYSICALLY NONLINEAR PROBLEMS (REVIEW)

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The studies on the theory of deformation and short- and long-term damage of physically nonlinear homogeneous and composite materials are systematized. In the case of short-term damage, a single microdamage is modeled by an empty quasispherical pore occurring in place of a microvolume damaged by the Huber–Mises criterion. The ultimate microstrength is assumed to be a random function of coordinates. In the case of long-term damage, the damage criterion for a single microvolume is characterized by its stress-rupture strength determined by the dependence of the time to brittle fracture on the difference between the equivalent stress and its limit, which is the ultimate strength. The equation of porosity balance at an arbitrary time and the equations relating macrostresses and macrostrains constitute a closed system. Algorithms of calculating microdamage and macrostresses as functions of time and macrostrains are developed. The effect of nonlinearity on the curves is studied

Keywords: composite material, stochastic struture, physical nonlinearity, deformation, short-term damage, long-term damage, porosity, effective deformation characteristics

Introduction. Occurrence and development of dispersed microdamages in materials under loading commonly lead to the formation and development of main cracks, which are a cause of failure of materials and structural members. Physically, the damage of a material may be considered as dispersed defects such as microcracks, microvoids, or destroyed microvolumes. They reduce the effective or bearing portion of the material that resists loads.

There are three approaches to the mathematical modeling of the damage of materials. The first approach proceeds from the microinhomogeneity of the elastic and strength properties of a material, resulting in dispersed microdamages under loading, which are modeled by microcracks or micropores [3, 7, 13, 17, 19, 20, 27–34, 40–46, 50, 51, 55–100]. The damage equations are derived from the theory of deformation of structurally inhomogeneous materials and certain failure criteria for microvolumes of the material. The second appreoach formally introduces a damage parameter as a measure of discontinuity of the material but do not indicate its physical meaning and postulates an evolutionary equation that relates the damage rate and the applied stress [1, 8–10, 15, 16, 25, 26]. The third approach describes damage by thermodynamic (rather than structural) parameters, which contribute, together with stresses and strains, to the laws of thermodynamics. This gives formal relationships among stresses, strains, and damage parameters [2, 5, 18, 23, 24, 52, 53].

When subjected to increasing load, many homogeneous and composite materials show a nonlinear relationship between macrostresses and macrostrains. This may be due to the physically nonlinear deformation of the components [14] and the formation of dispersed microdamages [33] occurring as microcracks or micropores in place of destroyed microvolumes [55, 56, 66]. The former type of nonlinearity is typical for composites with plastic metal matrix and polymer matrix at high temperatures. The latter type of nonlinearity is typical for materials with brittle components such as polymeric composites at low temperatures, carbon-matrix composites, ceramic composites, etc. Actually, both nonlinearities are manifested simultaneously. Therefore, it is of interest to study the coupled processes of physically nonlinear deformation and damage of homogeneous and composite materials.

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Predicting the effective deformation properties of composites whose components show nonlinear stress–strain behavior involves solving a physically nonlinear problem of elasticity for a microinhomogeneous body, which is very difficult to solve compared with the linear problem, especially for regular structures [6]. If the structure is stochastic, we can use the ergodic property [36, 37, 49] to replace the averaging of the solution over a macrovolume with preliminary statistical averaging at one point, which considerably simplifies the formulation and solution of the problem. The singular [49] or one-point [36] approximation allows us to solve the nonlinear problem only for composites with quasispherical inclusions or composites reinforced with unidirectional infinite fibers. With the method of conditional moments [37], it is possible to determine the effective properties of physically nonlinear composites with arbitrarily shaped reinforcement by solving a system of nonlinear algebraic equations for strains averaged over the components. This allows studying the nonlinear deformation properties of materials with soft metal or polymer matrix reinforced with quasispherical solid particles [43].

It is obvious that the first informal approach provides the most adequate modeling of real damage processes. Proceeding from the stochastic inhomogeneity of microstrength peculiar to real materials and described by probability distributions, we can explain and model short-term (instantaneous) damage [55, 56], which occurs upon the application of load, and long-term damage, which is the accumulation of microdamages after the application of load [57]. The real damage of a material is generally a combination of short- and long-term damage.

The stochastic equations of elasticity of porous materials whose skeleton is physically nonlinear underlie the mathematical theory of coupled processes of deformation and damage of physically nonlinear materials. The damage of a material is modeled by dispersed microvolumes destroyed to become randomly arranged micropores. A microdamage of a single microvolume is characterized by its ultimate strength according to the Huber–Mises failure criterion or by its stress-rupture strength described by a fractional or exponential power function, which is determined by the dependence of the time to brittle fracture on the difference between the equivalent stress and its limit (ultimate strength according to the Huber–Mises criterion). The ultimate microstrength is assumed to be a random function of coordinates whose one-point distribution is described by a power function on some interval or by the Weibull function. The effective elastic properties and the stress–strain state of a physically nonlinear material with randomly arranged microdamages are determined from the stochastic equations of elasticity of physically nonlinear porous materials. We will derive the equation of damage (porosity) balance at an arbitrary time from the properties of the distribution functions and ergodicity of the random field of ultimate microstrength and the dependence of the time to brittle failure for a microvolume on its stress state and ultimate microstrength. The macrostress–macrostrain relations for a physically nonlinear porous material and the porosity balance equation form a closed-loop system describing the joint processes of physically nonlinear deformation and microdamage. We will use an iteration method to develop algorithms for calculating the macrostresses and microdamage as functions of macrostrains and time and to plot the respective curves. The influence of nonlinearity on the deformation and microdamage of materials will be analyzed.

The present review systematizes the studies on the theory of deformation and short- and long-term damage of physically nonlinear composites of stochastic structure performed at the S. P. Timoshenko Institute of Mechanics over the period from 1993 through 2010.

1. Nonlinear Deformation of Materials.

1.1. Nonlinear Deformation of Dispersion-Reinforced Materials: Problem Formulation. Dispersion-reinforced materials which are composites reinforced with uniformly distributed small quasispherical or quasispheroidal solid particles are very popular. A composite of stochastic structure with perfectly bonded (continuity of forces and displacements at the interface) physically nonlinear components can be represented as a microinhomogeneous elastic medium. The relationship between microstresses σ_{ij} and microstrains ε_{ij} at an arbitrary point of a composite can be expressed as

$$\sigma_{ij} = \lambda_{ijmn}(\varepsilon_{\alpha\beta})\varepsilon_{mn}, \tag{1.1}$$

where λ_{ijmn} is the stiffness tensor deterministically dependent on the strains $\varepsilon_{\alpha\beta}$ is a statistically homogeneous random function of coordinates x_r.

If a macrovolume (which is a volume much greater than inhomogeneties) of a composite is subject to homogeneous macrostresses and macrostrains, the stresses σ_{ij} and strains ε_{ij} are ergodic statistically homogeneous random functions. Their expectations $\langle\sigma_{ij}\rangle$ and $\langle\varepsilon_{ij}\rangle$ at an arbitrary point are equal to the macrostresses and macrostrains, respectively. Substituting (1.1) into the equilibrium equations

$$\sigma_{ij,j} = 0 \tag{1.2}$$

and using the kinematic equations

$$\varepsilon_{ij} = u_{(i,j)} \equiv \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.3}$$

we obtain physically and statistically nonlinear equilibrium equations for displacements:

$$[\lambda_{ijmn}(\varepsilon_{\alpha\beta})u_{m,n}]_{,j} = 0. \tag{1.4}$$

Since a macrovolume is much greater than microinhomogeneities, it may be considered to be an infinite domain V [35, 36, 48]. Let us represent random fields of stresses, strains, and displacements as sums of population means and fluctuations:

$$\sigma_{ij} = \langle\sigma_{ij}\rangle + \sigma_{ij}^0, \quad \varepsilon_{ij} = \langle\varepsilon_{ij}\rangle + \varepsilon_{ij}^0, \quad u_i = \langle\varepsilon_{ij}\rangle x_j + u_i^0. \tag{1.5}$$

Then Eq. (1.4) becomes:

$$\lambda_{ijmn}^c u_{m,nj}^0 + \{[\lambda_{ijmn}(\varepsilon_{\alpha\beta}) - \lambda_{ijmn}^c]\varepsilon_{mn}\}_{,j} = 0, \tag{1.6}$$

where λ_{ijmn}^c is some stiffness tensor with components independent of the coordinates. The boundary condition for the infinitely distant boundary S is as follows, according (1.5):

$$u_i^0\big|_S = 0. \tag{1.7}$$

Using a tensor Green's function satisfying the equation

$$\lambda_{ijmn}^c G_{mk,jn}(x_r^{(1)} - x_r^{(2)}) + \delta(x_r^{(1)} - x_r^{(2)})\delta_{ik} = 0, \tag{1.8}$$

we reduce the boundary-value problem (1.6), (1.7) to an integral equation for the strain tensor:

$$\varepsilon_{ij}^{(1)} = \langle\varepsilon_{ij}\rangle + K_{ijpq}(x_r^{(1)} - x_r^{(2)})[\lambda_{pqmn}^{(2)}(\varepsilon_{\alpha\beta}^{(2)}) - \lambda_{pqmn}^c]\varepsilon_{mn}^{(2)}, \tag{1.9}$$

where the integral operator K_{ijpq} is defined by

$$K_{ijpq}(x_r^{(1)} - x_r^{(2)})\varphi^{(2)} = \int_{V^{(2)}} G_{(ip,j)q}(x_r^{(1)} - x_r^{(2)})(\varphi^{(2)} - \langle\varphi\rangle)\,dV^{(2)}, \tag{1.10}$$

where the superscript in parentheses denotes a point in space.

The nonlinear stress–strain relation (1.1) is referred to an arbitrary point of the composite, which is in one of its components. If the point is in the kth component, then

$$\sigma_{ij}^k = \lambda_{ijmn}^k(\varepsilon_{\alpha\beta}^k)\varepsilon_{mn}^k. \tag{1.11}$$

The stresses σ_{ij}^k and strains ε_{ij}^k in the kth components can be represented as

$$\sigma_{ij}^k = \langle\sigma_{ij}^k\rangle + \sigma_{ij}^{k0}, \quad \varepsilon_{ij}^k = \langle\varepsilon_{ij}^k\rangle + \varepsilon_{ij}^{k0}, \tag{1.12}$$

where $\langle\sigma_{ij}^k\rangle, \langle\varepsilon_{ij}^k\rangle$ are the average stresses and strains over the kth component; $\sigma_{ij}^{k0}, \varepsilon_{ij}^{k0}$ are the respective fluctuations within the kth component. If we neglect the fluctuations of stresses and strains within the component, then the nonlinear relation (1.11) becomes:

$$\langle\sigma_{ij}^k\rangle = \lambda_{ijmn}^k(\langle\varepsilon_{\alpha\beta}^k\rangle)\langle\varepsilon_{mn}^k\rangle. \tag{1.13}$$

Averaging (1.13) over a macrovolume, we obtain an expression for the macrostresses:

$$\langle\sigma_{ij}\rangle=\sum_{k=1}^{2}c_k\lambda^k_{ijmn}(\langle\varepsilon^k_{\alpha\beta}\rangle)\langle\varepsilon^k_{mn}\rangle, \tag{1.14}$$

where c_k is the volume fraction of the kth component.

Let us average (1.9) using conditional density $f(\varepsilon^{(1)}_{ij},\varepsilon^{(2)}_{ij},\lambda^{(2)}_{ijmn}|\nu^{(1)})$ (distribution density of the strains at the points $x^{(1)}_r,x^{(2)}_r$ and the elastic moduli at the point $x^{(2)}_r$ provided that the point $x^{(1)}_r$ is in the νth component). Then, neglecting the fluctuations of strains within the component, we obtain a system of nonlinear algebraic equations for the average strains in the component:

$$\langle\varepsilon^{\nu}_{ij}\rangle=\langle\varepsilon_{ij}\rangle+\sum_{k=1}^{2}K^{\nu k}_{ijpq}[\lambda^k_{pqmn}(\langle\varepsilon^k_{\alpha\beta}\rangle)-\lambda^c_{pqmn}]\langle\varepsilon^k_{mn}\rangle \quad (\nu=1,2), \tag{1.15}$$

where the matrix operator $K^{\nu k}_{ijpq}$ is defined by

$$K^{\nu k}_{ijpq}=K_{ijpq}(x^{(1)}_r-x^{(2)}_r)p_{\nu k}(x^{(1)}_r-x^{(2)}_r), \tag{1.16}$$

$p_{\nu k}(x^{(1)}_r-x^{(2)}_r)=f(^{(2)}_{k}|^{(1)}_{\nu})$ is the probability of transition from the point $x^{(1)}_r$ in the νth component to the point $x^{(2)}_r$ in the kth component. Determining the average strains $\langle\varepsilon^k_{ij}\rangle$ as functions of $\langle\varepsilon_{ij}\rangle$ from (1.15) and substituting them into (1.14), we obtain the nonlinear relation between macrostresses and macrostrains.

Let us consider a composite with an isotropic matrix and isotropic unidirectional quasispheroidal inclusions:

$$\lambda^k_{ijmn}=\lambda_k\delta_{ij}\delta_{mn}+2\mu_k I_{ijmn} \quad (k=1,2), \quad \lambda^c_{ijmn}=\lambda_c\delta_{ij}\delta_{mn}+2\mu_c I_{ijmn},$$

$$p_{\nu k}=c_k+(\delta_{\nu k}-c_k)\exp\left(-\sqrt{n_1^2(x_1^2+x_2^2)+n_2^2x_3^2}\right), \tag{1.17}$$

where $\lambda_k,\mu_k,\lambda_c,\mu_c$ are the elastic moduli of the components and reference body; $I_{ijmn}=(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})/2$ is a unit tensor; n_1 and n_2 are the reciprocal semiaxes of quasispheroidal inclusions. In this case, operator (1.16) becomes:

$$K^{\nu k}_{ijpq}=(\delta_{\nu k}-c_k)\{a_1\delta_{ij}\delta_{pq}+a_2I_{ijpq}+a_3[\delta_{ij}\delta_{3p}\delta_{3q}+\delta_{i3}\delta_{j3}(\delta_{pq}-2\delta_{3p}\delta_{3q})]$$

$$+a_4\delta_{i3}\delta_{j3}\delta_{3p}\delta_{3q}+a_5(I_{i3pq}\delta_{j3}+I_{j3pq}\delta_{i3}-2\delta_{i3}\delta_{j3}\delta_{3p}\delta_{3q})\},$$

$$a_1=\frac{(\lambda_c+\mu_c)(1-s_1-s_2)}{8\mu_c(\lambda_c+2\mu_c)}, \quad a_2=-\frac{(\lambda_c+3\mu_c)(1-s_1)+(\lambda_c+\mu_c)s_2}{4\mu_c(\lambda_c+2\mu_c)},$$

$$a_3=\frac{(\lambda_c+\mu_c)(s_1+5s_2-1)}{8\mu_c(\lambda_c+2\mu_c)}, \quad a_5=\frac{\mu_c-(2\lambda_c+5\mu_c)s_1+5(\lambda_c+\mu_c)s_2}{4\mu_c(\lambda_c+2\mu_c)},$$

$$a_4=\frac{\lambda_c+5\mu_c-(\lambda_c+13\mu_c)s_1-5(\lambda_c+2\mu_c)s_2}{8\mu_c(\lambda_c+2\mu_c)}, \quad s_1=\frac{1}{1-k^2}(1-s),$$

$$s_2=\frac{1-(1+2k^2)s_1}{2(1-k^2)}, \quad k=\frac{n_1}{n_2}, \quad s=\begin{cases}-\dfrac{k}{\sqrt{k^2-1}}\ln\left(k-\sqrt{k^2-1}\right), & k\ge 1,\\ \dfrac{k}{\sqrt{1-k^2}}\arcsin\sqrt{1-k^2}, & k\le 1.\end{cases} \tag{1.18}$$

If $k=0,\infty,1$, we obtain the expressions of the operator for laminated, unidirectional fibrous, and particulate materials, respectively.

Let the bulk strains and stresses of the inclusions and matrix are related linearly, i.e., the bulk moduli $K_\nu = \lambda_\nu + 2\mu_\nu / 3$ are independent of the strains, and the deviatoric stresses $\langle \sigma_{ij}^\nu \rangle'$ and strains $\langle \varepsilon_{ij}^\nu \rangle'$ are nonlinearly related by

$$\langle \sigma_{ij}^\nu \rangle' = 2\mu_\nu (J_\nu) \langle \varepsilon_{ij}^\nu \rangle', \quad J_\nu = [\langle \varepsilon_{ij}^\nu \rangle' \langle \varepsilon_{ij}^\nu \rangle']^{1/2} \quad (\nu = 1, 2). \tag{1.19}$$

Using formulas (1.15), (1.17)–(1.19), we find the average strains:

$$\langle \varepsilon_{ij}^\nu \rangle = \langle \varepsilon_{ij} \rangle + (-1)^{\nu+1} c_{3-\nu} \{[\lambda_1(J_1) - \lambda_2(J_2)] Q_{ijkk} \langle \varepsilon_{pp} \rangle \delta_{ij}$$

$$+ 2[\mu_1(J_1) - \mu_2(J_2)] Q_{ijmn} \langle \varepsilon_{mn} \rangle \}, \tag{1.20}$$

where the transversely isotropic tensor Q_{ijmn} has the following nonzero elements:

$$Q_{1111} + Q_{1122} = Q_{2222} + Q_{2211}$$

$$= -\frac{1}{2\Delta} \{(\lambda_c + \mu_c)(\lambda_c + 2\mu_c + \lambda' + \mu')s_2 + \mu_c[(\lambda_c + 2\mu_c)(1 - s_1) + (\lambda' + 2\mu')s_1(1 - s_1)]\},$$

$$Q_{1133} = Q_{2233} = Q_{3311} = Q_{3322} = \frac{1}{2\Delta}[(\lambda_c + \mu_c)(\lambda_c + 2\mu_c + \lambda')s_2 + \lambda' \mu_c s_1 (1 - s_1)],$$

$$Q_{3333} = \frac{1}{\Delta}\{\mu_c(\lambda' + \mu')s_1^2 - (\lambda_c + 2\mu_c + \lambda' + \mu')[(\lambda_c + \mu_c)s_2 + \mu_c s_1]\},$$

$$Q_{1212} = Q_{2121} = Q_{2112} = Q_{1221} = -\frac{1}{4} \frac{(\lambda_c + \mu_c)s_2 + (\lambda_c + 3\mu_c)(1 - s)}{2\mu_c(\lambda_c + 2\mu_c) + \mu'[(\lambda_c + \mu_c)s_2 + (\lambda_c + 3\mu_c)(1 - s_1)]},$$

$$Q_{1313} = Q_{3131} = Q_{3113} = Q_{1331} = Q_{2323} = Q_{3232} = Q_{3223} = Q_{2332}$$

$$= -\frac{1}{4} \frac{(\lambda_c + 2\mu_c)(1 + s_1)s_2 - 4(\lambda_c + \mu_c)s_2}{2\mu_c(\lambda_c + 2\mu_c) + \mu'[(\lambda_c + 2\mu_c)(1 + s_1)s_2 - 4(\lambda_c + \mu_c)s_2]},$$

$$(\Delta = \mu_c(\lambda_c + 2\mu_c)[\lambda_c + 2\mu_c + \lambda' + \mu'(1 + s_1)] + \mu'\{3(\lambda_c + \mu_c)(\lambda_c + 3\mu_c)s_2$$

$$+ (3\lambda' + 2\mu')[(\lambda_c + \mu_c)s_2 + \mu_c s_1(1 + s_1)]\},$$

$$\lambda' = c_1\lambda_2(J_2) + c_2\lambda_1(J_1) - \lambda_c, \quad \mu' = c_1\mu_2(J_2) + c_2\mu_1(J_1) - \mu_c). \tag{1.21}$$

Substituting (1.17), (1.20)–(1.22) into (1.14), we arrive at the relation between the macrostresses and macrostrains:

$$\langle \sigma_{ij} \rangle = (\lambda_{11}^* - \lambda_{12}^*) \langle \varepsilon_{ij} \rangle + (\lambda_{12}^* \langle \varepsilon_{rr} \rangle + \lambda_{13}^* \langle \varepsilon_{33} \chi \rangle) \delta_{ij},$$

$$\langle \sigma_{33} \rangle = \lambda_{13}^* \langle \varepsilon_{rr} \rangle + \lambda_{33}^* \langle \varepsilon_{33} \rangle, \quad \langle \sigma_{i3} \rangle = 2\lambda_{44}^* \langle \varepsilon_{i3} \rangle \quad (i, k, r = 1, 2), \tag{1.23}$$

where the effective elastic moduli are defined by the following formulas [43]:

$$\frac{\lambda_{11}^* + \lambda_{12}^*}{2} = c_1[\lambda_1(J_1) + \mu_1(J_1)] + c_2[\lambda_2(J_2) + \mu_2(J_2)] - \frac{c_1 c_2}{\Delta}\{[\lambda_1(J_1) - \lambda_2(J_2)]^2$$

$$\times [\mu_c(\lambda_c + 2\mu_c) + 3\mu' A] + 2[\lambda_1(J_1) - \lambda_2(J_2)][\mu_1(J_1) - \mu_2(J_2)][\mu_c(\lambda_c + 2\mu_c)(1 - s_1) + 4\mu' A]$$

$$+ [\mu_1(J_1) - \mu_2(J_2)]^2 [\mu_c(\lambda_c + 2\mu_c)(1 - s_1) + (\lambda_c + \mu_c)(\lambda_c + 2\mu_c)s_2 + (\lambda' + 2\mu')A]\},$$

$$\frac{\lambda_{11}^* - \lambda_{12}^*}{2} = c_1\mu_1(J_1) + c_2\mu_2(J_2) - \frac{c_1 c_2 [\mu_1(J_1) - \mu_2(J_2)]^2 [(\lambda_c + 3\mu_c)(1 - s_1) + (\lambda_c + 3\mu_c)s_2]}{2\mu_c(\lambda_c + 2\mu_c) + \mu'[(\lambda_c + 3\mu_c)(1 - s_1) + (\lambda_c + 3\mu_c)s_2]},$$

$$\lambda_{13}^* = c_1\lambda_1(J_1) + c_2\lambda_2(J_2) - \frac{c_1 c_2}{\Delta}\{[\lambda_1(J_1) - \lambda_2(J_2)]^2[\mu_c(\lambda_c + 2\mu_c) + 3\mu' A]$$

$$+ [\lambda_1(J_1) - \lambda_2(J_2)][\mu_1(J_1) - \mu_2(J_2)][\mu_c(\lambda_c + 2\mu_c)(1 - s_1) + 4\mu' A]$$

$$- 2[\mu_1(J_1) - \mu_2(J_2)]^2[(\lambda_c + \mu_c)(\lambda_c + 2\mu_c)s_2 + \lambda' A],$$

$$\lambda_{33}^* = c_1[\lambda_1(J_1) + 2\mu_1(J_1)] + c_2[\lambda_2(J_2) + 2\mu_2(J_2)]$$

$$- \frac{c_1 c_2}{\Delta}\Big\{[\lambda_1(J_1) - \lambda_2(J_2)]^2[\mu_c(\lambda_c + 2\mu_c) + 3\mu' A]$$

$$+ 4[\lambda_1(J_1) - \lambda_2(J_2)][\mu_1(J_1) - \mu_2(J_2)][\mu_c(\lambda_c + 2\mu_c)s_1 + \mu' A]$$

$$+ 4[\mu_1(J_1) - \mu_2(J_2)]^2[\mu_c(\lambda_c + 2\mu_c)s_1 + (\lambda_c + \mu_c)(\lambda_c + 2\mu_c)s_2 + (\lambda' + \mu')A]\Big\},$$

$$\lambda_{44}^* = c_1\mu_1(J_1) + c_2\mu_2(J_2) - \frac{c_1 c_2[\mu_1(J_1) - \mu_2(J_2)]^2[(\lambda_c + 2\mu_c)(1 + s_1) - 4(\lambda_c + 2\mu_c)s_2]}{2\mu_c(\lambda_c + 2\mu_c) + \mu'[(\lambda_c + 2\mu_c)(1 + s_1) - 4(\lambda_c + 2\mu_c)s_2]},$$

$$A = \mu_c s_1(1 - s_1) + (\lambda_c + \mu_c)s_2. \tag{1.24}$$

Since the solution is approximate due to the neglect of the fluctuations of the strains within the component, these formulas, include the elastic moduli λ_c, μ_c of the reference body. It is reasonable to choose their values considering the coupling of the components and keeping them close to the real or experimental values [36, 37] or values obtained by other methods [6, 49]. If the inclusions are stiffer than the matrix, then

$$\lambda_c = \left(\frac{c_1}{K_1} + \frac{c_2}{K_2}\right)^{-1} - \frac{2}{3}\mu_c, \quad \mu_c = \left[\frac{c_1}{\mu_1(J_1)} + \frac{c_2}{\mu_2(J_2)}\right]^{-1}. \tag{1.25}$$

If the matrix is stiffer than the inclusions, then

$$\lambda_c = (c_1 K_1 + c_2 K_2) - 2\mu_c / 3, \quad \mu_c = c_1\mu_1(J_1) + c_2\mu_2(J_2). \tag{1.26}$$

The effective elastic constants (1.24)–(1.26) appearing in the constitutive equations (1.23) depend on the deviatoric average strain $\langle\varepsilon_{ij}^2\rangle'$ of the matrix; therefore, they should be expressed in terms of the given macrostrains $\langle\varepsilon_{ij}\rangle$. To this end, we use formulas (1.23) and

$$\langle\sigma_{ij}\rangle' = c_1\langle\sigma_{ij}^1\rangle' + c_2\langle\sigma_{ij}^2\rangle', \quad \langle\varepsilon_{ij}\rangle' = c_1\langle\varepsilon_{ij}^1\rangle' + c_2\langle\varepsilon_{ij}^2\rangle',$$

$$\langle\sigma_{ij}^\nu\rangle = \lambda_\nu(J_\nu)\langle\varepsilon_{pp}^\nu\rangle + 2\mu_\nu(J_\nu)\langle\varepsilon_{ij}^\nu\rangle \quad (i, j, p = 1, 2, 3, \ \nu = 1, 2), \tag{1.27}$$

whence we find

$$\langle\varepsilon_{ij}^\nu\rangle = (-1)^{(\nu+1)}\frac{\lambda_{11}^* - \lambda_{12}^* - 2\mu_{3-\nu}(J_{3-\nu})}{2c_\nu[\mu_2(J_2) - \mu_1(J_1)]}\langle\varepsilon_{ij}\rangle$$

$$+ \frac{2[\lambda_{13}^* - \lambda_{3-\nu}(J_{3-\nu})][\mu_2(J_2) - \mu_1(J_1)] - [\lambda_{33}^* - \lambda_{13}^* - 2\mu_{3-\nu}(J_{3-\nu})][\lambda_2(J_2) - \lambda_1(J_1)]}{6(-1)^{(\nu+1)}c_\nu(K_1 - K_2)[\mu_2(J_2) - \mu_1(J_1)]}\langle\varepsilon_{33}\rangle\delta_{ij},$$

$$\langle\varepsilon_{33}^\nu\rangle = \frac{2[\lambda_{13}^* - \lambda_{3-\nu}(J_{3-\nu})][\mu_2(J_2) - \mu_1(J_1)] - [\lambda_{11}^* + \lambda_{12}^* - 2\lambda_{13}^* - 2\mu_{3-\nu}(J_{3-\nu})][\lambda_2(J_2) - \lambda_1(J_1)]}{6(-1)^{(\nu+1)}c_\nu(K_1 - K_2)[\mu_2(J_2) - \mu_1(J_1)]}\langle\varepsilon_{rr}\rangle$$

$$+\frac{2[\lambda_{33}^{*}-\lambda_{3-\nu}(J_{3-\nu})-2\mu_{3-\nu}(J_{3-\nu})][\mu_2(J_2)-\mu_1(J_1)]+2[\lambda_{33}^{*}-\lambda_{13}^{*}-2\mu_{3-\nu}(J_{3-\nu})][\lambda_2(J_2)-\lambda_1(J_1)]}{6(-1)^{(\nu+1)}c_\nu(K_1-K_2)[\mu_2(J_2)-\mu_1(J_1)]}\langle\varepsilon_{33}\rangle$$

$$\langle\varepsilon_{i3}^{\nu}\rangle=(-1)^{(\nu+1)}\frac{\lambda_{44}^{*}-\mu_{3-\nu}(J_{3-\nu})}{c_\nu[\mu_2(J_2)-\mu_1(J_1)]}\langle\varepsilon_{i3}\rangle\quad(i,j,r=1,2,\ \nu=1,2). \tag{1.28}$$

Determining the invariants J_1 and J_2 as functions of macrostrains from the nonlinear equations (1.28) and substituting them into (1.23)–(1.26), we obtain a nonlinear relationship between macrostresses and macrostrains.

The nonlinear system of equations (1.23)–(1.28) can be solved numerically using the following iterative method. The nth approximation of the effective elastic moduli is determined by the formulas

$$\frac{\lambda_{11}^{*(n)}+\lambda_{12}^{*(n)}}{2}=c_1[\lambda_1(J_{1(n)})+\mu_1(J_{1(n)})]+c_2[\lambda_2(J_{2(n)})+\mu_2(J_{2(n)})]$$

$$-\frac{c_1c_2}{\Delta_{(n)}}\{[\lambda_1(J_{1(n)})-\lambda_2(J_{2(n)})]^2[\mu_{c(n)}(\lambda_{c(n)}+2\mu_{c(n)})+3\mu'_{(n)}A_{(n)}]$$

$$+2[\lambda_1(J_{1(n)})-\lambda_2(J_{2(n)})][\mu_1(J_{1(n)})-\mu_2(J_{2(n)})][\mu_{c(n)}(\lambda_{c(n)}+2\mu_{c(n)})(1-s_1)+4\mu'_{(n)}A_{(n)}]$$

$$+[\mu_1(J_{1(n)})-\mu_2(J_{2(n)})]^2[\mu_{c(n)}(\lambda_{c(n)}+2\mu_{c(n)})(1-s_1)$$

$$+(\lambda_{c(n)}+\mu_{c(n)})(\lambda_{c(n)}+2\mu_{c(n)})s_2+(\lambda'_{(n)}+2\mu'_{(n)})A_{(n)}]\},$$

$$\frac{\lambda_{11}^{*(n)}-\lambda_{12}^{*(n)}}{2}=c_1\mu_1(J_{1(n)})+c_2\mu_2(J_{2(n)})$$

$$-\frac{c_1c_2[\mu_1(J_{1(n)})-\mu_2(J_{2(n)})]^2[(\lambda_{c(n)}+3\mu_{c(n)})(1-s_1)+(\lambda_{c(n)}+3\mu_{c(n)})s_2]}{2\mu_{c(n)}(\lambda_{c(n)}+2\mu_{c(n)})+\mu'_{(n)}[(\lambda_{c(n)}+3\mu_{c(n)})(1-s_1)+(\lambda_{c(n)}+3\mu_{c(n)})s_2]},$$

$$\lambda_{13}^{*(n)}=c_1\lambda_1(J_{1(n)})+c_2\lambda_2(J_{2(n)})-\frac{c_1c_2}{\Delta_{(n)}}\{[\lambda_1(J_{1(n)})-\lambda_2(J_{2(n)})]^2[\mu_{c(n)}(\lambda_{c(n)}+2\mu_{c(n)})+3\mu'_{(n)}A_{(n)}]$$

$$+[\lambda_1(J_{1(n)})-\lambda_2(J_{2(n)})][\mu_1(J_{1(n)})-\mu_2(J_{2(n)})][\mu_{c(n)}(\lambda_{c(n)}+2\mu_{c(n)})(1+s_1)+4\mu'_{(n)}A_{(n)}]$$

$$-2[\mu_1(J_{1(n)})-\mu_2(J_{2(n)})]^2[(\lambda_{c(n)}+\mu_{c(n)})(\lambda_{c(n)}+2\mu_{c(n)})s_2+\lambda'_{(n)}A_{(n)}],$$

$$\lambda_{33}^{*(n)}=c_1[\lambda_1(J_{1(n)})+2\mu_1(J_{1(n)})]+c_2[\lambda_2(J_{2(n)})+2\mu_2(J_{2(n)})]$$

$$-\frac{c_1c_2}{\Delta_{(n)}}\{[\lambda_1(J_{1(n)})-\lambda_2(J_{2(n)})]^2[\mu_{c(n)}(\lambda_{c(n)}+2\mu_{c(n)})+3\mu'_{(n)}A_{(n)}]$$

$$+4[\lambda_1(J_{1(n)})-\lambda_2(J_{2(n)})][\mu_1(J_{1(n)})-\mu_2(J_{2(n)})][\mu_{c(n)}(\lambda_{c(n)}+2\mu_{c(n)})s_1+\mu'_{(n)}A_{(n)}]$$

$$+4[\mu_1(J_{1(n)})-\mu_2(J_{2(n)})]^2[\mu_{c(n)}(\lambda_{c(n)}+2\mu_{c(n)})s_1$$

$$+(\lambda_{c(n)}+\mu_{c(n)})(\lambda_{c(n)}+2\mu_{c(n)})s_2+(\lambda'_{(n)}+\mu'_{(n)})A_{(n)}]\}, \tag{1.29}$$

$$\lambda_{44}^{*(n)}=c_1\mu_1(J_{1(n)})+c_2\mu_2(J_{2(n)})$$

$$-\frac{c_1 c_2 [\mu_1(J_{1(n)}) - \mu_2(J_{2(n)})]^2 [(\lambda_{c(n)} + 2\mu_{c(n)})(1+s_1) - 4(\lambda_{c(n)} + 2\mu_{c(n)})s_2]}{2\mu_{c(n)}(\lambda_{c(n)} + 2\mu_{c(n)}) + \mu'_{(n)}[(\lambda_{c(n)} + 2\mu_{c(n)})(1+s_1) - 4(\lambda_{c(n)} + 2\mu_{c(n)})s_2]}$$

$$\Big(\lambda'_{(n)} = c_1\lambda_2(J_{2(n)}) + c_2\lambda_1(J_{1(n)}) - \lambda_{c(n)}, \quad \mu'_{(n)} = c_1\mu_2(J_{2(n)}) + c_2\mu_1(J_{1(n)}) - \mu_{c(n)},$$

$$\Delta_{(n)} = \mu_{c(n)}(\lambda_{c(n)} + 2\mu_{c(n)})[\lambda_{c(n)} + 2\mu_{c(n)} + \lambda'_{(n)} + \mu'_{(n)}(1+s_1)]$$

$$+\mu'_{(n)}\{3(\lambda_{c(n)} + \mu_{c(n)})(\lambda_{c(n)} + 3\mu_{c(n)})s_2 + (3\lambda'_{(n)} + 2\mu'_{(n)})[(\lambda_{c(n)} + \mu_{c(n)})s_2 + \mu_{c(n)}s_1(1+s_1)]\}, \tag{1.30}$$

$$A_{(n)} = \mu_{c(n)}s_1(1-s_1) + (\lambda_{c(n)} + \mu_{c(n)})s_2,$$

and

$$\lambda_{c(n)} = (c_1K_1 + c_2K_2) - 2\mu_{c(n)}/3, \quad \mu_{c(n)} = c_1\mu_1(J_{1(n)}) + c_2\mu_2(J_{2(n)}) \tag{1.31}$$

if the matrix is stiffer than the inclusions and

$$\lambda_{c(n)} = \left(\frac{c_1}{K_1} + \frac{c_2}{K_2}\right)^{-1} - \frac{2}{3}\mu_{c(n)}, \quad \mu_{c(n)} = \left[\frac{c_1}{\mu_1(J_{1(n)})} + \frac{c_2}{\mu_2(J_{2(n)})}\right]^{-1} \tag{1.32}$$

otherwise. The average strains $\langle\varepsilon_{ij}^{\nu}\rangle^{(n+1)}$ are determined in terms of the macrostrains $\langle\varepsilon_{ij}\rangle$ by the formulas

$$\langle\varepsilon_{ij}^{\nu}\rangle^{(n+1)} = (-1)^{(\nu+1)}\frac{\lambda_{11}^{*(n)} - \lambda_{12}^{*(n)} - 2\mu_{3-\nu}(J_{(3-\nu)(n)})}{2c_\nu[\mu_2(J_{2(n)}) - \mu_1(J_{1(n)})]}\langle\varepsilon_{ij}\rangle$$

$$+\frac{2[\lambda_{12}^{*(n)} - \lambda_{3-\nu}(J_{(3-\nu)(n)})][\mu_2(J_{2(n)}) - \mu_1(J_{1(n)})] - [\lambda_{11}^{*(n)} + \lambda_{13}^{*(n)} - 2\lambda_{12}^{*(n)} - 2\mu_{3-\nu}(J_{(3-\nu)(n)})][\lambda_2(J_{2(n)}) - \lambda_1(J_{1(n)})]}{6(-1)^{(\nu+1)}c_\nu(K_1 - K_2)[\mu_1(J_{1(n)}) - \mu_2(J_{2(n)})]}\langle\varepsilon_{rr}\rangle\delta_{ij}$$

$$+\frac{2[\lambda_{13}^{*(n)} - \lambda_{3-\nu}(J_{(3-\nu)(n)})][\mu_2(J_{2(n)}) - \mu_{3-\nu}(J_{1(n)})] - [\lambda_{33}^{*(n)} - \lambda_{13}^{*(n)} - 2\mu_{3-\nu}(J_{(3-\nu)(n)})][\lambda_2(J_{2(n)}) - \lambda_1(J_{1(n)})]}{6(-1)^{(\nu+1)}c_\nu(K_1 - K_2)[\mu_1(J_{1(n)}) - \mu_2(J_{2(n)})]}\langle\varepsilon_{33}\rangle\delta_{ij},$$

$$\langle\varepsilon_{33}^{\nu}\rangle^{(n+1)} =$$

$$= \frac{2[\lambda_{13}^{*(n)} - \lambda_{3-\nu}(J_{(3-\nu)(n)})][\mu_2(J_{2(n)}) - \mu_1(J_{1(n)})] - [\lambda_{11}^{*(n)} + \lambda_{12}^{*(n)} - 2\lambda_{13}^{*(n)} - 2\mu_{3-\nu}(J_{(3-\nu)(n)})][\lambda_2(J_{2(n)}) - \lambda_1(J_{1(n)})]}{6(-1)^{(\nu+1)}c_\nu(K_1 - K_2)[\mu_1(J_{1(n)}) - \mu_2(J_{2(n)})]}\langle\varepsilon_{rr}\rangle$$

$$+\frac{2[\lambda_{33}^{*(n)} - \lambda_{3-\nu}(J_{(3-\nu)(n)}) - 2\mu_{3-\nu}(J_{(3-\nu)(n)})][\mu_2(J_{2(n)}) - \mu_1(J_{1(n)})] + 2[\lambda_{33}^{*(n)} - \lambda_{13}^{*(n)} - 2\mu_{3-\nu}(J_{(3-\nu)(n)})][\lambda_2(J_{2(n)}) - \lambda_1(J_{1(n)})]}{6(-1)^{(\nu+1)}c_\nu(K_1 - K_2)[\mu_1(J_{1(n)}) - \mu_2(J_{2(n)})]}\langle\varepsilon_{33}\rangle,$$

$$\langle\varepsilon_{i3}^{\nu}\rangle^{(n+1)} = (-1)^{(\nu+1)}\frac{\lambda_{44}^{*(n)} - \mu_{3-\nu}(J_{(3-\nu)(n)})}{c_\nu[\mu_2(J_{2(n)}) - \mu_1(J_{1(n)})]}\langle\varepsilon_{i3}\rangle \quad (i, j, r = 1, 2, \ \nu = 1, 2), \tag{1.33}$$

$$J_{k(0)} = 0 \quad (n = 0, 1, 2, \ldots).$$

It is assumed that at small strains, the nonlinear stress–strain curves of the inclusions and matrix have linear segments with shear moduli $\mu_1(0)$ and $\mu_2(0)$, respectively.

1.2. Nonlinear Deformation of Particulate Composites. Let us consider a particulate composite with physically nonlinear isotropic inclusions and matrix. Denote the bulk and shear moduli of the inclusions and matrix by K_1, μ_2 and K_2, μ_2, respectively, and the volume fractions of the inclusions and matrix by c_1 and c_2, respectively. For a particulate composite, the macrostress–macrostrain relationship (1.23) becomes:

$$\langle \sigma_{ij} \rangle = (K^* - 2\mu^* / 3) \langle \varepsilon_{pp} \rangle \delta_{ij} + 2\mu^* \langle \varepsilon_{ij} \rangle. \tag{1.34}$$

The effective elastic constants (1.24)–(1.26) for a particulate composite are defined by the following formulas [11, 36, 37, 39]:

$$K^* = \langle K \rangle - \frac{c_1 c_2 (K_1 - K_2)^2}{c_1 K_2 + c_2 K_1 + n_c}, \quad \mu^* = \langle \mu \rangle - \frac{c_1 c_2 (\mu_1 - \mu_2)^2}{c_1 \mu_2 + c_2 \mu_1 + m_c}, \quad \lambda^* = K^* - 2\mu^* / 3 \tag{1.35}$$

$$\left(n_c = \frac{4}{3} \mu_c, \; m_c = \frac{\mu_c (9K_c + 8\mu_c)}{6(K_c + 2\mu_c)}, \quad K_c = \lambda_c + 2\mu_c / 3 \right). \tag{1.36}$$

Since the solution is approximate because of the neglect of the fluctuations of the parameters within the components, expressions (1.35) and (1.36) include the elastic moduli K_c, μ_c chosen considering the coupling of the components and keeping them as close as possible to the real values. An analysis of various alternatives for K_c, μ_c shows [39] that for a particulate composite it is expedient to set

$$K_c = c_1 K_1 + c_2 K_2, \quad \mu_c = c_1 \mu_1 + c_2 \mu_2 \tag{1.37}$$

if the matrix is stiffer than the inclusions and

$$K_c = \frac{K_1 K_2}{c_1 K_2 + c_2 K_1}, \quad \mu_c = \frac{\mu_1 \mu_2}{c_1 \mu_2 + c_2 \mu_1} \tag{1.38}$$

otherwise.

If the composite components are physically nonlinear (1.19), then the elastic moduli (1.36) are functions of the invariants J_1, J_2, and, consequently, of the average strains $\langle \varepsilon_{ij}^{\nu} \rangle$ $(\nu = 1, 2)$. Expressing the average strains in the components in terms of the average strains $\langle \varepsilon_{ij} \rangle$ in the composite, we obtain the effective moduli as functions of the average strains in the composite. For a particulate composite, the following expressions derive from (1.28) [11, 36, 37, 39]:

$$\langle \varepsilon_{ij}^{\nu} \rangle = (-1)^{\nu+1} \left[\frac{2\mu^* (\mu_1 - \mu_2)(K^* - K_{3-\nu}) - 3K^* (K_1 - K_2)(\mu^* - \mu_{3-\nu})}{6 c_\nu \mu^* (\mu_1 - \mu_2)(K_1 - K_2)} \langle \varepsilon_{rr} \rangle \delta_{ij} + \frac{\mu^* - \mu_{3-\nu}}{c_\nu (\mu_1 - \mu_2)} \langle \varepsilon_{ij} \rangle \right]. \tag{1.39}$$

Substituting expressions (1.39) into (1.35), (1.36), we obtain expressions for the effective moduli in terms of the average strains $\langle \varepsilon_{ij} \rangle$.

The effective moduli of a particulate composite with physically nonlinear components can be found using an iterative algorithm similar to (1.29)–(1.33). The nth approximation of the effective moduli $K^{*(n)}$ and $\mu^{*(n)}$ is defined by

$$K^{*(n)} = c_1 K_1 + c_2 K_2 - \frac{c_1 c_2 (K_1 - K_2)^2}{c_1 K_2 + c_2 K_1 + n_c^{(n)}},$$

$$\mu^{*(n)} = c_1 \mu_1 (J_{1(n)}) + c_2 \mu_2 (J_{2(n)}) - \frac{c_1 c_2 [\mu_1 (J_{1(n)}) - \mu_2 (J_{2(n)})]^2}{c_1 \mu_2 (J_{2(n)}) + c_2 \mu_1 (J_{1(n)}) + m_c^{(n)}},$$

$$\lambda^{*(n)} = K^{*(n)} - \frac{2}{3} \mu^{*(n)}, \tag{1.40}$$

$$\left(n_c^{(n)} = \frac{4}{3} \mu_c^{(n)}, \quad m_c^{(n)} = \frac{\mu_c^{(n)} (9K_c + 8\mu_c^{(n)})}{6(K_c + 2\mu_c^{(n)})}, \right. \tag{1.41}$$

and

$$K_c = c_1 K_1 + c_2 K_2, \quad \mu_c^{(n)} = c_1 \mu_1 (J_{1(n)}) + c_2 \mu_2 (J_{2(n)}) \tag{1.42}$$

if the matrix is stiffer than the inclusions and

$$K_c = \frac{K_1 K_2}{c_1 K_2 + c_2 K_1}, \quad \mu_c^{(n)} = \frac{\mu_1 (J_{1(n)}) \mu_2 (J_{2(n)})}{c_1 \mu_2 (J_{2(n)}) + c_2 \mu_1 (J_{1(n)})} \tag{1.43}$$

otherwise. The average strains $\langle \varepsilon_{ij}^{\nu} \rangle^{(n+1)}$ are determined in terms of the macrostrains $\langle \varepsilon_{ij} \rangle$ by the formulas

$$\langle \varepsilon_{ij}^{\nu} \rangle^{(n+1)} = (-1)^{\nu+1} \left\{ \frac{\mu^{*(n)} - \mu_{3-\nu} (J_{(3-\nu)(n)})}{c_\nu [\mu_1 (J_{1(n)}) - \mu_2 (J_{2(n)})]} \langle \varepsilon_{ij} \rangle \right.$$

$$\left. + \frac{2\mu^{*(n)} [\mu_1 (J_{1(n)}) - \mu_2 (J_{2(n)})](K^{*(n)} - K_{3-\nu}) - 3K^{*(n)} (K_1 - K_2)[\mu^{*(n)} - \mu_{3-\nu} (J_{(3-\nu)(n)})]}{6 c_\nu \mu^* [\mu_1 (J_{1(n)}) - \mu_2 (J_{2(n)})](K_1 - K_2)} \langle \varepsilon_{rr} \rangle \delta_{ij} \right\}. \tag{1.44}$$

It is assumed that at small strains, the nonlinear stress–strain curves of the components have linear segments with shear moduli $\mu_1(0)$ and $\mu_2(0)$, respectively.

Let us study, as an example, the nonlinear deformation of a particulate composite with linear elastic inclusions and nonlinear elastic matrix, with the bulk strains of the matrix being linear (i.e., the bulk modulus $K_2 = \lambda_2 + 2\mu_2 / 3$ does not depend on the strains) and the shear strains described by a linear hardening diagram (i.e., the deviatoric stresses $\langle \sigma_{ij}^2 \rangle'$ and strains $\langle \varepsilon_{ij}^2 \rangle'$ are nonlinearly related):

$$\sigma_{rr}^2 = K_2 \varepsilon_{rr}^2, \quad \sigma_{ij}'^2 = 2\mu_2 (J_2) \varepsilon_{ij}'^2, \tag{1.45}$$

where $\mu_2(J_2)$ is the shear modulus described by the function

$$\mu_2 (J_2) = \begin{cases} \mu_2, & T_2 \le T_{20}, \\ \mu_2' + (1 - \mu_2' / \mu_2^2) T_{20} / (2J_2), & T_2 \ge T_{20}, \end{cases} \tag{1.46}$$

$$\Big(J_2 = [\langle \varepsilon_{ij}^2 \rangle' \langle \varepsilon_{ij}^2 \rangle']^{1/2}, \quad T_2 = [\langle \sigma_{ij}^2 \rangle' \langle \sigma_{ij}^2 \rangle']^{1/2}, \quad T_{20} = \sigma_{20} \sqrt{2/3}, \tag{1.47}$$

$\langle \sigma_{ij}^2 \rangle', \langle \varepsilon_{ij}^2 \rangle'$ are the average deviatoric stresses and strains in the matrix; σ_{20} is the limit of proportionality of the matrix.

The above formulas were used to analyze effective nonlinear stress–strain curves of a particulate composite for different volume fractions of the components and given macroparameters

$$\langle \varepsilon_{11} \rangle \ne 0, \quad \langle \sigma_{22} \rangle = \langle \sigma_{33} \rangle = 0. \tag{1.48}$$

According to (1.34), the macrostress $\langle \sigma_{11} \rangle$ is related to the macrostrain $\langle \varepsilon_{11} \rangle$ by

$$\langle \sigma_{11} \rangle = \frac{3K^* \mu^*}{K^* + \mu^* / 3} \langle \varepsilon_{11} \rangle. \tag{1.49}$$

The composite has linear elastic inclusions made of aluminoborosilicate glass with the following characteristics [11]:

$$K_1 = 38.89 \text{ GPa}, \quad \mu_1 = 29.17 \text{ GPa} \tag{1.50}$$

and epoxy matrix described by the linear-hardening diagram (1.45)–(1.47) with the following constants [11, 22]:

$$K_2 = 3.33 \text{ GPa}, \quad \mu_{20} = 1.11 \text{ GPa}, \quad \mu_2' = 0.331 \text{ GPa} \tag{1.51}$$

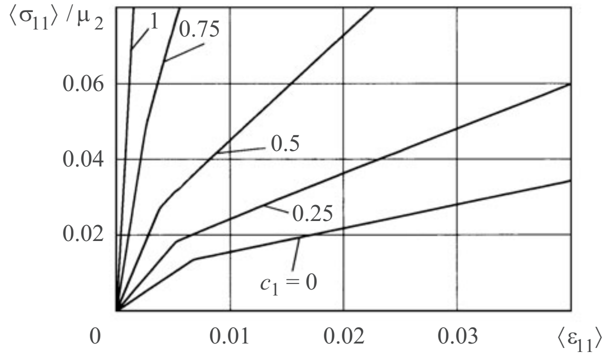

Fig. 1.1

and the yield stress

$$\sigma_{20} = 0.015 \text{ GPa}. \tag{1.52}$$

Figure 1.1 shows the macrostress $\langle\sigma_{11}\rangle/\mu_2$ as a function of the macrostrain $\langle\varepsilon_{11}\rangle$ in a particulate composite for different values of c_1. As is seen, the physical nonlinearity of the matrix has a significant effect on the stress–strain curves for all values of $c_1 < 1$. The curve of the material with linear-hardening matrix consists of two linear segments.

1.3. Nonlinear Deformation of Laminated Composites. Let us consider a laminated material with nonlinear elastic isotropic components. The bulk and shear moduli of the νth component are denoted by K_ν and μ_ν, and the volume fraction of the νth component by c_ν. The macrostrains $\langle\varepsilon_{ij}\rangle$ and macrostresses $\langle\sigma_{ij}\rangle$ in the composite are related, according to (1.23), by

$$\langle\sigma_{ij}\rangle = (\lambda_{11}^* - \lambda_{12}^*)\langle\varepsilon_{ij}\rangle + (\lambda_{12}^*\langle\varepsilon_{rr}\rangle + \lambda_{13}^*\langle\varepsilon_{33}\rangle)\delta_{ij},$$

$$\langle\sigma_{33}\rangle = \lambda_{13}^*\langle\varepsilon_{rr}\rangle + \lambda_{33}^*\langle\varepsilon_{33}\rangle, \quad \langle\sigma_{i3}\rangle = 2\lambda_{44}^*\langle\varepsilon_{i3}\rangle \quad (i, j, r = 1, 2). \tag{1.53}$$

For a laminated composite, the effective moduli $\lambda_{11}^*, \lambda_{12}^*, \lambda_{13}^*, \lambda_{33}^*, \lambda_{44}^*$ are defined, according to (1.24)–(1.26), by the following formulas [11, 36, 37, 39]:

$$\lambda_{11}^* = \left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1}\cdot\left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle^{2} + 4\left\langle\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right\rangle,$$

$$\lambda_{12}^* = \left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1}\cdot\left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle^{2} + 2\left\langle\frac{\lambda\mu}{\lambda+2\mu}\right\rangle,$$

$$\lambda_{13}^* = \left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1}\left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle, \quad \lambda_{33}^* = \left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1}, \quad \lambda_{44}^* = \left\langle\frac{1}{\mu}\right\rangle^{-1} \tag{1.54}$$

$$(\lambda_\nu = K_\nu - 2\mu_\nu/3, \quad \langle f\rangle = c_1 f_1 + c_2 f_2, \quad \nu = 1, 2). \tag{1.55}$$

If the composite components are physically nonlinear (1.19), then the elastic moduli (1.54), (1.55) are functions of the invariants J_1, J_2, and, consequently, of the average strains $\langle\varepsilon_{ij}^\nu\rangle$ $(\nu = 1, 2)$. Expressing the average strains in the components in terms of the average strains $\langle\varepsilon_{ij}\rangle$ in the composite, we obtain the effective moduli as functions of the average strains in the composite. For a laminated composite, the following expressions derive from (1.28) [11, 36, 37, 39]:

$$\langle\varepsilon_{ij}^\nu\rangle = \langle\varepsilon_{ij}\rangle, \quad \langle\varepsilon_{i3}^\nu\rangle = \frac{1}{\mu_\nu}\cdot\left\langle\frac{1}{\mu}\right\rangle^{-1}\langle\varepsilon_{i3}\rangle,$$

$$\langle \varepsilon_{33}^{\nu} \rangle = \frac{1}{\lambda_{\nu} + 2\mu_{\nu}} \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1} \cdot \left[\left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle - \lambda_{\nu} \left\langle \frac{1}{\lambda + 2\mu} \right\rangle \right] \langle \varepsilon_{33} \rangle + \langle \varepsilon_{rr} \rangle \quad (\nu = 1, 2). \tag{1.56}$$

Substituting expressions (1.56) into (1.54), (1.55), we obtain expressions for the effective moduli in terms of the average strains $\langle \varepsilon_{ij} \rangle$.

The effective moduli of a laminated composite with physically nonlinear components can be found using an iterative algorithm similar to (1.29)–(1.33) or (1.40)–(1.44). The *n*th approximation of the effective elastic moduli is determined by the formulas

$$\lambda_{11}^{*(n)} = \left\langle \frac{1}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle^{-1} \cdot \left\langle \frac{\lambda(J_{(n)})}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle^{2} + 4 \left\langle \frac{\mu(J_{(n)})[\lambda(J_{(n)}) + \mu(J_{(n)})]}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle,$$

$$\lambda_{12}^{*(n)} = \left\langle \frac{1}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle^{-1} \cdot \left\langle \frac{\lambda(J_{(n)})}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle^{2} + 2 \left\langle \frac{\lambda(J_{(n)})\mu(J_{(n)})}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle,$$

$$\lambda_{13}^{*(n)} = \left\langle \frac{1}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle^{-1} \cdot \left\langle \frac{\lambda(J_{(n)})}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle,$$

$$\lambda_{33}^{*(n)} = \left\langle \frac{1}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle^{-1}, \qquad \lambda_{44}^{*(n)} = \left\langle \frac{1}{\mu(J_{(n)})} \right\rangle^{-1} \tag{1.57}$$

$$(\lambda_{\nu}(J_{\nu(n)}) = K_{\nu} - 2\mu_{\nu} / 3(J_{\nu(n)}) \quad (\nu = 1, 2), \quad \langle f(J_{(n)}) \rangle = c_1 f_1(J_{1(n)}) + c_2 f_2(J_{2(n)})). \tag{1.58}$$

The average strains $\langle \varepsilon_{ij}^{\nu} \rangle^{(n+1)}$ are determined in terms of the macrostrains $\langle \varepsilon_{ij} \rangle$ by the formulas

$$\langle \varepsilon_{ij}^{\nu(n+1)} \rangle = \langle \varepsilon_{ij} \rangle, \quad \langle \varepsilon_{i3}^{\nu(n+1)} \rangle = \frac{1}{\mu_{\nu}(J_{\nu(n)})} \cdot \left\langle \frac{1}{\mu(J_{(n)})} \right\rangle^{-1} \langle \varepsilon_{i3} \rangle,$$

$$\langle \varepsilon_{33}^{\nu(n+1)} \rangle = \frac{1}{\lambda_{\nu}(J_{\nu(n)}) + 2\mu_{\nu}(J_{\nu(n)})} \cdot \left\langle \frac{1}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle^{-1}$$

$$\times \left[\left\langle \frac{\lambda(J_{(n)})}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle - \lambda_{\nu}(J_{\nu(n)}) \left\langle \frac{1}{\lambda(J_{(n)}) + 2\mu(J_{(n)})} \right\rangle \right] \langle \varepsilon_{33} \rangle + \langle \varepsilon_{rr} \rangle \qquad (\nu = 1, 2). \tag{1.59}$$

It is assumed that at small strains, the nonlinear stress–strain curves of the components have linear segments with shear moduli $\mu_1(0)$ and $\mu_2(0)$, respectively.

Let us study, as an example, the nonlinear deformation of a laminated composite with linear elastic reinforcement and nonlinear elastic matrix, with the bulk strains being linear and the shear strains described by linear-hardening diagram (1.46), (1.47).

The composite has aluminoborosilicate glass reinforcement with characteristics (1.50) and epoxy matrix with constants (1.51), linear-hardening diagram (1.45)–(1.47), and yield stress (1.52).

If

$$\langle \varepsilon_{33} \rangle \neq 0, \quad \langle \sigma_{11} \rangle = \langle \sigma_{22} \rangle = 0, \tag{1.60}$$

then, according to (1.53), the macrostress $\langle \sigma_{33} \rangle$ is related to the macrostrain $\langle \varepsilon_{33} \rangle$ by

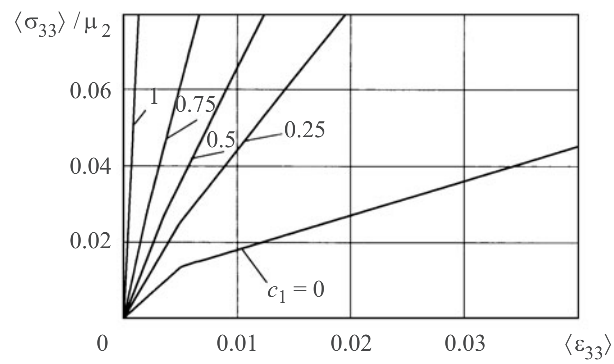

Fig. 1.2

$$\langle\sigma_{33}\rangle = \frac{1}{\lambda_{11}^* + \lambda_{12}^*}[(\lambda_{11}^* + \lambda_{12}^*)\lambda_{33}^* - 2(\lambda_{13}^*)^2]\langle\varepsilon_{33}\rangle. \tag{1.61}$$

Figure 1.2 shows the macrostress $\langle\sigma_{33}\rangle / \mu_2$ as a function of the macrostrain $\langle\varepsilon_{33}\rangle$ in a laminated composite for different values of c_1. It can be seen that the physical nonlinearity of the matrix has a significant effect on the stress–strain curve ($\langle\sigma_{33}\rangle / \mu_2$ versus $\langle\varepsilon_{33}\rangle$) for all values of $c_1 < 1$. The curve of the material with linear-hardening matrix consists of two linear segments.

1.4. Nonlinear Deformation of Fibrous Composites. Let us consider a unidirectional fiber-reinforced material with nonlinear elastic isotropic components. Denote the bulk and shear moduli of the fibers and matrix by K_1, μ_1 and K_2, μ_2, respectively, and the volume fractions of the fibers and matrix by c_1 and c_2, respectively. The macrostrains $\langle\varepsilon_{ij}\rangle$ and macrostresses $\langle\sigma_{ij}\rangle$ in the composite are related, according to (1.23), by (1.53).

For a fibrous composite, the effective moduli $\lambda_{11}^*, \lambda_{12}^*, \lambda_{13}^*, \lambda_{33}^*, \lambda_{44}^*$ are defined, according to (1.24)–(1.26), by the following formulas [11, 36, 37, 39]:

$$\frac{\lambda_{11}^* + \lambda_{12}^*}{2} = c_1(\lambda_1 + \mu_1) + c_2(\lambda_2 + \mu_2) - \frac{c_1 c_2 (\lambda_1 + \mu_1 - \lambda_2 - \mu_2)^2}{c_1(\lambda_2 + \mu_2) + c_2(\lambda_1 + \mu_1) + \mu_c},$$

$$\frac{\lambda_{11}^* - \lambda_{12}^*}{2} = c_1\mu_1 + c_2\mu_2 - \frac{c_1 c_2 (\mu_1 - \mu_2)^2}{c_1\mu_2 + c_2\mu_1 + \dfrac{\mu_c(\lambda_c + \mu_c)}{\lambda_c + 3\mu_c}},$$

$$\lambda_{13}^* = c_1\lambda_1 + c_2\lambda_2 - \frac{c_1 c_2 (\lambda_1 + \mu_1 - \lambda_2 - \mu_2)(\lambda_1 - \lambda_2)}{c_1(\lambda_2 + \mu_2) + c_2(\lambda_1 + \mu_1) + \mu_c},$$

$$\lambda_{33}^* = c_1(\lambda_1 + 2\mu_1) + c_2(\lambda_2 + 2\mu_2) - \frac{c_1 c_2 (\lambda_1 - \lambda_2)^2}{c_1(\lambda_2 + \mu_2) + c_2(\lambda_1 + \mu_1) + \mu_c},$$

$$\lambda_{44}^* = c_1\mu_1 + c_2\mu_2 - \frac{c_1 c_2 (\mu_1 - \mu_2)^2}{c_1\mu_2 + c_2\mu_1 + \mu_c} \quad (\lambda_\nu = K_\nu - 2\mu_\nu / 3, \nu = 1, 2), \tag{1.62}$$

$$(\mu_c = c_1\mu_1 + c_2\mu_2, \quad \lambda_c = c_1\lambda_1 + c_2\lambda_2 \tag{1.63}$$

if the matrix is stiffer than the fibers and

$$\mu_c = \left(\frac{c_1}{\mu_1} + \frac{c_2}{\mu_2}\right)^{-1}, \quad \lambda_c = \left(\frac{c_1}{K_1} + \frac{c_2}{K_2}\right)^{-1} - \frac{2}{3}\mu_c \tag{1.64}$$

if the fibers are stiffer than the matrix.

If the composite components are physically nonlinear (1.19), then the elastic moduli (1.62)–(1.64) are functions of the invariants J_1, J_2, and, consequently, of the average strains $\langle \varepsilon_{ij}^1 \rangle, \langle \varepsilon_{ij}^2 \rangle$. Expressing the average strains in the components in terms of the average strains $\langle \varepsilon_{ij} \rangle$ in the composite, we obtain the effective moduli as functions of the average strains in the composite. For a fibrous composite, the following expressions derive from (1.28) [11, 36, 37, 39]:

$$\langle \varepsilon_{ij}^{\nu} \rangle = (-1)^{\nu+1} \left[\frac{\lambda_{11}^* - \lambda_{12}^* - 2\mu_{3-\nu}}{2c_\nu(\mu_2 - \mu_1)} \langle \varepsilon_{ij} \rangle \right.$$

$$+ \frac{2(\lambda_{12}^* - \lambda_{3-\nu})(\mu_2 - \mu_1) - (\lambda_{11}^* + \lambda_{13}^* - 2\lambda_{12}^* - 2\mu_{3-\nu})(\lambda_2 - \lambda_1)}{6c_\nu(K_2 - K_1)(\mu_2 - \mu_1)} \langle \varepsilon_{rr} \rangle \delta_{ij}$$

$$\left. + \frac{2(\lambda_{13}^* - \lambda_{3-\nu})(\mu_2 - \mu_1) - (\lambda_{33}^* - \lambda_{13}^* - 2\mu_{3-\nu})(\lambda_2 - \lambda_1)}{6c_\nu(K_2 - K_1)(\mu_2 - \mu_1)} \langle \varepsilon_{33} \rangle \delta_{ij} \right],$$

$$\langle \varepsilon_{33}^{\nu} \rangle = (-1)^{\nu+1} \left[\frac{2(\lambda_{13}^* - \lambda_{3-\nu})(\mu_2 - \mu_1) - (\lambda_{11}^* + \lambda_{12}^* - 2\lambda_{13}^* - 2\mu_{3-\nu})(\lambda_2 - \lambda_1)}{6c_\nu(K_2 - K_1)(\mu_2 - \mu_1)} \langle \varepsilon_{rr} \rangle \right.$$

$$\left. + \frac{(\lambda_{33}^* - \lambda_{3-\nu} - 2\mu_{3-\nu})(\mu_2 - \mu_1) - (\lambda_{33}^* - \lambda_{13}^* - 2\mu_{3-\nu})(\lambda_2 - \lambda_1)}{3c_\nu(K_2 - K_1)(\mu_2 - \mu_1)} \langle \varepsilon_{33} \rangle \right],$$

$$\langle \varepsilon_{i3}^{\nu} \rangle = (-1)^{\nu+1} \frac{\lambda_{44}^* - \mu_{3-\nu}}{c_\nu(\mu_2 - \mu_1)} \langle \varepsilon_{i3} \rangle \quad (i, j, r = 1,\ 2). \tag{1.65}$$

Substituting expressions (1.65) into (1.62)–(1.64), we obtain expressions for the effective moduli in terms of the average strains $\langle \varepsilon_{ij} \rangle$.

The effective moduli of a fibrous composite with physically nonlinear components can be found using an iterative algorithm similar to (1.29)–(1.33), or (1.40)–(1.44), or (1.57)–(1.59). The nth approximation of the effective elastic moduli is determined by the formulas

$$\frac{\lambda_{11}^{*(n)} + \lambda_{12}^{*(n)}}{2} = c_1[\lambda_1(J_{1(n)}) + \mu_1(J_{1(n)})] + c_2[\lambda_2(J_{2(n)}) + \mu_2(J_{2(n)})]$$

$$- \frac{c_1 c_2[\lambda_1(J_{1(n)}) + \mu_1(J_{1(n)}) - \lambda_2(J_{2(n)}) - \mu_2(J_{2(n)})]^2}{c_1[\lambda_2(J_{2(n)}) + \mu_2(J_{2(n)})] + c_2[\lambda_1(J_{1(n)}) + \mu_1(J_{1(n)})] + \mu_c^{(n)}},$$

$$\frac{\lambda_{11}^{*(n)} - \lambda_{12}^{*(n)}}{2} = c_1\mu_1(J_{1(n)}) + c_2\mu_2(J_{2(n)}) - \frac{c_1 c_2[\mu_1(J_{1(n)}) - \mu_2(J_{2(n)})]^2}{c_1\mu_2(J_{2(n)}) + c_2\mu_1(J_{1(n)}) + \dfrac{\mu_c^{(n)}(\lambda_c^{(n)} + \mu_c^{(n)})}{\lambda_c^{(n)} + 3\mu_c^{(n)}}},$$

$$\lambda_{13}^{*(n)} = c_1\lambda_1(J_{1(n)}) + c_2\lambda_2(J_{2(n)})$$

$$- \frac{c_1 c_2[\lambda_1(J_{1(n)}) + \mu_1(J_{1(n)}) - \lambda_2(J_{2(n)}) - \mu_2(J_{2(n)})][\lambda_1(J_{1(n)}) - \lambda_2(J_{2(n)})]}{c_1[\lambda_2(J_{2(n)}) + \mu_2(J_{2(n)})] + c_2[\lambda_1(J_{1(n)}) + \mu_1(J_{1(n)})] + \mu_c^{(n)}},$$

$$\lambda_{33}^{*(n)} = c_1[\lambda_1(J_{1(n)}) + 2\mu_1(J_{1(n)})] + c_2[\lambda_2(J_{2(n)}) + 2\mu_2(J_{2(n)})]$$

$$-\frac{c_1c_2[\lambda_1(J_{1(n)})-\lambda_2(J_{2(n)})]^2}{c_1[\lambda_2(J_{2(n)})+\mu_2(J_{2(n)})]+c_2[\lambda_1(J_{1(n)})+\mu_1(J_{1(n)})]+\mu_c^{(n)}}, \tag{1.66}$$

$$\lambda_{44}^{*(n)}=c_1\mu_1(J_{1(n)})+c_2\mu_2(J_{2(n)})-\frac{c_1c_2[\mu_1(J_{1(n)})-\mu_2(J_{2(n)})]^2}{c_1\mu_2(J_{2(n)})+c_2\mu_1(J_{1(n)})+\mu_c^{(n)}}$$

$$(\mu_c^{(n)}=c_1\mu_1(J_{1(n)})+c_2\mu_2(J_{2(n)}),\quad \lambda_c^{(n)}=c_1\lambda_1(J_{1(n)})+c_2\lambda_2(J_{2(n)}) \tag{1.67}$$

if the matrix is stiffer than the fibers and

$$\mu_c^{(n)}=\left[\frac{c_1}{\mu_1(J_{1(n)})}+\frac{c_2}{\mu_2(J_{2(n)})}\right]^{-1},\quad \lambda_c^{(n)}=\left(\frac{c_1}{K_1}+\frac{c_2}{K_2}\right)^{-1}-\frac{2}{3}\mu_c^{(n)} \tag{1.68}$$

if the fibers are stiffer than the matrix.

The average strains $\langle\varepsilon_{ij}^{\nu}\rangle^{(n+1)}$ are determined in terms of the macrostrains $\langle\varepsilon_{ij}\rangle$ by the formulas

$$\langle\varepsilon_{ij}^{\nu}\rangle^{(n)}=(-1)^{\nu+1}\left\{\frac{\lambda_{11}^{*(n)}-\lambda_{12}^{*(n)}-2\mu_{3-\nu}(J_{(3-\nu)(n)})}{2c_\nu[\mu_2(J_{2(n)})-\mu_1(J_{1(n)})]}\langle\varepsilon_{ij}\rangle\right.$$

$$+\frac{2[\lambda_{12}^{*(n)}-\lambda_{3-\nu}(J_{(3-\nu)(n)})][\mu_2(J_{2(n)})-\mu_1(J_{1(n)})]-[\lambda_{11}^{*(n)}+\lambda_{13}^{*(n)}-2\lambda_{12}^{*(n)}-2\mu_{3-\nu}(J_{(3-\nu)(n)})][\lambda_2(J_{2(n)})-\lambda_1(J_{1(n)})]}{6c_\nu(K_2-K_1)[\mu_2(J_{2(n)})-\mu_1(J_{1(n)})]}\langle\varepsilon_{rr}\rangle\delta_{ij}$$

$$\left.+\frac{2[\lambda_{13}^{*(n)}-\lambda_{3-\nu}(J_{(3-\nu)(n)})][\mu_2(J_{2(n)})-\mu_1(J_{1(n)})]-[\lambda_{33}^{*(n)}-\lambda_{13}^{*(n)}-2\mu_{3-\nu}(J_{(3-\nu)(n)})][\lambda_2(J_{2(n)})-\lambda_1(J_{1(n)})]}{6c_\nu(K_2-K_1)[\mu_2(J_{2(n)})-\mu_1(J_{1(n)})]}\langle\varepsilon_{33}\rangle\delta_{ij}\right\},$$

$$\langle\varepsilon_{33}^{\nu}\rangle^{(n)}=(-1)^{\nu+1}$$

$$\times\left\{\frac{2[\lambda_{13}^{*(n)}-\lambda_{3-\nu}(J_{(3-\nu)(n)})][\mu_2(J_{2(n)})-\mu_1(J_{1(n)})]-[\lambda_{11}^{*(n)}+\lambda_{12}^{*(n)}-2\lambda_{13}^{*(n)}-2\mu_{3-\nu}(J_{(3-\nu)(n)})][\lambda_2(J_{2(n)})-\lambda_1(J_{1(n)})]}{6c_\nu(K_2-K_1)[\mu_2(J_{2(n)})-\mu_1(J_{1(n)})]}\langle\varepsilon_{rr}\rangle\right.$$

$$\left.+\frac{[\lambda_{33}^{*(n)}-\lambda_{3-\nu}(J_{(3-\nu)(n)})-2\mu_{3-\nu}(J_{(3-\nu)(n)})][\mu_2(J_{2(n)})-\mu_1(J_{1(n)})]-[\lambda_{33}^{*(n)}-\lambda_{13}^{*(n)}-2\mu_{3-\nu}(J_{(3-\nu)(n)})][\lambda_2(J_{2(n)})-\lambda_1(J_{1(n)})]}{3c_\nu(K_2-K_1)[\mu_2(J_{2(n)})-\mu_1(J_{1(n)})]}\langle\varepsilon_{33}\rangle\right\}$$

$$\langle\varepsilon_{i3}^{\nu}\rangle^{(n)}=(-1)^{\nu+1}\frac{\lambda_{44}^{*(n)}-\mu_{3-\nu}(J_{(3-\nu)(n)})}{c_\nu[\mu_2(J_{2(n)})-\mu_1(J_{1(n)})]}\langle\varepsilon_{i3}\rangle\quad (i,j,r=1,2). \tag{1.69}$$

It is assumed that at small strains, the nonlinear stress–strain curves of the components have linear segments with shear moduli $\mu_1(0)$ and $\mu_2(0)$, respectively.

The foregoing can easily be generalized by assuming that the matrix is isotropic and physically nonlinear, and the fibers are linear elastic transversely isotropic and normal to the isotropy plane x_1x_2. The elastic moduli of fibers are denoted by λ_{11}^1, $\lambda_{12}^1, \lambda_{13}^1, \lambda_{33}^1, \lambda_{44}^1$. As in the case of a material with isotropic components, the macrostresses $\langle\varepsilon_{ij}\rangle$ are related to the macrostrains $\langle\sigma_{ij}\rangle$ by (1.53), where the effective elastic moduli λ_{pq}^* are defined by the following formulas [11, 36, 37, 39]:

$$\lambda_{11}^*+\lambda_{12}^*=c_1(\lambda_{11}^1+\lambda_{12}^1)+2c_2(\lambda_2+\mu_2)-\frac{c_1c_2(\lambda_{11}^1+\lambda_{12}^1-2\lambda_2-2\mu_2)^2}{2c_1(\lambda_2+\mu_2)+c_2(\lambda_{11}^1+\lambda_{12}^1)+2m},$$

$$\lambda_{11}^* - \lambda_{12}^* = c_1(\lambda_{11}^1 - \lambda_{12}^1) + 2c_2\mu_2 - \frac{c_1 c_2 (\lambda_{11}^1 - \lambda_{12}^1 - 2\mu_2)^2}{2c_1\mu_2 + c_2(\lambda_{11}^1 - \lambda_{12}^1) + 2n},$$

$$\lambda_{13}^* = c_1\lambda_{13}^* + c_2\lambda_2 - \frac{c_1 c_2 (\lambda_{11}^1 + \lambda_{12}^1 - 2\lambda_2 - 2\mu_2)(\lambda_{13}^1 - \lambda_{2p})}{2c_1(\lambda_2 + \mu_2) + c_2(\lambda_{11}^1 + \lambda_{12}^1) + 2m}, \tag{1.70}$$

$$\lambda_{33}^* = c_1\lambda_{33}^* + c_2(\lambda_2 + 2\mu_2) - \frac{2c_1 c_2 (\lambda_{13}^1 - \lambda_2)^2}{2c_1(\lambda_2 + \mu_2) + c_2(\lambda_{11}^1 + \lambda_{12}^1) + 2m},$$

$$\lambda_{44}^* = c_1\lambda_{44}^1 + c_2\mu_2 - \frac{c_1 c_2 (\lambda_{44}^1 - \mu_2)^2}{c_1\mu_2 + c_2\lambda_{44}^1 + s}$$

$$\Big(2m = c_1(\lambda_{11}^1 - \lambda_{12}^1) + 2c_2\mu_2, \quad 2n = c_1(\lambda_{11}^1 + \lambda_{12}^1) + 2c_2(\lambda_2 + \mu_2), \quad s = c_1\lambda_{44}^1 + 2c_2\mu_2 \tag{1.71}$$

if the matrix is stiffer than the fibers and

$$2m = \left(\frac{c_1}{\lambda_{11}^1 - \lambda_{12}^1} + \frac{c_2}{2\mu_2}\right)^{-1}, \quad 2n = \left[\frac{c_1}{\lambda_{11}^1 + \lambda_{12}^1} + \frac{c_2}{2(\lambda_2 + \mu_2)}\right]^{-1}, \quad s = \left(\frac{c_1}{\lambda_{44}^1} + \frac{c_2}{2\mu_2}\right)^{-1} \tag{1.72}$$

if the fibers are stiffer than the matrix.

If the composite components are physically nonlinear (1.19) ($\nu = 2$), then the elastic moduli (1.70)–(1.72) are functions of the invariant J_2, and, consequently, of the average strains $\langle \varepsilon_{ij}^2 \rangle$. Expressing the average strains in the matrix in terms of the average strains $\langle \varepsilon_{ij} \rangle$ in the composite, we obtain the effective moduli as functions of the average strains in the matrix [11, 36, 37, 39]:

$$\langle \varepsilon_{ij}^2 \rangle = \frac{\lambda_{11}^* - \lambda_{12}^* - \lambda_{11}^1 + \lambda_{12}^1}{c_2(2\mu_2 - \lambda_{11}^1 + \lambda_{12}^1)}\langle \varepsilon_{ij} \rangle - \frac{1}{\Delta_2}\{[(\lambda_{11}^* - \lambda_{11}^1)a_1 - (\lambda_{12}^* - \lambda_{12}^1)a_2 - (\lambda_{13}^* - \lambda_{13}^1)a_3]\langle \varepsilon_{rr} \rangle$$

$$+ [(\lambda_{13}^* - \lambda_{13}^1)(a_1 - a_2) - (\lambda_{33}^* - \lambda_{33}^1)a_3]\langle \varepsilon_{33} \rangle\}\delta_{ij},$$

$$\langle \varepsilon_{33}^2 \rangle = -\frac{1}{\Delta_2}\{[(\lambda_{13}^* - \lambda_{13}^1)a_4 - (\lambda_{11}^* + \lambda_{12}^* - \lambda_{11}^1 - \lambda_{12}^1)a_3]\langle \varepsilon_{rr} \rangle$$

$$+ [(\lambda_{33}^* - \lambda_{33}^1)a_4 - 2(\lambda_{13}^* - \lambda_{13}^1)a_3]\langle \varepsilon_{33} \rangle\}, \tag{1.73}$$

$$\langle \varepsilon_{i3}^2 \rangle = \frac{\lambda_{44}^* - \lambda_{44}^1}{c_2(\mu_2 - \lambda_{44}^1)}\langle \varepsilon_{i3} \rangle \qquad (i, j, r = 1, 2),$$

$$\Delta_2 = c_2(\lambda_{11}^1 - \lambda_{12}^1 - 2\mu_2) \times [(\lambda_{11}^1 + \lambda_{12}^1 - 2\lambda_2 - 2\mu_2)(\lambda_{33}^1 - \lambda_2 - 2\mu_2) - 2(\lambda_{13}^1 - \lambda_2)^2],$$

$$a_1 = (\lambda_{13}^1 - \lambda_2)^2 - (\lambda_{12}^1 - \lambda_2)(\lambda_{33}^1 - \lambda_2 - 2\mu_2),$$

$$a_2 = (\lambda_{13}^1 - \lambda_2)^2 - (\lambda_{11}^1 - \lambda_2 - 2\mu_2)(\lambda_{33}^1 - \lambda_2 - 2\mu_2),$$

$$a_3 = (\lambda_{13}^1 - \lambda_2)(\lambda_{11}^1 - \lambda_{12}^1 - 2\mu_2), \quad a_4 = (\lambda_{11}^1 + \lambda_{12}^1 - 2\lambda_2 - 2\mu_2)(\lambda_{11}^1 - \lambda_{12}^1 - 2\mu_2). \tag{1.74}$$

The effective moduli of a fibrous composite with physically nonlinear components can be found using an iterative algorithm similar to (1.66)–(1.69). The nth approximation of the effective elastic moduli is determined by the formulas

$$\lambda_{11}^{*(n)} + \lambda_{12}^{*(n)} = c_1(\lambda_{11}^1 + \lambda_{12}^1) + 2c_2[\lambda_2(J_{2(n)}) + \mu_2(J_{2(n)})]$$

$$-\frac{c_1 c_2[\lambda_{11}^1 + \lambda_{12}^1 - 2\lambda_2(J_{2(n)}) - 2\mu_2(J_{2(n)})]^2}{2c_1[\lambda_2(J_{2(n)}) + \mu_2(J_{2(n)})] + c_2(\lambda_{11}^1 + \lambda_{12}^1) + 2m^{(n)}},$$

$$\lambda_{11}^{*(n)} - \lambda_{12}^{*(n)} = c_1(\lambda_{11}^1 - \lambda_{12}^1) + 2c_2\mu_2(J_{2(n)}) - \frac{c_1 c_2[\lambda_{11}^1 - \lambda_{12}^1 - 2\mu_2(J_{2(n)})]^2}{2c_1\mu_2(J_{2(n)}) + c_2(\lambda_{11}^1 - \lambda_{12}^1) + \dfrac{2m^{(n)}n^{(n)}}{n^{(n)} + 2m^{(n)}}},$$

$$\lambda_{13}^{*(n)} = c_1\lambda_{13}^1 + c_2\lambda_2(J_{2(n)}) - \frac{c_1 c_2[\lambda_{11}^1 + \lambda_{12}^1 - 2\lambda_2(J_{2(n)}) - 2\mu_2(J_{2(n)})][\lambda_{13}^1 - \lambda_2(J_{2(n)})]}{2c_1[\lambda_2(J_{2(n)}) + \mu_2(J_{2(n)})] + c_2(\lambda_{11}^1 + \lambda_{12}^1) + 2m^{(n)}},$$

$$\lambda_{33}^{*(n)} = c_1\lambda_{33}^1 + c_2[\lambda_2(J_{2(n)}) + 2\mu_2(J_{2(n)})] - \frac{2c_1 c_2[\lambda_{13}^1 - \lambda_2(J_{2(n)})]^2}{2c_1[\lambda_2(J_{2(n)}) + \mu_2(J_{2(n)})] + c_2(\lambda_{11}^1 + \lambda_{12}^1) + 2m^{(n)}},$$

$$\lambda_{44}^{*(n)} = c_1\lambda_{44}^1 + c_2\mu_2(J_{2(n)}) - \frac{c_1 c_2[\lambda_{44}^1 - \mu_2(J_{2(n)})]^2}{c_1\mu_2(J_{2(n)}) + c_2\lambda_{44}^1 + s^{(n)}} \tag{1.75}$$

$$\Big(2m^{(n)} = c_1(\lambda_{11}^1 - \lambda_{12}^1) + 2c_{22}(J_{2(n)})$$

$$2n^{(n)} == c_1(\lambda_{11}^1 + \lambda_{12}^1) + 2c_2[\lambda_2(J_{2(n)}) + \mu_2(J_{2(n)})], \quad s^{(n)} = c_1\lambda_{44}^1 + c_2\mu_2(J_{2(n)}) \tag{1.76}$$

if the matrix is stiffer than the fibers and

$$2m^{(n)} = \left[\frac{c_1}{\lambda_{11}^1 - \lambda_{12}^1} + \frac{c_2}{2\mu_2(J_{2(n)})}\right]^{-1}, \quad 2n^{(n)} = \left\{\frac{c_1}{\lambda_{11}^1 + \lambda_{12}^1} + \frac{c_2}{2[\lambda_2(J_{2(n)}) + \mu_2(J_{2(n)})]}\right\}^{-1},$$

$$s^{(n)} = \left[\frac{c_1}{\lambda_{44}^1} + \frac{c_2}{\mu_2(J_{2(n)})}\right]^{-1} \tag{1.77}$$

if the fibers are stiffer than the matrix.

The average strains $\langle \varepsilon_{ij}^{\nu} \rangle^{(n+1)}$ are determined in terms of the macrostrains $\langle \varepsilon_{ij} \rangle$ by the formulas

$$\langle \varepsilon_{ij}^2 \rangle^{(n)} = \frac{\lambda_{11}^{*(n)} - \lambda_{12}^{*(n)} - \lambda_{11}^1 + \lambda_{12}^1}{c_2[2\mu_2(J_{2(n)}) - \lambda_{11}^1 + \lambda_{12}^1]} \langle \varepsilon_{ij} \rangle$$

$$-\frac{1}{\Delta_2^{(n)}}\{[(\lambda_{11}^{*(n)} - \lambda_{11}^1)a_1^{(n)} - (\lambda_{12}^{*(n)} - \lambda_{12}^1)a_2^{(n)} - (\lambda_{13}^{*(n)} - \lambda_{13}^1)a_3^{(n)}]\langle \varepsilon_{rr} \rangle$$

$$+[(\lambda_{13}^{*(n)} - \lambda_{13}^1)(a_1^{(n)} - a_2^{(n)}) - (\lambda_{33}^{*(n)} - \lambda_{33}^1)a_3^{(n)}]\langle \varepsilon_{33} \rangle\}\delta_{ij},$$

$$\langle \varepsilon_{33}^2 \rangle^{(n)} = -\frac{1}{\Delta_2^{(n)}}\{[(\lambda_{13}^{*(n)} - \lambda_{13}^1)a_4^{(n)} - (\lambda_{11}^{*(n)} + \lambda_{12}^{*(n)} - \lambda_{11}^1 - \lambda_{12}^1)a_3^{(n)}]\langle \varepsilon_{rr} \rangle$$

$$+[(\lambda_{33}^{*(n)}-\lambda_{33}^{1})a_4^{(n)}-2(\lambda_{13}^{*(n)}-\lambda_{13}^{1})a_3^{(n)}]\langle\varepsilon_{rr}\rangle \tag{1.78}$$

$$\langle\varepsilon_{i3}^{2}\rangle^{(n)}=\frac{\lambda_{44}^{*(n)}-\lambda_{44}^{1}}{c_2[\mu_2(J_{2(n)})-\lambda_{44}^{1}]}\langle\varepsilon_{i3}\rangle \quad (i,j,r=1,2)$$

$$\Big(\Delta_2^{(n)}=c_2[\lambda_{11}^{1}-\lambda_{12}^{1}-2\mu_2(J_{2(n)})]$$

$$\times\{[\lambda_{11}^{1}+\lambda_{12}^{1}-2\lambda_2(J_{2(n)})-2\mu_2(J_{2(n)})][\lambda_{33}^{1}-\lambda_{2p}^{(n)}-2\mu_2(J_{2(n)})]-2[\lambda_{13}^{1}-\lambda_2(J_{2(n)})]^2\},$$

$$a_1^{(n)}=[\lambda_{13}^{1}-\lambda_2(J_{2(n)})]^2-[\lambda_{12}^{1}-\lambda_2(J_{2(n)})][\lambda_{33}^{1}-\lambda_{2p}^{(n)}-2\mu_2(J_{2(n)})],$$

$$a_2^{(n)}=[\lambda_{13}^{1}-\lambda_2(J_{2(n)})]^2-[\lambda_{11}^{1}-\lambda_{2p}^{(n)}-2\mu_2(J_{2(n)})][\lambda_{33}^{1}-\lambda_2(J_{2(n)})-2\mu_2(J_{2(n)})],$$

$$a_3^{(n)}=[\lambda_{13}^{1}-\lambda_2(J_{2(n)})][\lambda_{11}^{1}-\lambda_{12}^{1}-2\mu_2(J_{2(n)})],$$

$$a_4^{(n)}=[\lambda_{11}^{1}+\lambda_{12}^{1}-2\lambda_2(J_{2(n)})-2\mu_2(J_{2(n)})][\lambda_{11}^{1}-\lambda_{12}^{1}-2\mu_2(J_{2(n)})]\Big). \tag{1.79}$$

It is assumed that at small strains, the nonlinear stress–strain curves of the matrix have a linear segment with shear modulus $\mu_2(0)$.

Let us study, as an example, the nonlinear deformation of a laminated composite with linear elastic reinforcement and nonlinear elastic matrix, with the bulk strains being linear and the shear strains described by linear-hardening diagram (1.45)–(1.47).

The components of the composite are high-modulus carbon fibers with the following characteristics [22]:

$$E_1^1=8\text{ GPa},\quad E_3^1=226\text{ GPa},\quad \nu_{12}^1=0.3,\quad \nu_{13}^1=0.2,\quad G_{13}^1=60\text{ GPa}, \tag{1.80}$$

volume fraction $c_1=0, 0.25, 0.5, 0.75, 1.0$ and epoxy matrix with constants (1.51), linear-hardening diagram (1.46), (1.47), and yield stress (1.52), where E_1^1 and E_3^1, ν_{12}^1 and ν_{13}^1, G_{12}^1 and G_{13}^1 are the transverse and longitudinal Young's moduli, Poisson's ratios, and shear moduli of fibers related to $\lambda_{11}^1, \lambda_{12}^1, \lambda_{13}^1, \lambda_{33}^1, \lambda_{44}^1$ by the formulas

$$\lambda_{11}^1+\lambda_{12}^1=E_1^1E_3^1\left[E_3^1\left(2-\frac{E_1^1}{2G_{12}^1}\right)-2E_1^1(\nu_{13}^1)^2\right]^{-1},\quad \lambda_{11}^1-\lambda_{12}^1=G_{12}^1,$$

$$\lambda_{13}^1=\nu_{13}^1(\lambda_{11}^1+\lambda_{12}^1),\quad \lambda_{33}^1=(\lambda_{11}^1+\lambda_{12}^1)\frac{E_3^1}{E_1^1}\left(2-\frac{E_1^1}{2G_{12}^1}\right),\quad \lambda_{44}^1=G_{13}^1. \tag{1.81}$$

If

$$\langle\varepsilon_{11}\rangle\neq 0,\quad \langle\sigma_{22}\rangle=\langle\sigma_{33}\rangle=0, \tag{1.82}$$

then, according to (1.53), the macrostress $\langle\sigma_{11}\rangle$ is related to the macrostrain $\langle\varepsilon_{11}\rangle$ by

$$\langle\sigma_{11}\rangle=\frac{\lambda_{11}^*-\lambda_{12}^*}{\lambda_{11}^*\lambda_{33}^*-(\lambda_{13}^*)^2}[(\lambda_{11}^*+\lambda_{12}^*)\lambda_{33}^*-2(\lambda_{13}^*)^2]\langle\varepsilon_{11}\rangle. \tag{1.83}$$

Figure 1.3 shows the macrostress $\langle\sigma_{11}\rangle/\mu_2$ as a function of the macrostrain $\langle\varepsilon_{11}\rangle$ in a fibrous composite for different values of c_1. It can be seen that the physical nonlinearity of the matrix has a significant effect on the stress–strain curve

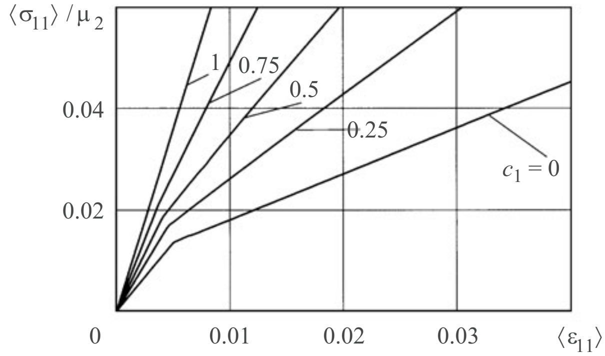

Fig. 1.3

($\langle \sigma_{11} \rangle / \mu_2$ versus $\langle \varepsilon_{11} \rangle$) for all values of $c_1 < 1$. The curve of the material with linear-hardening matrix consists of two linear segments.

2. Short-Term Microdamage of Materials during Nonlinear Deformation.

2.1. Homogeneous Material. Let us consider the physically nonlinear deformation of an isotropic material described by the dependence of the bulk (K) and shear (μ) moduli on strains and accompanied by microdamage. The microdamage of the material is modeled by randomly arranged quasispherical micropores occurring in those microvolumes where the stresses exceed the ultimate microstrength.

The macrostresses and macrostrains are related by (1.34), where the effective moduli K^* and μ^* are functions of porosity p and macrostrains $\langle \varepsilon_{ij} \rangle$.

The effective moduli of a porous physically nonlinear material can be determined using an iterative algorithm [38]. The nth approximation of the effective moduli $K^{*(n)}$ and $\mu^{*(n)}$ is defined by

$$K^{*(n)} = \frac{4K\mu(J^1_{(n)})(1-p)^2}{3Kp + 4\mu(J^1_{(n)})(1-p)}, \qquad \mu^{*(n)} = \frac{[9K + 8\mu(J^1_{(n)})]\mu(J^1_{(n)})(1-p)^2}{3K(3-p) + 4\mu(J^1_{(n)})(2+p)}, \tag{2.1}$$

where $J^1_{(n)} = [(\langle \varepsilon^1_{ij} \rangle')^{(n)} (\langle \varepsilon^1_{ij} \rangle')^{(n)}]^{1/2}$ is the nth approximation of the second invariant of the deviatoric average-strain tensor $\langle \varepsilon^1_{ij} \rangle^{(n)}$ in the undamaged portion of the material. At the $(n+1)$th iteration, these strains $\langle \varepsilon_{ij} \rangle$ are determined in terms of the macrostrains $\langle \varepsilon \rangle$ by the formulas

$$\langle \varepsilon^1_{ij} \rangle^{(n+1)} = \frac{1}{(1-p)} \left[\frac{K^{*(n)}}{K} V_{ij\alpha\beta} + \frac{\mu^{*(n)}}{\mu(J^1_{(n)})} D_{ij\alpha\beta} \right] \langle \varepsilon_{\alpha\beta} \rangle, \tag{2.2}$$

where $V_{ij\alpha\beta}$ and $D_{ij\alpha\beta}$ are the volumetric and deviatoric components of the unit tensor $I_{ij\alpha\beta}$,

$$I_{ij\alpha\beta} = V_{ij\alpha\beta} + D_{ij\alpha\beta}, \quad V_{ij\alpha\beta} = 1/3\delta_{ij}\delta_{\alpha\beta}, \quad D_{ij\alpha\beta} = 1/2(\delta_{\alpha j}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha} - 2/3\delta_{ij}\delta_{\alpha\beta}). \tag{2.3}$$

Given macrostrains $\langle \varepsilon_{ij} \rangle$, the effective moduli are determined as the limits of the iterative process

$$K^* = \lim_{n\to\infty} K^{*(n)}, \quad \mu^* = \lim_{n\to\infty} \mu^{*(n)}. \tag{2.4}$$

We will use the Huber–Mises criterion [15] as a condition for the formation of a microdamage in a microvolume of the undamaged portion of the material:

$$I^1_\sigma = k, \tag{2.5}$$

where $I_\sigma^1 = (\langle\sigma_{ij}^1\rangle'\langle\sigma_{ij}^1\rangle')^{1/2}$ is the second invariant of the deviatoric average-stress tensor $\langle\sigma_{ij}^1\rangle'$ in the undamaged portion of the material; k is the ultimate microstrength, which is a random function of coordinates. Since the average stresses $\langle\sigma_{ij}^1\rangle'$ in the undamaged portion are related to the macrostresses $\langle\sigma_{ij}\rangle$ as follows [33]:

$$\langle\sigma_{ij}^1\rangle = \frac{1}{1-p}\langle\sigma_{ij}\rangle, \tag{2.6}$$

a failure criterion in terms of macrostresses follows from (2.5):

$$\frac{I_\sigma}{1-p} = k, \tag{2.7}$$

where $I_\sigma = (\langle\sigma_{ij}\rangle'\langle\sigma_{ij}\rangle')^{1/2}$ is the second invariant of the deviatoric macrostress tensor. If the macrostrains are given, then, according to (1.34), (2.7), we obtain a failure criterion in terms of macrostrains:

$$\frac{2\mu^*(p,\langle\varepsilon_{ij}\rangle}{1-p} I_\varepsilon = k, \tag{2.8}$$

where $I_\varepsilon = (\langle\varepsilon_{ij}\rangle'\langle\varepsilon_{ij}\rangle')^{1/2}$ is the second invariant of the deviatoric macrostrain tensor.

The one-point distribution function $F(k)$, where k is the ultimate microstrength for the undamaged portion of the material, can be approximated by a power function on some interval

$$F(k) = \begin{cases} 0, & k < k_0, \\ \left(\dfrac{k-k_0}{k_1-k_0}\right)^n, & k_0 \le k \le k_1, \\ 1, & k > k_1, \end{cases} \tag{2.9}$$

or by the Weibull function

$$F(k) = \begin{cases} 0, & k < k_0, \\ 1-\exp[-m(k-k_0)^n], & k \ge k_0, \end{cases} \tag{2.10}$$

where k_0 is the minimum value of the ultimate microstrength; k_1, m, n are deterministic constants describing the behavior of the distribution function and determined by fitting experimental microstrength scatter or stress–strain curves.

Assume that the random field of ultimate microstrength k is statistically homogeneous in real materials, and its correlation scale and the size of single microdamages and the distances between them are negligible compared with the macrovolume. Then the random field k and the distribution of macrostresses in the material under uniform loading are ergodic, and the distribution function $F(k)$ defines the fraction of the undamaged portion of the material in which the ultimate microstrength is less than k. Therefore, if the stresses $\langle\sigma_{ij}^1\rangle$ are nonzero, the function $F(I_\sigma^1)$ defines, according to (2.5), (2.9), and (2.10), the content of damaged microvolumes of the skeleton. Since the damaged microvolumes are modeled by pores, we can write the porosity balance equation [55]:

$$p = p_0 + (1-p_0)F(I_\sigma^1), \tag{2.11}$$

where p_0 is the initial porosity.

If the homogeneous macrostresses $\langle\sigma_{ij}\rangle$ are given, then, according to (2.7), the porosity balance equation (2.11) becomes

$$p = p_0 + (1-p_0)F\left(\frac{I_\sigma}{1-p}\right). \tag{2.12}$$

If the macrostrains $\langle \varepsilon_{ij} \rangle$ are given, then, according to (2.8), we have

$$p = p_0 + (1 - p_0) F\left(\frac{2\mu^*(p, \langle \varepsilon_{ij} \rangle)}{1 - p} I_\varepsilon \right). \tag{2.13}$$

Thus, the microdamage of a material with given macrostresses does not depend, according to (2.12), on its elastic properties, including physical nonlinearity. If the macrostrains are given, the microdamage is described by the more complicated equation (2.13) which contains the effective shear modulus μ^* depending on porosity and macrostrains $\langle \varepsilon_{ij} \rangle$.

Equations (1.34) and (2.13) form a closed system describing the coupled processes of statistically homogeneous physically nonlinear deformation and damage. Physical nonlinearity affects the way pores form during deformation, and the porosity of the material has an effect on its stress–strain curve. This is why the nonlinearity of the stress–strain curve is determined by the physical nonlinearity of the material and the increase in the porosity during physically nonlinear deformation.

To describe the coupled processes of physically nonlinear deformation and damage, it is necessary to find the macrostrain-dependent effective elastic moduli of the porous material with the iterative algorithm (2.1)–(2.3) and to determine the porosity from Eq. (2.13) also with an iterative method. At the nth step of the iterative process (2.1)–(2.3), Eq. (2.13) is represented as

$$f^{(n)}(p) \equiv p - p_0 - (1 - p_0) F\left(\frac{2\mu^{*(n)}(p, \langle \varepsilon_{ij} \rangle)}{1 - p} I_\varepsilon \right) = 0. \tag{2.14}$$

Then the root p of Eq. (2.14) at the mth step of some iterative process can be expressed as

$$p^{(m,n)} = A f^{(n)}(p^{(m-1)}), \tag{2.15}$$

where A is an operator on the function $f^{(n)}(p)$.

The root is found as follows:

$$p = \lim_{\substack{m \to \infty \\ n \to \infty}} p^{(m,n)}. \tag{2.16}$$

Formulas (1.34), (2.4), (2.16) provide the solution to the problem posed, i.e., macrodeformation ($\langle \sigma_{ij} \rangle$ versus $\langle \varepsilon_{ij} \rangle$) and microdamage ($p$ versus $\langle \varepsilon_{ij} \rangle$) curves for a physically nonlinear material.

Let us analyze, as an example, the coupled processes of nonlinear deformation and microdamage of a material with bulk strains being linear and shear strains described by a linear-hardening diagram:

$$\sigma_{rr} = K\varepsilon_{rr}, \quad \sigma'_{ij} = 2\mu(J)\varepsilon'_{ij}, \tag{2.17}$$

where the bulk modulus K does not depend on the strains, and the shear modulus $\mu(J)$ is described by

$$\mu(J) = \begin{cases} \mu_0, & T \le T_0, \\ \mu' + \left(1 - \dfrac{\mu'}{\mu_0}\right)\dfrac{T_0}{2J}, & T \ge T_0, \end{cases} \tag{2.18}$$

$$(J = (\varepsilon'_{ij}\varepsilon'_{ij})^{1/2}, \quad T = (\sigma'_{ij}\sigma'_{ij})^{1/2}, \quad T_0 = \sigma_{20}\sqrt{2/3}, \tag{2.19}$$

where ε'_{ij} and σ'_{ij} are the strain and stress deviators; σ_{20} is the tensile proportional limit assumed to be independent of the coordinates; μ_0 and μ' are material constants.

To describe the coupled processes of physically nonlinear deformation and short-term damage of a homogeneous material with given macrostrains, we will use an algorithm based on the secant method [4]. This theory was used to study the coupled processes of nonlinear deformation and microdamage of a homogeneous material described by the linear-hardening diagram (2.17), (2.18) with the following constants [12]:

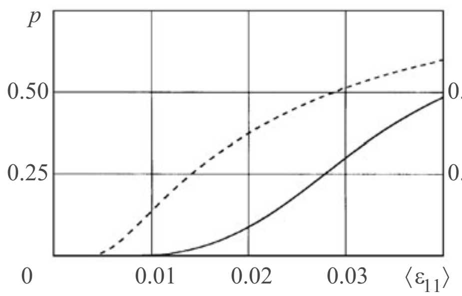

Fig. 2.1

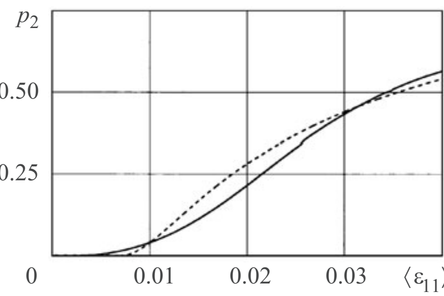

Fig. 2.2

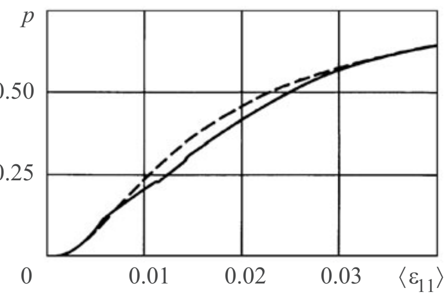

Fig. 2.3

$$K = 3.33 \text{ GPa}, \quad \mu_0 = 1.11 \text{ GPa}, \quad \mu' = 0.331 \text{ GPa} \tag{2.20}$$

and the following proportional limits and minimum tensile microstrength ($\sigma_p = k_0\sqrt{3/2}$):

$$\sigma_0 = 0.003 \text{ GPa}, \quad \sigma_p = 0.011 \text{ GPa}, \tag{2.21}$$

$$\sigma_0 = \sigma_p = 0.007 \text{ GPa}, \tag{2.22}$$

$$\sigma_0 = 0.015 \text{ GPa}, \quad \sigma_p = 0.003. \tag{2.23}$$

Figures 2.1–2.3 show, by solid lines, the porosity p as a function of the macrostrain $\langle\varepsilon_{11}\rangle$ for the linear-hardening material with proportional limits and minimum tensile microstrength (2.21)–(2.23), respectively. For comparison, the figures show (by dashed lines) p versus $\langle\varepsilon_{11}\rangle$ for the linear elastic material. As is seen, the physical nonlinearity of the material has a significant effect on its microdamage, especially for $\sigma_p > \sigma_0$, i.e., when the proportional limit is less than the minimum tensile microstrength. When $\sigma_p > \sigma_0$, microdamage in the linear-hardening material begins at higher macrostrains than in the linear elastic material. When $\sigma_p < \sigma_0$, microdamage in the linear elastic material begins at the same macrostrain as in the physically nonlinear material, but develops more intensively at the initial stage (i.e., its porosity is higher than in the physically nonlinear material at the same macrostrain).

Figures 2.4–2.6 show, by solid lines, the macrostress $\langle\sigma_{11}\rangle$ as a function of the macrostrain $\langle\varepsilon_{11}\rangle$ for the linear-hardening material with microdamage for (2.21)–(2.23), respectively. For comparison, the figures show $\langle\sigma_{11}\rangle$ versus $\langle\varepsilon_{11}\rangle$ for the linear elastic material with microdamage (dashed lines) and the linear-hardening material without microdamage (dotted lines). As is seen, physical nonlinearity has a significant effect on the stress–strain behavior of the material, especially for $\sigma_p > \sigma_0$. The stress–strain curve of the linear elastic material with microdamage consists of linear and nonlinear segments. The curve of the linear-hardening material without microdamage consists of two linear segments. The stress–strain curve of the linear-hardening material with microdamage consists of two linear and one nonlinear segments. Comparing the linear elastic and linear-hardening materials subject to microdamage shows that at the initial stage of deformation ($\langle\varepsilon_{11}\rangle < 0.02$), the macrostress in the former is higher than in the latter, given the same macrostrain.

2.2. Particulate Material. Let us consider the physically nonlinear deformation of a particulate composite described by the dependence of the bulk (K_ν) and shear μ_ν ($\nu = 1, 2$) moduli on strains and accompanied by microdamage. The microdamage of the composite components is modeled by randomly arranged quasispherical micropores occurring in those microvolumes where the stresses exceed the ultimate microstrength. The porosity of the inclusions and matrix is denoted by p_1 and p_2, respectively. Denote the bulk and shear moduli of the porous inclusions and matrix by K_1, μ_1 and K_2, μ_2, respectively, and their volume fractions by c_1 and c_2, respectively. The macrostresses $\langle\sigma_{ij}\rangle$ and macrostrains $\langle\varepsilon_{ij}\rangle$ are related by (1.34), where the effective moduli K^* and μ^* are functions of porosities p_1, p_2 and macrostrains $\langle\varepsilon_{ij}\rangle$.

The effective moduli of a physically nonlinear particulate composite with porous components can be determined using the following iterative algorithm. The nth approximation of the effective moduli $K^{*(n)}$ and $\mu^{*(n)}$ is determined, according to (1.40)–(1.43) in terms of the nth approximation of the respective moduli of the inclusions ($K_{1p}^{(n)}, \mu_{1p}^{(n)}$) and matrix ($K_{2p}^{(n)}, \mu_{2p}^{(n)}$) [11, 36, 37, 39] as

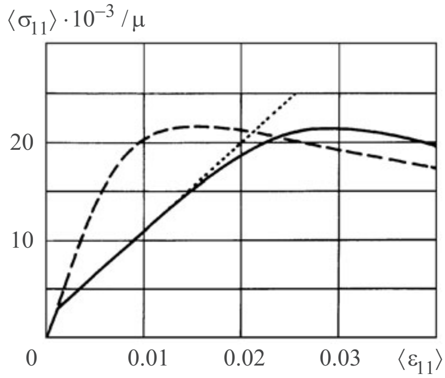

Fig. 2.4

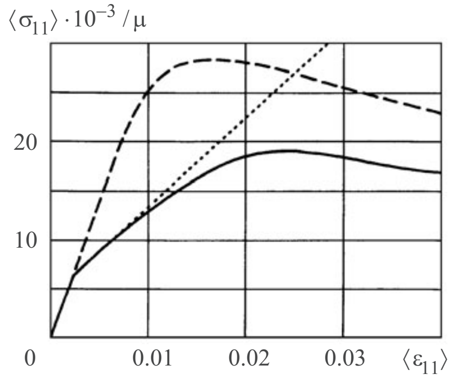

Fig. 2.5

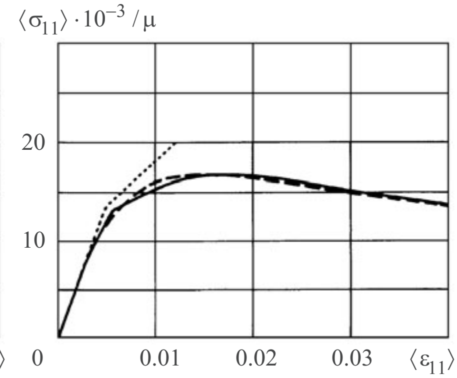

Fig. 2.6

$$K^{*(n)} = c_1 K_{1p} + c_2 K_{2p} - c_1 c_2 \frac{(K_{1p} - K_{2p})^2}{c_1 K_{2p} + c_2 K_{1p} + n_c^{(n)}},$$

$$\mu^{*(n)} = c_1 \mu_{1p}^{(n)} + c_2 \mu_{2p}^{(n)} - c_1 c_2 \frac{(\mu_{1p}^{(n)} - \mu_{2p}^{(n)})^2}{c_1 \mu_{2p}^{(n)} + c_2 \mu_{1p}^{(n)} + m_c^{(n)}}, \tag{2.24}$$

$$\left(n_c^{(n)} = \frac{4}{3} \mu_c^{(n)},\ m_c^{(n)} = \frac{\mu_c^{(n)} (9K_c + 8\mu_c^{(n)})}{6(K_c + 2\mu_c^{(n)})}, \right. \tag{2.25}$$

and

$$K_c = c_1 K_{1p} + c_2 K_{2p}, \quad \mu_c^{(n)} = c_1 \mu_{1p}^{(n)} + c_2 \mu_{2p}^{(n)} \tag{2.26}$$

if the porous matrix is stiffer than the porous inclusions and

$$K_c = \frac{K_{1p} K_{2p}}{c_1 K_{2p} + c_2 K_{1p}}, \quad \mu_c^{(n)} = \frac{\mu_{1p}^{(n)} \mu_{2p}^{(n)}}{c_1 \mu_{2p}^{(n)} + c_2 \mu_{1p}^{(n)}} \tag{2.27}$$

otherwise. The *n*th approximation of the effective moduli of porous inclusions, $K_{1p}^{(n)}, \mu_{1p}^{(n)}$, and porous matrix, $K_{2p}^{(n)}, \mu_{2p}^{(n)}$, is defined by the following formulas [35, 38, 39]:

$$K_{\nu p}^{(n)} = \frac{4K_\nu \mu_\nu (J_{\nu(n)}^1)(1 - p_\nu)^2}{3K_\nu p_\nu + 4\mu_\nu (J_{\nu(n)}^1)(1 - p_\nu)},$$

$$\mu_{\nu p}^{(n)} = \frac{[9K_\nu + 8\mu_\nu (J_{\nu(n)}^1)]\mu_\nu (J_{\nu(n)}^1)(1 - p_\nu)^2}{3K_\nu (3 - p_\nu) + 4\mu_\nu (J_{\nu(n)}^1)(2 + p_\nu)} \quad (\nu = 1, 2), \tag{2.28}$$

where $J_{\nu(n)}^1$ is the *n*th approximation of the second invariant of the deviatoric average-strain tensor $\langle \varepsilon_{ij}^{1\nu} \rangle^{(n)}$ in the undamaged portion of the inclusions or matrix. The $(n+1)$th approximation is related to the *n*th approximation $\langle \varepsilon_{ij}^{\nu} \rangle^{(n)}$ of the average strains in the components by

$$\langle \varepsilon_{ij}^{1\nu} \rangle^{(n+1)} = \frac{1}{(1-p_\nu)} \left\{ \frac{\mu_{\nu p}^{(n)}}{\mu_\nu (J_{\nu(n)}^1)} \langle \varepsilon_{ij}^\nu \rangle^{(n)} + \frac{1}{3} \left[\frac{K_{\nu p}^{(n)}}{K_\nu} - \frac{\mu_{\nu p}^{(n)}}{\mu_\nu (J_{\nu(n)}^1)} \right] \langle \varepsilon_{rr}^\nu \rangle^{(n)} \delta_{ij} \right\} \quad (\nu = 1, 2). \tag{2.29}$$

The average strains $\langle \varepsilon_{ij}^\nu \rangle^{(n)}$ are determined in terms of the macrostrains $\langle \varepsilon_{ij} \rangle$ by the following formulas [11, 36, 37, 39]:

$$\langle \varepsilon_{ij}^\nu \rangle^{(n)} = (-1)^{\nu+1} \left[\frac{\mu^{*(n)} - \mu_{(3-\nu)p}^{(n)}}{c_\nu (\mu_{1p}^{(n)} - \mu_{2p}^{(n)})} \langle \varepsilon_{ij} \rangle \right.$$

$$\left. + \frac{2\mu^{*(n)} (\mu_{1p}^{(n)} - \mu_{2p}^{(n)})(K^{*(n)} - K_{(3-\nu)p}^{(n)}) - 3K^{*(n)} (K_{1p}^{(n)} - K_{2p}^{(n)})(\mu^{*(n)} - \mu_{(3-\nu)p}^{(n)})}{6c_\nu \mu^{*(n)} (\mu_{1p}^{(n)} - \mu_{2p}^{(n)})(K_{1p}^{(n)} - K_{2p}^{(n)})} \langle \varepsilon_{rr} \rangle \delta_{ij} \right]. \tag{2.30}$$

Given macrostrains $\langle \varepsilon_{ij} \rangle$, the effective moduli are determined as the limits of the iterative process

$$K^* = \lim_{n \to \infty} K^{*(n)}, \quad \mu^* = \lim_{n \to \infty} \mu^{*(n)}. \tag{2.31}$$

We will use the Huber–Mises criterion [15] as a condition for the formation of a microdamage in a microvolume of the undamaged portion of the components:

$$I_\sigma^{1\nu} = k_\nu \quad (\nu = 1, 2), \tag{2.32}$$

where $I_\sigma^{1\nu} = (\langle \sigma_{ij}^{1\nu} \rangle' \langle \sigma_{ij}^{1\nu} \rangle')^{1/2}$ is the second invariant of the deviatoric average-stress tensor $\langle \sigma_{ij}^{1\nu} \rangle'$ in the undamaged portion of the kth component; k_ν is the ultimate microstrength, which is a random function of coordinates. Since the invariant of the average-stress deviator $I_\sigma^{1\nu}$ is related to the invariant of the average-strain deviator $I_\varepsilon^k = (\langle \varepsilon_{ij}^\nu \rangle' \langle \varepsilon_{ij}^\nu \rangle')^{1/2}$ as

$$I_\sigma^{1\nu} = \frac{2\mu_{\nu p}}{1 - p_\nu} I_\varepsilon^\nu, \tag{2.33}$$

and I_ε^ν is related to the invariant $I_\varepsilon = (\langle \varepsilon_{ij} \rangle' \langle \varepsilon_{ij} \rangle')^{1/2}$ for the whole composite by

$$I_\varepsilon^\nu = (-1)^{\nu+1} \frac{\mu^* - \mu_{(3-\nu)p}}{c_\nu (\mu_{1p} - \mu_{2p})} I_\varepsilon, \tag{2.34}$$

the failure criterion (2.32) can be expressed in terms of macrostrains as

$$(-1)^{\nu+1} \frac{2\mu_{\nu p} (\mu^* - \mu_{(3-\nu)p})}{c_\nu (1 - p_\nu)(\mu_{1p} - \mu_{2p})} I_\varepsilon = k_\nu \quad (\nu = 1, 2). \tag{2.35}$$

The one-point distribution function $F_\nu(k_\nu)$ for some microvolume in the undamaged portion of a component can be approximated by a power function on some interval

$$F_\nu(k_\nu) = \begin{cases} 0, & k_\nu < k_{\nu 0}, \\ \left(\dfrac{k_\nu - k_{\nu 0}}{k_{1\nu} - k_{\nu 0}} \right)^{n_\nu}, & k_{\nu 0} \le k_\nu \le k_{\nu 1}, \\ 1, & k_\nu > k_{\nu 1}, \end{cases} \tag{2.36}$$

or by the Weibull function

$$F_\nu(k_\nu) = \begin{cases} 0, & k_\nu < k_{\nu 0}, \\ 1 - \exp[-m_\nu (k_\nu - k_{\nu 0})^{n_\nu}], & k_\nu \ge k_{\nu 0}, \end{cases} \tag{2.37}$$

where $k_{\nu 0}$ is the minimum value of ultimate microstrength in a component; $k_{\nu 1}, m_\nu, n_\nu$ are deterministic constants describing the behavior of the distribution function and determined by fitting experimental microstrength scatter or stress–strain curves.

Assume that the random field of ultimate microstrength k_ν is statistically homogeneous in real materials, and its correlation scale and the size of single microdamages and the distances between them are negligible compared with the macrovolume. Then the random field k_ν and the distribution of macrostresses in the component under uniform loading are ergodic, and the distribution function $F_\nu(k_\nu)$ defines the fraction of the undamaged portion of the component in which the ultimate microstrength is less than k_ν. Therefore, if the stresses $\langle \sigma_{ij}^{1\nu} \rangle$ are nonzero, the function $F_\nu(I_\sigma^{1\nu})$ defines, according to (2.32), (2.36), and (2.37), the content of instantaneously damaged microvolumes of the skeleton of the component. Since the damaged microvolumes are modeled by pores, we can write a porosity balance equation [55, 58, 62]:

$$p_\nu = p_{\nu 0} + (1 - p_{\nu 0}) F_\nu (I_\sigma^{1\nu}), \tag{2.38}$$

where $p_{\nu 0}$ is initial porosity.

With (2.34) and (2.35), the porosity balance equation (2.36) takes the following form in the macrostrain space:

$$p_\nu = p_{\nu 0} + (1 - p_{\nu 0}) F_\nu \left[(-1)^{\nu+1} \frac{2\mu_{\nu p} (\mu^* - \mu_{(3-\nu)p})}{c_\nu (1 - p_\nu)(\mu_{1p} - \mu_{2p})} I_\varepsilon \right], \tag{2.39}$$

where μ^* and μ_{1p}, μ_{2p} are defined by (2.24)–(2.28).

Equations (1.34), (2.24)–(2.30), (2.36) (or (2.37)), (2.39) form a closed-form system describing the coupled processes of statistically homogeneous physically nonlinear deformation and damage of a particulate composite. The physical nonlinearity of its components affects the way pores form during deformation, and the porosity of the components has an effect on its stress–strain curve. This is why the nonlinearity of the stress–strain curve of the particulate composite is determined by the physical nonlinearity of its components and the increase in the porosity during physically nonlinear deformation.

To describe the coupled processes of physically nonlinear deformation and damage of a particulate composite with given macrostrains, it is necessary to find the macrostrain-dependent effective elastic moduli of the composite with porous components with the iterative algorithm (2.24)–(2.30) and to determine the porosity from Eq. (2.39) also with an iterative method. At the nth step of the iterative process (2.24)–(2.30), Eq. (2.39) is represented as

$$f_\nu^{(n)} \equiv p_\nu - p_{\nu 0} - (1 - p_{\nu 0}) F_\nu \left[(-1)^{\nu+1} \frac{2\mu_{\nu p}^{(n)} (\mu^{*(n)} - \mu_{(3-\nu)p}^{(n)})}{c_\nu (1 - p_\nu)(\mu_{1p}^{(n)} - \mu_{2p}^{(n)})} I_\varepsilon \right]. \tag{2.40}$$

Then the root p_ν of Eq. (2.40) at the mth step of some iterative process can be expressed as

$$p_\nu^{(m,n)} = A_\nu f_\nu^{(n)} (p_\nu^{(m-1)}), \tag{2.41}$$

where A_ν is an operator on the function $f_\nu^{(n)}(p_\nu)$.

The root is found as follows:

$$p_\nu = \lim_{\substack{m \to \infty \\ n \to \infty}} p_\nu^{(m,n)}. \tag{2.42}$$

Formulas (1.34), (2.24)–(2.31), (2.36) (or (2.37)), (2.40), (2.42) provide the solution to the problem posed, i.e., macrodeformation ($\langle \sigma_{ij} \rangle$ versus $\langle \varepsilon_{ij} \rangle$) and microdamage ($p_\nu$ versus $\langle \varepsilon_{ij} \rangle$) curves for a particulate composite with physically nonlinear components.

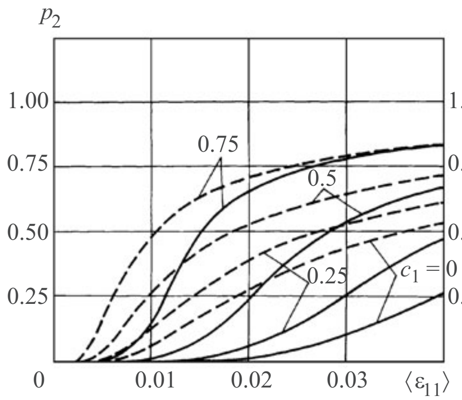

Fig. 2.7

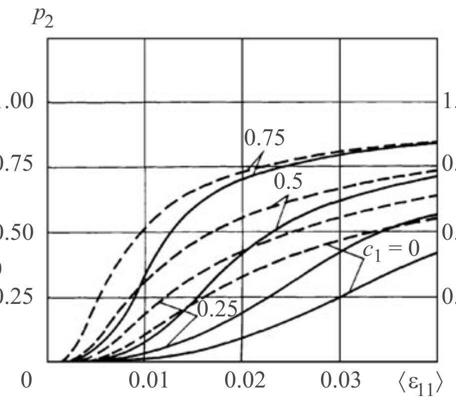

Fig. 2.8

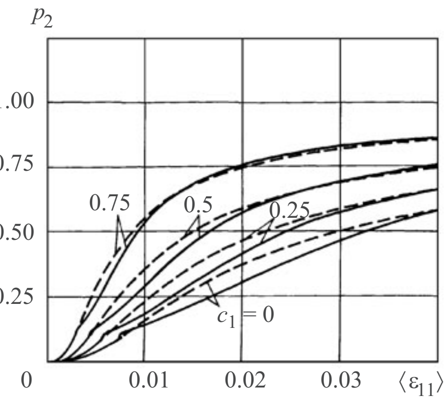

Fig. 2.9

Let us study, as an example, the coupled processes of nonlinear deformation and microdamage of a particulate composite with linear elastic inclusions and nonlinear elastic matrix with bulk strains being linear and shear strains described by a linear-hardening diagram (1.45)–(1.47).

To describe the coupled processes of physically nonlinear deformation and short-term damage of a particulate composite with given macrostrains, we will use an algorithm based on the secant method [4]. The above theory was used to study the coupled processes of nonlinear deformation and microdamage of a particulate composite with microdamaged matrix for Weibull distribution and macroparameters (1.48). According to (1.34), the macrostress $\langle \sigma_{11} \rangle$ is related to the macrostrain $\langle \varepsilon_{11} \rangle$ by (1.49). In the porosity balance equation (2.39), we use

$$I_\varepsilon = \sqrt{\frac{2}{3}} \frac{3K^* \langle \varepsilon_{11} \rangle}{2(K^* + 1/3\mu^*)} \tag{2.43}$$

which is equivalent to (1.48).

The inclusions are linear elastic particles with characteristics (1.50). The matrix is described by linear-hardening diagram (1.45)–(1.47) with constants (1.51) and the following yield stresses and minimum tensile microstrength $\sigma_{2p} = k_{20}\sqrt{3/2}$:

$$\sigma_{20} = 0.003 \text{ GPa}, \qquad \sigma_{2p} = 0.011 \text{ GPa}, \tag{2.44}$$

$$\sigma_{20} = \sigma_{2p} = 0.007 \text{ GPa}, \tag{2.45}$$

$$\sigma_{20} = 0.015 \text{ GPa}, \qquad \sigma_{2p} = 0.003 \text{ GPa}. \tag{2.46}$$

Figures 2.7–2.9 show, by solid lines, the porosity p_2 as a function of the macrostrain $\langle \varepsilon_{11} \rangle$ for a particulate composite with linear-hardening matrix with proportional limits and minimum tensile microstrength (2.44)–(2.46), respectively. For comparison, the figures show (dashed lines) p_2 versus $\langle \varepsilon_{11} \rangle$ for a particulate composite with linear elastic matrix. As is seen, the physical nonlinearity of the matrix has a significant effect on its microdamage, especially for $\sigma_{2p} > \sigma_{20}$, i.e., when the proportional limit is less than the minimum tensile microstrength. The figures demonstrate that when $\sigma_{2p} > \sigma_{20}$, microdamage in the particulate composite with linear-hardening matrix begins at higher macrostrain than in the composite with linear elastic matrix for all values of c_1. When $\sigma_{2p} < \sigma_{20}$, microdamage in the composite with linear elastic matrix begins at the same macrostrain as in the composite with physically nonlinear matrix, but develops more intensively at the initial stage, especially for $c_1 < 0.5$ (i.e., its porosity is higher than in the physically nonlinear material at the same macrostrain).

Figures 2.10–2.12 show, by solid lines, the macrostress $\langle \sigma_{11} \rangle$ as a function of the macrostrain $\langle \varepsilon_{11} \rangle$ for a particulate composite with linear-hardening matrix with microdamage for (2.44)–(2.46), respectively. For comparison, the same figures show $\langle \sigma_{11} \rangle$ versus $\langle \varepsilon_{11} \rangle$ for the linear matrix with microdamage (dashed lines) and the linear-hardening matrix without microdamage (dotted lines). As is seen, the physical nonlinearity of the matrix has a significant effect on the stress–strain curves

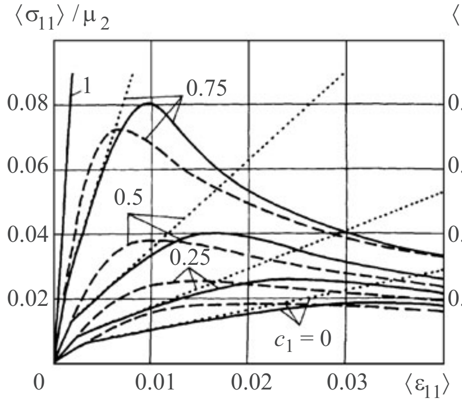

Fig. 2.10

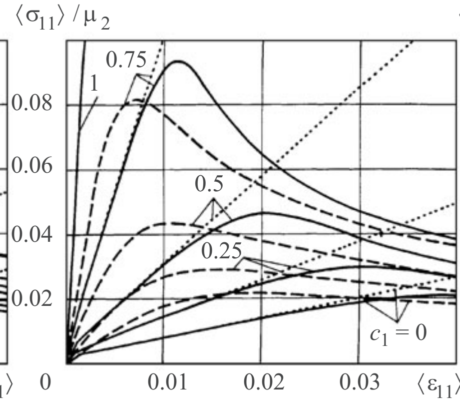

Fig. 2.11

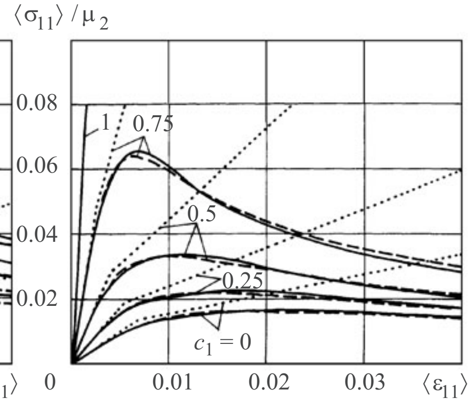

Fig. 2.12

for all values of $c_1 < 1$, especially for $\sigma_{2p} > \sigma_{20}$. The stress–strain curve for the composite with linear elastic matrix with microdamage consists of linear and nonlinear segments. The stress–strain curve of the material with linear-hardening matrix without microdamage consists of two linear segments. The stress–strain curve of the material with linear-hardening matrix with microdamage consists of two linear and one nonlinear segments. Comparing the materials with linear elastic and linear-hardening matrix subject to microdamage shows that the macrostress in the latter is higher than in the former, especially for $c_1 \ge 0.5$.

2.3. Laminated Material. Let us consider the physically nonlinear deformation of a laminated composite with isotropic components described by the dependence of the bulk (K_ν) and shear (μ_ν, $\nu = 1,2$) moduli on strains and accompanied by microdamage. The microdamage of the composite components is modeled by randomly arranged quasispherical micropores occurring in those microvolumes where the stresses exceed the ultimate microstrength. Denote the bulk and shear moduli of the skeleton of the νth component by K_ν, μ_ν, its porosity by p_ν, and the volume fraction of the porous νth component by c_ν. The macrostresses $\langle \sigma_{ij} \rangle$ and macrostrains $\langle \varepsilon_{ij} \rangle$ are related by (1.53), where the effective moduli $\lambda^*_{11}, \lambda^*_{12}, \lambda^*_{13}, \lambda^*_{33}, \lambda^*_{44}$ are functions of the porosities p_ν $(\nu = 1,2)$ and macrostrains $\langle \varepsilon_{ij} \rangle$. The effective moduli of a physically nonlinear laminated composite with porous components can be determined using the following iterative algorithm. The *n*th approximation of the effective moduli $\lambda^{*(n)}_{11}, \lambda^{*(n)}_{12}, \lambda^{*(n)}_{13}, \lambda^{*(n)}_{33}, \lambda^{*(n)}_{44}$ is determined, according to (1.57), (1.58), in terms of the *n*th approximation $\lambda^{(n)}_{\nu p}, \mu^{(n)}_{\nu p}$ $(\nu = 1,2)$ of the respective moduli of the porous components [11, 36, 37, 39] as

$$\lambda^{*(n)}_{11} = \left\langle \frac{1}{\lambda^{(n)}_p + 2\mu^{(n)}_p} \right\rangle^{-1} \left\langle \frac{\lambda^{(n)}_p}{\lambda^{(n)}_p + 2\mu^{(n)}_p} \right\rangle^2 + \left\langle \frac{\mu^{(n)}_p (\lambda^{(n)}_p + \mu^{(n)}_p)}{\lambda^{(n)}_p + 2\mu^{(n)}_p} \right\rangle,$$

$$\lambda^{*(n)}_{12} = \left\langle \frac{1}{\lambda^{(n)}_p + 2\mu^{(n)}_p} \right\rangle^{-1} \left\langle \frac{\lambda^{(n)}_p}{\lambda^{(n)}_p + 2\mu^{(n)}_p} \right\rangle^2 + 2\left\langle \frac{\lambda^{(n)}_p \mu^{(n)}_p}{\lambda^{(n)}_p + 2\mu^{(n)}_p} \right\rangle, \quad \lambda^{*(n)}_{44} = \left\langle \frac{1}{\mu^{(n)}_p} \right\rangle^{-1},$$

$$\lambda^{*(n)}_{13} = \left\langle \frac{1}{\lambda^{(n)}_p + 2\mu^{(n)}_p} \right\rangle^{-1} \left\langle \frac{\lambda^{(n)}_p}{\lambda^{(n)}_p + 2\mu^{(n)}_p} \right\rangle, \quad \lambda^{*(n)}_{33} = \left\langle \frac{1}{\lambda^{(n)}_p + 2\mu^{(n)}_p} \right\rangle^{-1}, \tag{2.47}$$

where

$$\langle \varphi^{(n)}_p \rangle = c_1 \varphi^{(n)}_{1p} + c_2 \varphi^{(n)}_{2p}, \tag{2.48}$$

φ is an arbitrary function.

The nth approximation $K_{\nu p}^{(n)}, \mu_{\nu p}^{(n)}, \lambda_{\nu p}^{(n)}$ $(\lambda_{\nu p}^{(n)} = K_{\nu p}^{(n)} - 2\mu_{\nu p}^{(n)} / 3)$ of the effective moduli of the porous νth component is defined by formulas (2.28), where $\langle \varepsilon_{ij}^{1\nu} \rangle^{(n)}$ is the nth approximation of the average strains in the undamaged portion of the νth component. They are related to the nth approximation $\langle \varepsilon_{ij}^{\nu} \rangle^{(n)}$ of the average strains in the components by (2.29). The average strains $\langle \varepsilon_{ij}^{\nu} \rangle^{(n)}$ are determined in terms of the macrostrains $\langle \varepsilon_{ij} \rangle$ by the following formulas [11, 36, 37, 39]:

$$\langle \varepsilon_{ij}^{\nu} \rangle^{(n)} = \langle \varepsilon_{ij} \rangle, \quad \langle \varepsilon_{i3}^{\nu} \rangle^{(n)} = \frac{1}{\mu_{\nu p}^{(n)}} \left\langle \frac{1}{\mu_p^{(n)}} \right\rangle^{-1} \langle \varepsilon_{i3} \rangle, \tag{2.49}$$

$$\langle \varepsilon_{33}^{\nu} \rangle^{(n)} = \frac{1}{\lambda_{\nu p}^{(n)} + 2\mu_{\nu p}^{(n)}} \left\langle \frac{1}{\lambda_p^{(n)} + 2\mu_p^{(n)}} \right\rangle^{-1} \left[\left\langle \frac{\lambda_p^{(n)}}{\lambda_p^{(n)} + 2\mu_p^{(n)}} \right\rangle - \lambda_{\nu p}^{(n)} \left\langle \frac{1}{\lambda_p^{(n)} + 2\mu_p^{(n)}} \right\rangle \right] \langle \varepsilon_{rr} \rangle + \langle \varepsilon_{33} \rangle$$

$$(i, \nu = 1, 2).$$

We will use the Huber–Mises criterion (2.32) as a condition for the formation of a microdamage in a microvolume of the undamaged portion of the components: The one-point distribution function $F_\nu(k_\nu)$ for some microvolume in the undamaged portion of the νth component can be approximated by a power function on some interval (2.36) or by Weibull function (2.37).

Following the same line of reasoning as in 2.2, we can write the porosity balance equation for the matrix [55, 59, 63] in the form (2.38), where the average stresses $\langle \sigma_{ij}^{1\nu} \rangle$ in the undamaged portion of the νth component are related to the macrostrains $\langle \varepsilon_{ij} \rangle$ as follows [11, 36, 37, 39]:

$$\langle \sigma_{ij}^{1\nu} \rangle = \frac{1}{1 - p_\nu}$$

$$\times \left[2\mu_{\nu p} \langle \varepsilon_{ij} \rangle + \frac{\lambda_{\nu p}}{\lambda_{\nu p} + 2\mu_{\nu p}} \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle^{-1} \left(\left\langle \frac{\lambda_p}{\lambda_p + 2\mu_p} \right\rangle + 2\mu_{\nu p} \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle \right) \langle \varepsilon_{rr} \rangle \delta_{ij} \right.$$

$$\left. + \frac{\lambda_{\nu p}}{\lambda_{\nu p} + 2\mu_{\nu p}} \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle^{-1} \langle \varepsilon_{33} \rangle \delta_{ij} \right],$$

$$\langle \sigma_{33}^{1\nu} \rangle = \frac{1}{1 - p_\nu} \left[\left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle^{-1} \left(\left\langle \frac{\lambda_p}{\lambda_p + 2\mu_p} \right\rangle \langle \varepsilon_{rr} \rangle + \langle \varepsilon_{33} \rangle \right) \right],$$

$$\langle \sigma_{i3}^{1\nu} \rangle = \frac{2}{1 - p_\nu} \left\langle \frac{1}{\mu_p} \right\rangle^{-1} \langle \varepsilon_{i3} \rangle \quad (i, j, r, \nu = 1, 2), \tag{2.50}$$

and the effective moduli $\lambda_{\nu p}, \mu_{\nu p}$ are defined by (2.28).

Equations (1.53), (2.28), (2.47)–(2.49), (2.36) (or (2.37)), (2.38), (2.50) form a closed-form system describing the coupled processes of statistically homogeneous physically nonlinear deformation and damage of a laminated material. The physical nonlinearity of its components affects the way pores form during deformation, and the porosity of the components has an effect on its stress–strain curve. This is why the nonlinearity of the stress–strain curve of the laminated composite is determined by the physical nonlinearity of its components and the increase in the porosity during physically nonlinear deformation.

To describe the coupled processes of physically nonlinear deformation and damage of a laminated composite with given macrostrains, it is necessary to find the macrostrain-dependent effective elastic moduli of the composite with porous components with the iterative algorithm (2.28), (2.47)–(2.49) and to determine the porosity from Eq. (2.36) (or (2.37)), (2.38), (2.50), also with an iterative method. Using formulas (1.53), (2.28), (2.47)–(2.49), (2.36) (or (2.37)), (2.38), (2.50), we represent Eq. (2.38) at the nth step of the iterative process (2.28), (2.47)–(2.49) in the form

$$f_k^{(n)} \equiv p_k - p_{k0} - (1 - p_{k0}) F_k (I_\sigma^{1k(n)}), \tag{2.51}$$

where [11, 36, 37, 39]

$$I_\sigma^{1k(n)} = (\langle \sigma_{ij}^{1k(n)} \rangle' \langle \sigma_{ij}^{1k(n)} \rangle')^{1/2}, \tag{2.52}$$

$$\langle \sigma_{ij}^{1\nu(n)} \rangle = \frac{1}{1 - p_\nu}$$

$$\times \left[2\mu_{\nu p}^{(n)} \langle \varepsilon_{ij} \rangle + \frac{\lambda_{\nu p}^{(n)}}{\lambda_{\nu p}^{(n)} + 2\mu_{\nu p}^{(n)}} \left\langle \frac{1}{\lambda_p^{(n)} + 2\mu_p^{(n)}} \right\rangle^{-1} \left(\left\langle \frac{\lambda_p^{(n)}}{\lambda_p^{(n)} + 2\mu_p^{(n)}} \right\rangle + 2\mu_{\nu p}^{(n)} \left\langle \frac{1}{\lambda_p^{(n)} + 2\mu_p^{(n)}} \right\rangle \right) \right.$$

$$\left. \times \langle \varepsilon_{rr} \rangle \delta_{ij} + \frac{\lambda_{\nu p}^{(n)}}{\lambda_{\nu p}^{(n)} + 2\mu_{\nu p}^{(n)}} \left\langle \frac{1}{\lambda_p^{(n)} + 2\mu_p^{(n)}} \right\rangle^{-1} \langle \varepsilon_{33} \rangle \delta_{ij} \right],$$

$$\langle \sigma_{33}^{1\nu(n)} \rangle = \frac{1}{1 - p_\nu} \left[\left\langle \frac{1}{\lambda_p^{(n)} + 2\mu_p^{(n)}} \right\rangle^{-1} \left(\left\langle \frac{\lambda_p^{(n)}}{\lambda_p^{(n)} + 2\mu_p^{(n)}} \right\rangle \langle \varepsilon_{rr} \rangle + \langle \varepsilon_{33} \rangle \right) \right],$$

$$\langle \sigma_{i3}^{1\nu(n)} \rangle = \frac{1}{1 - p_\nu} \left\langle \frac{1}{\mu_p^{(n)}} \right\rangle^{-1} \langle \varepsilon_{i3} \rangle \qquad (i, j, r, \nu = 1, 2). \tag{2.53}$$

The secant method [4] is used to develop an iterative algorithm for determining the volume fraction of microdamage in the components and the deformation characteristics of the composite, i.e., formulas (1.53), (2.28), (2.47)–(2.49), (2.36) (or (2.37)), (2.38), (2.50) define macrodeformation ($\langle \sigma_{ij} \rangle$ versus $\langle \varepsilon_{ij} \rangle$) and microdamage ($p_\nu$ versus $\langle \varepsilon_{ij} \rangle$) diagrams for a laminated composite with physically nonlinear components.

Let us analyze, as an example, the coupled processes of nonlinear deformation and microdamage of a laminated composite with linear-hardening component with microdamage and linear elastic component without microdamage with bulk strains being linear and shear strains described by linear-hardening diagram (1.45)–(1.47).

The above theory was used to study the coupled processes of nonlinear deformation and microdamage of a laminated composite with microdamaged matrix for Weibull distribution and various cases of loading. The composite has aluminoborosilicate glass reinforcement with characteristics (1.50) and epoxy matrix with linear-hardening diagram (1.45)–(1.47) with constants (1.51), proportional limits and minimum tensile microstrength $\sigma_{2p} = k_{20}\sqrt{3/2}$ (2.52)–(2.54).

Given macroparameters (1.60), the macrostress $\langle \sigma_{33} \rangle$ is related to the macrostrain $\langle \varepsilon_{33} \rangle$ by (1.61), according to (1.53), In the porosity balance equation (1.45)–(1.47), we use

$$\langle \varepsilon_{11} \rangle = \langle \varepsilon_{22} \rangle = -\frac{\lambda_{13}^*}{\lambda_{11}^* + \lambda_{12}^*} \langle \varepsilon_{33} \rangle. \tag{2.54}$$

The physical nonlinearity of the matrix of the composite has a significant effect on its microdamage when $\sigma_{2p} > \sigma_{20}$ and has a noticeable but not so significant effect when $\sigma_{2p} < \sigma_{20}$. When $\sigma_{2p} \geq \sigma_{20}$, microdamage in the laminated composite with linear-hardening matrix begins at higher macrostrains than in the composite with linear elastic matrix for all values of c_1. When $\sigma_{2p} < \sigma_{20}$, microdamage in the composite with linear elastic matrix begins at the same macrostrain as in the composite with physically nonlinear matrix, but develops more intensively at the initial stage (i.e., its porosity is higher than in the material with physically nonlinear matrix at the same macrostrain).

The physical nonlinearity of the matrix has a significant effect on the stress–strain curves for all volume fractions of the components, especially for $\sigma_{2p} > \sigma_{20}$. The stress–strain curve for the composite with linear elastic matrix with microdamage consists of linear and nonlinear segments. The stress–strain curve of the material with linear-hardening matrix without microdamage consists of two linear segments. The stress–strain curve of the material with linear-hardening matrix with microdamage consists of two linear and one nonlinear segments. Comparing the materials with linear elastic and linear-hardening matrix subject to microdamage shows that the macrostress in the latter is higher than in the former, especially for $\sigma_{2p} > \sigma_{20}$.

2.4. Fibrous Material. Let us consider the physically nonlinear deformation of a unidirectional fibrous material with transversely isotropic fibers and isotropic matrix described by the dependence of the bulk (K_2) and shear (μ_2) moduli on strains and accompanied by microdamage in the matrix during loading. The microdamage of the matrix is modeled by randomly arranged quasispherical micropores occurring in those microvolumes where the stresses exceed the ultimate microstrength. Let the fibers be transversely isotropic and normal to the isotropy plane x_1x_2. Denote the elastic moduli of the fibers by $\lambda_{11}^1, \lambda_{12}^1, \lambda_{13}^1$, λ_{33}^1, λ_{44}^1, the bulk and shear moduli of the skeleton of the matrix by K_2, μ_2, its porosity by p_2, and the volume fractions of fibers and porous matrix by c_1 and c_2, respectively. The macrostresses $\langle\sigma_{ij}\rangle$ and macrostrains $\langle\varepsilon_{ij}\rangle$ are related by (1.53), where the effective moduli $\lambda_{11}^*, \lambda_{12}^*, \lambda_{13}^*, \lambda_{33}^*, \lambda_{44}^*$ are functions of the porosity p_2 of the matrix and the macrostrains $\langle\varepsilon_{ij}\rangle$.

The effective moduli of a physically nonlinear fibrous composite with porous matrix can be determined using the following iterative algorithm. The *n*th approximation of the effective moduli $\lambda_{11}^{*(n)}, \lambda_{12}^{*(n)}, \lambda_{13}^{*(n)}, \lambda_{33}^{*(n)}, \lambda_{44}^{*(n)}$ is determined in terms of the *n*th approximation of the respective moduli of the fibers (λ_{11}^1, λ_{12}^1, λ_{13}^1, λ_{33}^1, λ_{44}^1) and porous matrix ($\lambda_{2p}^{(n)}, \mu_{2p}^{(n)}$ ($\lambda_{2p}^{(n)} = K_{2p}^{(n)} - 2\mu_{2p}^{(n)}/3$)) [11, 36, 37, 39] as

$$\lambda_{11}^{*(n)} + \lambda_{12}^{*(n)} = c_1(\lambda_{11}^1 + \lambda_{12}^1) + 2c_2(\lambda_{2p}^{(n)} + \mu_{2p}^{(n)}) - \frac{c_1c_2(\lambda_{11}^1 + \lambda_{12}^1 - 2\lambda_{2p}^{(n)} - 2\mu_{2p}^{(n)})^2}{2c_1(\lambda_{2p}^{(n)} + \mu_{2p}^{(n)}) + c_2(\lambda_{11}^1 + \lambda_{12}^1) + 2m^{(n)}},$$

$$\lambda_{11}^{*(n)} - \lambda_{12}^{*(n)} = c_1(\lambda_{11}^1 - \lambda_{12}^1) + 2c_2\mu_{2p}^{(n)} - \frac{c_1c_2(\lambda_{11}^1 - \lambda_{12}^1 - 2\mu_{2p}^{(n)})^2}{2c_1\mu_{2p}^{(n)} + c_2(\lambda_{11}^1 - \lambda_{12}^1) + \dfrac{2m^{(n)}n^{(n)}}{n^{(n)} + 2m^{(n)}}},$$

$$\lambda_{13}^{*(n)} = c_1\lambda_{13}^1 + c_2\lambda_{2p}^{(n)} - \frac{c_1c_2(\lambda_{11}^1 + \lambda_{12}^1 - 2\lambda_{2p}^{(n)} - 2\mu_{2p}^{(n)})(\lambda_{13}^1 - \lambda_{2p}^{(n)})}{2c_1(\lambda_{2p}^{(n)} + \mu_{2p}^{(n)}) + c_2(\lambda_{11}^1 + \lambda_{12}^1) + 2m^{(n)}}, \tag{2.55}$$

$$\lambda_{33}^{*(n)} = c_1\lambda_{33}^1 + c_2(\lambda_{2p}^{(n)} + 2\mu_{2p}^{(n)}) - \frac{2c_1c_2(\lambda_{13}^1 - \lambda_{2p}^{(n)})^2}{2c_1(\lambda_{2p}^{(n)} + \mu_{2p}^{(n)}) + c_2(\lambda_{11}^1 + \lambda_{12}^1) + 2m^{(n)}},$$

$$\lambda_{44}^{*(n)} = c_1\lambda_{44}^1 + c_2\mu_{2p}^{(n)} - \frac{c_1c_2(\lambda_{44}^1 - \mu_{2p}^{(n)})^2}{c_1\mu_{2p}^{(n)} + c_2\lambda_{44}^1 + s^{(n)}},$$

$$\left(2m^{(n)} = c_1(\lambda_{11}^1 - \lambda_{12}^1) + 2c_2\mu_{2p}^{(n)}, 2n^{(n)} = c_1(\lambda_{11}^1 + \lambda_{12}^1) + 2c_2(\lambda_{2p}^{(n)} + \mu_{2p}^{(n)}),\quad s^{(n)} = c_1\lambda_{44}^1 + c_2\mu_{2p}^{(n)}\right. \tag{2.56}$$

if the matrix is stiffer than the fibers and

$$2m^{(n)} = \left(\frac{c_1}{\lambda_{11}^1 - \lambda_{12}^1} + \frac{c_2}{2\mu_{2p}^{(n)}} \right)^{-1}, \quad 2n^{(n)} = \left(\frac{c_1}{\lambda_{11}^1 + \lambda_{12}^1} + \frac{c_2}{2(\lambda_{2p}^{(n)} + \mu_{2p}^{(n)})} \right)^{-1}, \quad s^{(n)} = \left(\frac{c_1}{\lambda_{44}^1} + \frac{c_2}{\mu_{2p}^{(n)}} \right)^{-1} \tag{2.57}$$

if the fibers are stiffer than the matrix.

The nth approximation $K_{2p}^{(n)}, \lambda_{2p}^{(n)}, \mu_{2p}^{(n)}$ of the effective moduli of the porous matrix is defined by formulas (2.28) ($\nu = 2$), where $\langle \varepsilon_{ij}^{12} \rangle^{(n)}$ is the nth approximation of the average strains in the undamaged portion of the matrix. They are related to the nth approximation $\langle \varepsilon_{ij}^{2} \rangle^{(n)}$ of the average strains in the matrix by (2.29). The average strains $\langle \varepsilon_{ij}^{2} \rangle^{(n)}$ are determined in terms of the macrostrains $\langle \varepsilon_{ij} \rangle$ by the following formulas [11, 36, 37, 39]:

$$\langle \varepsilon_{ij}^2 \rangle^{(n)} = \frac{\lambda_{11}^{*(n)} - \lambda_{12}^{*(n)} - \lambda_{11}^1 + \lambda_{12}^1}{c_2 (2\mu_{2p}^{(n)} - \lambda_{11}^1 + \lambda_{12}^1)} \langle \varepsilon_{ij} \rangle$$

$$- \frac{1}{\Delta_2^{(n)}} \{ [(\lambda_{11}^{*(n)} - \lambda_{11}^1) a_1^{(n)} - (\lambda_{12}^{*(n)} - \lambda_{12}^1) a_2^{(n)} - (\lambda_{13}^{*(n)} - \lambda_{13}^1) a_3^{(n)}] \langle \varepsilon_{rr} \rangle$$

$$+ [(\lambda_{13}^{*(n)} - \lambda_{13}^1)(a_1^{(n)} - a_2^{(n)}) - (\lambda_{33}^{*(n)} - \lambda_{33}^1) a_3^{(n)}] \langle \varepsilon_{33} \rangle \} \delta_{ij},$$

$$\langle \varepsilon_{33}^2 \rangle^{(n)} = - \frac{1}{\Delta_2^{(n)}} \{ [(\lambda_{13}^{*(n)} - \lambda_{13}^1) a_4^{(n)} - (\lambda_{11}^{*(n)} + \lambda_{12}^{*(n)} - \lambda_{11}^1 - \lambda_{12}^1) a_3^{(n)}] \langle \varepsilon_{rr} \rangle$$

$$+ [(\lambda_{33}^{*(n)} - \lambda_{33}^1) a_4^{(n)} - 2(\lambda_{13}^{*(n)} - \lambda_{13}^1) a_3^{(n)}] \langle \varepsilon_{rr} \rangle \}, \tag{2.58}$$

$$\langle \varepsilon_{i3}^2 \rangle^{(n)} = \frac{\lambda_{44}^{*(n)} - \lambda_{44}^1}{c_2 (\mu_{2p}^{(n)} - \lambda_{44}^1)} \langle \varepsilon_{i3} \rangle \quad (i, j, r = 1, 2),$$

$$\Big(\Delta_2^{(n)} = c_2 (\lambda_{11}^1 - \lambda_{12}^1 - 2\mu_{2p}^{(n)}) [(\lambda_{11}^1 + \lambda_{12}^1 - 2\lambda_{2p}^{(n)} - 2\mu_{2p}^{(n)})(\lambda_{33}^1 - \lambda_{2p}^{(n)} - 2\mu_{2p}^{(n)}) - 2(\lambda_{13}^1 - \lambda_{2p}^{(n)})^2],$$

$$a_1^{(n)} = (\lambda_{13}^1 - \lambda_{2p}^{(n)})^2 - (\lambda_{12}^1 - \lambda_{2p}^{(n)})(\lambda_{33}^1 - \lambda_{2p}^{(n)} - 2\mu_{2p}^{(n)}),$$

$$a_2^{(n)} = (\lambda_{13}^1 - \lambda_{2p}^{(n)})^2 - (\lambda_{11}^1 - \lambda_{2p}^{(n)} - 2\mu_{2p}^{(n)})(\lambda_{33}^1 - \lambda_{2p}^{(n)} - 2\mu_{2p}^{(n)}), \tag{2.59}$$

$$a_3^{(n)} = (\lambda_{13}^1 - \lambda_{2p}^{(n)})(\lambda_{11}^1 - \lambda_{12}^1 - 2\mu_{2p}^{(n)}),$$

$$a_4^{(n)} = (\lambda_{11}^1 + \lambda_{12}^1 - 2\lambda_{2p}^{(n)} - 2\mu_{2p}^{(n)})(\lambda_{11}^1 - \lambda_{12}^1 - 2\mu_{2p}^{(n)}),$$

and the effective moduli $K_{2p}^{(n)}, \lambda_{2p}^{(n)}, \mu_{2p}^{(n)}$ are defined by (2.28).

Given macrostrains $\langle \varepsilon_{ij} \rangle$, the effective moduli are determined as the limits of the iterative process

$$\lambda_{lm}^* = \lim_{n \to \infty} \lambda_{lm}^{*(n)}. \tag{2.60}$$

We will use the Huber–Mises criterion (2.32) ($\nu = 2$) as a condition for the formation of a microdamage in a microvolume of the undamaged portion of the matrix. The one-point distribution function $F_2(k_2)$ for some microvolume in the undamaged portion of the matrix can be approximated by a power function on some interval (2.36) or by Weibull function (2.37).

Since the damaged microvolumes are modeled by pores, we can write the porosity balance equation for the matrix in the form (2.38) ($\nu = 2$), where the average stresses $\langle \sigma_{ij}^{12} \rangle^{(n)}$ in the undamaged portion of the matrix are related to the macrostrains $\langle \varepsilon_{ij} \rangle$ by (2.29), (2.58), (2.59), and

$$\langle \sigma_{ij}^{12} \rangle^{(n)} = \frac{1}{1 - p_2} \left[\left(K_{2p}^{(n)} - \frac{2}{3} \mu_{2p}^{(n)} \right) \langle \varepsilon_{rr}^{2} \rangle^{(n)} \delta_{ij} + 2\mu_{2p}^{(n)} \langle \varepsilon_{ij}^{2} \rangle^{(n)} \right]. \tag{2.61}$$

Equations (1.53), (2.28), (2.55)–(2.57), (2.36) (or (2.37)), (2.38), (2.58), (2.59) (2.61) form a closed-form system describing the coupled processes of statistically homogeneous physically nonlinear deformation and damage of a fibrous material. The physical nonlinearity of its components affects the way pores form during deformation, and the porosity of the components has an effect on its stress–strain curve. This is why the nonlinearity of the stress–strain curve of the fibrous composite is determined by the physical nonlinearity of its components and the increase in the porosity during physically nonlinear deformation.

To describe the coupled processes of physically nonlinear deformation and damage of a fibrous composite with given macrostrains, it is necessary to find the macrostrain-dependent effective elastic moduli of the composite with porous components with the iterative algorithm (2.28), (2.55)–(2.57) and to determine the porosity from Eq. (2.36) (or (2.37)), (2.38), (2.58), (2.59), (2.61), also with an iterative method. Using formulas (1.53), (2.28), (2.55)–(2.57), (2.36) (or (2.37)), (2.38), (2.58), (2.59), (2.61), we represent Eq. (2.38) at the *n*th step of the iterative process (2.28), (2.55)–(2.57) in the form (2.52), (2.53). The secant method [4] is used to develop an iterative algorithm for determining the volume fraction of microdamage in the components and the deformation characteristics of the composite, i.e., formulas (1.53), (2.28), (2.55)–(2.57), (2.36) (or (2.37)), (2.38), (2.58), (2.59), (2.61) define macrodeformation ($\langle \sigma_{ij} \rangle$ versus $\langle \varepsilon_{ij} \rangle$) and microdamage ($p_\nu$ versus $\langle \varepsilon_{ij} \rangle$) diagrams for the composite components.

Let us study, as an example, the coupled processes of nonlinear deformation and microdamage of a fibrous composite with linear-hardening matrix with bulk strains being linear and shear strains described by linear-hardening diagram (1.45)–(1.47).

The above theory was used to study the coupled processes of nonlinear deformation and microdamage of a fibrous composite with microdamaged matrix for Weibull distribution and various cases of loading. The composite has high-modulus carbon fibers with characteristics (1.80) and epoxy matrix with linear-hardening diagram (1.45)–(1.47) with constants (1.51), proportional limits and minimum tensile microstrength $\sigma_p = k_{02}\sqrt{3/2}$ (2.44)–(2.46).

Given macroparameters (1.82), the macrostress $\langle \sigma_{11} \rangle$ is related to the macrostrain $\langle \varepsilon_{11} \rangle$ by (1.83), according to (1.53), In the porosity balance equation (2.38), we use

$$\langle \varepsilon_{22} \rangle = \frac{(\lambda_{13}^{*})^2 - \lambda_{12}^{*}\lambda_{33}^{*}}{\lambda_{11}^{*}\lambda_{33}^{*} - (\lambda_{13}^{*})^2} \langle \varepsilon_{11} \rangle, \quad \langle \varepsilon_{33} \rangle = \frac{(\lambda_{12}^{*} - \lambda_{11}^{*})\lambda_{13}^{*}}{\lambda_{11}^{*}\lambda_{33}^{*} - (\lambda_{13}^{*})^2} \langle \varepsilon_{11} \rangle. \tag{2.62}$$

Given macroparameters (1.62), the macrostress $\langle \sigma_{33} \rangle$ is related to the macrostrain $\langle \varepsilon_{33} \rangle$ by (1.63), according to (1.53), In the porosity balance equation (1.45)–(1.47), we use

An analysis shows that the physical nonlinearity of the matrix has a significant effect on the microdamage of the composite. The physical nonlinearity of the matrix of the composite has a significant effect on its microdamage when $\sigma_{2p} > \sigma_{20}$ and has a noticeable but not so significant effect when $\sigma_{2p} < \sigma_{20}$. When $\sigma_{2p} \geq \sigma_{20}$, microdamage in the fibrous composite with linear-hardening matrix begins at higher macrostrains than in the composite with linear elastic matrix for all values of c_1. When $\sigma_{2p} < \sigma_{20}$, microdamage in the composite with linear elastic matrix begins at the same macrostrain as in the composite with physically nonlinear matrix, but develops more intensively at the initial stage (i.e., its porosity is higher than in the material with physically nonlinear matrix at the same macrostrain).

The physical nonlinearity of the matrix has a significant effect on the stress–strain curves for all values of c_1, especially for $\sigma_{2p} \geq \sigma_{20}$. The stress–strain curve for the composite with linear elastic matrix with microdamage consists of linear and nonlinear segments. The stress–strain curve of the material with linear-hardening matrix without microdamage consists of two linear segments. The stress–strain curve of the material with linear-hardening matrix with microdamage consists of two linear and one nonlinear segments. Comparing the materials with linear elastic and linear-hardening matrix subject to microdamage shows that the macrostress in the latter is higher than in the former, especially for $\sigma_{2p} > \sigma_{20}$.

3. Long-Term Damage of Materials during Nonlinear Deformation.

3.1. Homogeneous Material. Let us consider the physically nonlinear deformation of an isotropic material described by the dependence of the bulk (K) and shear (μ) moduli on strains and accompanied by microdamage. The microdamage of the material is modeled by randomly arranged quasispherical micropores occurring in those microvolumes where the stresses exceed the ultimate microstrength.

The stresses and strains at an arbitrary point of a physically nonlinear porous material are related by

$$\sigma_{ij} = \lambda(\varepsilon_{\alpha\beta})\varepsilon_{rr}\delta_{ij} + 2\mu(\varepsilon_{\alpha\beta})\varepsilon_{ij} \quad (\lambda = K - 2/3\mu), \tag{3.1}$$

where the bulk (K) and shear (μ) moduli deterministically depending on the strains $\varepsilon_{\alpha\beta}$ are random functions of coordinates that have the values $K(\varepsilon^1_{\alpha\beta}), \mu(\varepsilon^1_{\alpha\beta})$ in the skeleton and $K = \mu = 0$ in pores, the index 1 referring to the skeleton.

If a macrovolume (which is a volume much greater than the pores and distances between them) is subject to homogeneous macrostresses and macrostrains, the microstresses σ_{ij} and microstrains ε_{ij} are ergodic statistically homogeneous random functions of coordinates. Their expectations $\langle\sigma_{ij}\rangle$ and $\langle\varepsilon_{ij}\rangle$ at an arbitrary point are equal to the macrostresses and macrostrains, respectively. Substituting (3.1) into the equilibrium equation

$$\sigma_{ij,j} = 0 \tag{3.2}$$

and using the kinematic equations

$$\varepsilon_{ij} = u_{(i,j)} \equiv \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{3.3}$$

we obtain an equation for the fluctuations of displacements:

$$\mu_c u^0_{i,rr} + (\lambda_c + \mu_c)u^0_{r,ri} = \{[\lambda(\varepsilon_{\alpha\beta}) - \lambda_c]\varepsilon_{rr}\delta_{ij} + 2[\mu(\varepsilon_{\alpha\beta}) - \mu_c]\varepsilon_{ij}\},_j, \tag{3.4}$$

where λ_c and μ_c are the constant elastic moduli of a reference body; the fluctuations of displacements are given by

$$u_i = \langle\varepsilon_{ij}\rangle x_j + u^0_i. \tag{3.5}$$

The boundary condition on a boundary s at infinity follows from (3.5):

$$u^0_i\Big|_s = 0. \tag{3.6}$$

Using Green's function satisfying the equation

$$\mu_c G_{in,rr}(x^{(1)}_r - x^{(2)}_r) + (\lambda_c + \mu_c)G_{rn,ri}(x^{(1)}_r - x^{(2)}_r) + \delta(x^{(1)}_r - x^{(2)}_r)\delta_{in} = 0, \tag{3.7}$$

we reduce the boundary-value problem (3.4), (3.6) to an integral equation for the strain tensor:

$$\varepsilon^{(1)}_{ij} = \langle\varepsilon_{ij}\rangle + K_{ijpq}(x^{(1)}_r - x^{(2)}_r)\{[\lambda^{(2)}(\varepsilon^{(2)}_{\alpha\beta}) - \lambda_c]\varepsilon^{(2)}_{rr}\delta_{pq} + 2[\mu^{(2)}(\varepsilon^{(2)}_{\alpha\beta}) - \mu_c]\varepsilon^{(2)}_{pq}\}, \tag{3.8}$$

where the integral operator K_{ijpq} is defined by

$$K_{ijpq}(x^{(1)}_r - x^{(2)}_r)\varphi^{(2)} = \int_{V^{(2)}} G_{(ip,j)q}(x^{(1)}_r - x^{(2)}_r)(\varphi^{(2)} - \langle\varphi\rangle)dV^{(2)}, \tag{3.9}$$

where the superscript in parentheses denotes a point in space.

The stresses σ^1_{ij} and strains ε^1_{ij} in the skeleton (undamaged portion of the material) can be represented as the sum

$$\sigma^1_{ij} = \langle\sigma^1_{ij}\rangle + \sigma^{10}_{ij}, \quad \varepsilon^1_{ij} = \langle\varepsilon^1_{ij}\rangle + \varepsilon^{10}_{ij}, \tag{3.10}$$

where $\langle\sigma^1_{ij}\rangle, \langle\varepsilon^1_{ij}\rangle$ are the average stresses and strains in the skeleton; $\sigma^{10}_{ij}, \varepsilon^{10}_{ij}$ are the respective fluctuations within the skeleton. If these fluctuations are neglected, the nonlinear relation (3.1) becomes

$$\langle\sigma^1_{ij}\rangle = \lambda(\langle\varepsilon^1_{\alpha\beta}\rangle)\langle\varepsilon^1_{rr}\rangle\delta_{ij} + 2\mu(\langle\varepsilon^1_{\alpha\beta}\rangle)\langle\varepsilon^1_{ij}\rangle, \tag{3.11}$$

whence follows an expression for the macrostresses:

$$\langle\sigma_{ij}\rangle = (1-p)\langle\sigma^1_{ij}\rangle = (1-p)[\lambda(\langle\varepsilon^1_{\alpha\beta}\rangle)\langle\varepsilon^1_{rr}\rangle\delta_{ij} + 2\mu(\langle\varepsilon^1_{\alpha\beta}\rangle)\langle\varepsilon^1_{ij}\rangle]. \tag{3.12}$$

To plot macrostresses versus macrostrains, it is necessary to determine $\langle\varepsilon^1_{ij}\rangle$ as a function of $\langle\varepsilon_{ij}\rangle$ and substitute it into (3.12). To this end, we will average Eq. (3.8) using conditional density $f(\varepsilon^{(1)}_{ij}, \varepsilon^{(2)}_{ij}|^{(1)}_{1})$ (distribution density of strains at points $x^{(1)}_r, x^{(2)}_r$ provided that the point $x^{(1)}_r, x^{(2)}_r$ is in the skeleton). Then, neglecting the fluctuations of strains within the skeleton, we obtain a system of nonlinear algebraic equations for the average strains in the skeleton [36, 37]:

$$\langle\varepsilon^1_{ij}\rangle = \langle\varepsilon_{ij}\rangle + K^{11}_{ijpq}\{[\lambda(\langle\varepsilon^1_{\alpha\beta}\rangle) - \lambda_c]\langle\varepsilon^1_{rr}\rangle\delta_{pq} + 2[\mu(\langle\varepsilon^1_{\alpha\beta}\rangle) - \mu_c]\langle\varepsilon^1_{pq}\rangle\}, \tag{3.13}$$

where the matrix operator K^{11}_{ijpq} is defined by

$$K^{11}_{ijpq} = K_{ijpq}(x^{(1)}_r - x^{(2)}_r)p_{11}(x^{(1)}_r - x^{(2)}_r), \tag{3.14}$$

where $p_{11}(x^{(1)}_r - x^{(2)}_r) = f(^{(2)}_{1}|^{(1)}_{1})$ is the probability of transition from the point $x^{(1)}_r, x^{(2)}_r$ to the point $x^{(2)}_r$ within the skeleton. If the pores are quasispherical and dispersed statistically isotropically, the transition probability is defined by

$$p_{11}(r) = 1 - p[1 - \varphi(r)], \quad r^2 = (x^{(2)}_i - x^{(1)}_i)(x^{(2)}_i - x^{(1)}_i), \tag{3.15}$$

where $\varphi(r)$ is a correlation coefficient such that $\varphi(0) = 1$, $\varphi(\infty) = 0$.

The macrostress–macrostrain relationship follows from (3.12)–(3.15):

$$\langle\sigma_{ij}\rangle = (K^* - 2\mu^*/3)\langle\varepsilon_{rr}\rangle\delta_{ij} + 2\mu^*\langle\varepsilon_{ij}\rangle, \tag{3.16}$$

where the effective moduli K^* and μ^* are functions of p and $\langle\varepsilon_{ij}\rangle$. The effective moduli of a porous physically nonlinear material can be determined using an iterative algorithm [11, 35, 39]. The effective moduli K^*, μ^* of a porous physically nonlinear material are expressed [11, 36, 37] in terms of those of its undamaged portion, K, μ, as

$$K^* = \frac{4K(\langle\varepsilon^1_{ij}\rangle)\mu(\langle\varepsilon^1_{ij}\rangle)(1-p)^2}{3K(\langle\varepsilon^1_{ij}\rangle)p + 4\mu(\langle\varepsilon^1_{ij}\rangle)(1-p)},$$

$$\mu^* = \frac{[9K(\langle\varepsilon^1_{ij}\rangle) + 8\mu(\langle\varepsilon^1_{ij}\rangle)]\mu(\langle\varepsilon^1_{ij}\rangle)(1-p)^2}{3K(\langle\varepsilon^1_{ij}\rangle)(3-p) + 4\mu(\langle\varepsilon^1_{ij}\rangle)(2+p)}, \tag{3.17}$$

where $\langle\varepsilon^1_{ij}\rangle$ are the average strains in the undamaged portion of the material. Since they are expressed in terms of the elastic moduli K, μ of its components, which, in turn, are functions of the average strains in the undamaged portion of the ν component, they can be determined using the following iterative algorithm. Their $(n+1)$th approximation is related to the nth approximation by

$$\langle\varepsilon^1_{ij}\rangle^{(n+1)} = \frac{1}{(1-p)}\left[\frac{K^{*(n)}}{K(\langle\varepsilon^1_{ij}\rangle^{(n)})}V_{ij\alpha\beta} + \frac{\mu^{*(n)}}{\mu(\langle\varepsilon^1_{ij}\rangle^{(n)})}D_{ij\alpha\beta}\right]\langle\varepsilon_{\alpha\beta}\rangle, \tag{3.18}$$

where $V_{ij\alpha\beta}$ and $D_{ij\alpha\beta}$ are the volumetric and deviatoric components of the unit tensor $I_{ij\alpha\beta}$,

$$I_{ij\alpha\beta}=V_{ij\alpha\beta}+D_{ij\alpha\beta},\quad V_{ij\alpha\beta}=\delta_{ij}\delta_{\alpha\beta}/3,\quad D_{ij\alpha\beta}=(\delta_{\alpha j}\delta_{j\beta}+\delta_{i\beta}\delta_{j\alpha}-2\delta_{ij}\delta_{\alpha\beta}/3)/2.$$

The zero-order approximation represents a physically linear material.

We will use the Huber–Mises criterion [15] as a condition for the formation of a microdamage in a microvolume of the undamaged portion of the material:

$$I^1_{\langle\sigma\rangle}=k, \tag{3.19}$$

where $I^1_{\langle\sigma\rangle}=(\langle\sigma^1_{ij}\rangle'\langle\sigma^1_{ij}\rangle')^{1/2}$ is the second invariant of the deviatoric average-stress tensor $\langle\sigma^1_{ij}\rangle'$ in the undamaged portion of the material; k is the ultimate microstrength, which is a random function of coordinates. Since the average stresses $\langle\sigma^1_{ij}\rangle$ in the undamaged portion are related to the macrostresses $\langle\sigma_{ij}\rangle$ as follows [35, 39]:

$$\langle\sigma^1_{ij}\rangle=\frac{1}{1-p}\langle\sigma_{ij}\rangle, \tag{3.20}$$

the invariant of the average-stress deviator $I^1_{\langle\sigma\rangle}$ is related to the invariant of the macrostress deviator $I_{\langle\sigma\rangle}=(\langle\sigma_{ij}\rangle'\langle\sigma_{ij}\rangle')^{1/2}$ and the invariant of the macrostrain deviator $I_{\langle\varepsilon\rangle}=(\langle\varepsilon_{ij}\rangle'\langle\varepsilon_{ij}\rangle')^{1/2}$ as

$$I^1_{\langle\sigma\rangle}=\frac{1}{1-p}I_{\langle\sigma\rangle}, \tag{3.21}$$

$$I^1_{\langle\sigma\rangle}=\frac{2\mu^*}{1-p}I_{\langle\varepsilon\rangle}. \tag{3.22}$$

A failure criterion in terms of macrostresses follows from (3.19), (3.21):

$$\frac{1}{1-p}I_{\langle\sigma\rangle}=k, \tag{3.23}$$

and a failure criterion in terms of macrostrains follows from (3.19), (3.22):

$$\frac{2\mu^*(p,\langle\varepsilon_{ij}\rangle)}{1-p}I_{\langle\varepsilon\rangle}=k. \tag{3.24}$$

If the invariant $I^1_{\langle\sigma\rangle}$ does not reach the limiting value k in some microvolume of the material, then, according to the stress-rupture criterion, failure will occur in some time τ_k, which depends on the difference between $I^1_{\langle\sigma\rangle}$ and k. In the general case, this dependence can be represented as some function:

$$\tau_k=\varphi(I^1_{\langle\sigma\rangle},k), \tag{3.25}$$

where $\varphi(k,k)=0$ and $\varphi(0,k)=\infty$ according to (3.19).

The one-point distribution function $F(k)$ for some microvolume in the undamaged portion of the material can be approximated by a power function on some interval

$$F(k)=\begin{cases}0, & k<k_0,\\ \left(\dfrac{k-k_0}{k_1-k_0}\right)^\beta, & k_0\le k\le k_1,\\ 1, & k>k_1\end{cases} \tag{3.26}$$

or by the Weibull function

$$F(k)=\begin{cases}0, & k<k_0,\\ 1-\exp[-m(k-k_0)^{\beta}], & k\ge k_0,\end{cases} \tag{3.27}$$

where k_0 is the minimum value of k from which failure begins in some volumes of the material; k_1, m, β are constants found from strength scatter fitting.

Assume that the random field of the ultimate microstrength k is statistically homogeneous, which is typical of real materials, and the microdamages and the distances between them are negligible compared with the inclusions and the distances between them. Then the distribution function $F(k)$ is ergodic because it defines the content of the undamaged portion of the material in which the ultimate microstrength is less than k. Therefore, if the stresses $\langle\sigma_{ij}^1\rangle$ are nonzero, the function $F(I_{\langle\sigma\rangle}^1)$ defines, according to (3.19), (3.26), and (3.27), the content of instantaneously destroyed microvolumes. Since the damaged microvolumes are modeled by pores, we can write a balance equation for destroyed microvolumes or porosity of the material subject to short-term damage [55]:

$$p=p_0+(1-p_0)F(I_{\langle\sigma\rangle}^1). \tag{3.28}$$

If the homogeneous macrostresses $\langle\sigma_{ij}\rangle$ are given, then, according to (3.21), the porosity balance equation (3.28) becomes

$$p=p_0+(1-p_0)F\left(\frac{1}{1-p}I_{\langle\sigma\rangle}\right). \tag{3.29}$$

If the macrostrains $\langle\varepsilon_{ij}\rangle$ are given, then, according to (3.22), we have

$$p=p_0+(1-p_0)F\left(\frac{2\mu^*(p,\langle\varepsilon_{ij}\rangle)}{1-p}I_{\langle\varepsilon\rangle}\right). \tag{3.30}$$

If the stresses $\langle\sigma_{ij}^1\rangle$ act for some time t, then, according to the stress-rupture criterion (3.25), those microvolumes are destroyed that have k such that

$$t\ge\tau_k=\varphi(I_{\langle\sigma\rangle}^1,k), \tag{3.31}$$

where $I_{\langle\sigma\rangle}^1$ is defined by (3.21), (3.22).

The time to brittle failure τ_k for real materials at low temperatures is finite beginning only from some value of $I_{\langle\sigma\rangle}^1>0$. In this case, the durability function $\phi(I_{\langle\sigma\rangle}^1,k)$ can be represented as follows [57]:

$$\phi(I_{\langle\sigma\rangle}^1,k)=\tau_0\left(\frac{k-I_{\langle\sigma\rangle}^1}{I_{\langle\sigma\rangle}^1-\gamma k}\right)^{n_1}\quad(\gamma k\le I_{\langle\sigma\rangle}^1\le k,\quad \gamma<1), \tag{3.32}$$

where some typical time τ_0, exponent n_1, and coefficient γ are determined from the fit of experimental durability curves.

Substituting (3.32) into (3.31), we arrive at the inequality

$$k\le I_{\langle\sigma\rangle}^1\frac{1+\bar{t}^{\,1/n_1}}{1+\gamma\bar{t}^{\,1/n_1}}\quad\left(\bar{t}=\frac{t}{\tau_0}\right). \tag{3.33}$$

Considering the definition of the distribution function $F(k)$, we conclude that the function $F[(I_{\langle\sigma\rangle}^1)\psi(\bar{t})]$, where

$$\psi(\bar{t}) = \frac{1+\bar{t}^{1/n_1}}{1+\gamma\bar{t}^{1/n_1}}, \quad (3.34)$$

defines the relative content of destroyed microvolumes in the undamaged portion of the material at the time $\bar{t}$. Then, in view of (3.28), the porosity balance equation for the material subject to long-term damage can be represented in the following form [57]:

$$p = p_0 + (1-p_0)F[(I^1_{\langle\sigma\rangle})\psi(\bar{t})] \quad (3.35)$$

or, in view of (3.21):

$$p = p_0 + (1-p_0)F\left[\frac{I_{\langle\sigma\rangle}}{1-p}\psi(\bar{t})\right], \quad (3.36)$$

where p is a function of dimensionless time $\bar{t}$, and $I_{\langle\sigma\rangle}$ is defined by (3.22).

If the time τ_k is finite for arbitrary values of $I^1_{\langle\sigma\rangle}$, which may be observed at high temperatures, then the durability function can be approximated by an exponential power function [57]:

$$\varphi(I^1_{\langle\sigma\rangle}, k) = \tau_0\left\{\exp m_1\left[(k/I^1_{\langle\sigma\rangle})^{n_1} - 1\right] - 1\right\}^{n_2}, \quad (3.37)$$

which has enough constants τ_0, m_1, n_1, n_2 to fit experimental curves. Substituting (3.37) into (3.31), we arrive at the inequality

$$k \le I^1_{\langle\sigma\rangle}\left[1 + \frac{1}{m_1}\ln(1+\bar{t}^{1/n_2})\right]^{1/n_1} \quad \left(\bar{t} = \frac{t}{\tau_0}\right). \quad (3.38)$$

Considering the definition of the distribution function $F(k)$, we conclude that the function $F[(I^1_{\langle\sigma\rangle})\psi(\bar{t})]$, where

$$\psi(\bar{t}) = \left[1 + \frac{1}{m_1}\ln(1+\bar{t}^{1/n_2})\right]^{1/n_1}, \quad (3.39)$$

defines the relative content of destroyed microvolumes in the undamaged portion of the material at the time $\bar{t}$. Then, in view of (3.20), the porosity balance equation (3.25) for a material subject to long-term damage can be represented in the form (3.35).

At $\bar{t} = 0$, the porosity balance equation (3.35) with (3.26), (3.27) defines the short-term (instantaneous) damage of the material. As time elapses, Eq. (3.35) with (3.26), (3.27), (3.34) (or (3.39)) defines its long-term damage, which consists of short-term damage and additional time-dependent damage.

Equations (3.16)–(3.18), (3.35), (3.26), (3.27), (3.34) (or (3.39)) form a closed-form system describing the coupled processes of statistically homogeneous physically nonlinear deformation and long-term damage. Physical nonlinearity affects the way pores form during deformation, and the porosity of the material has an effect on its stress–strain curve. This is why the nonlinearity of the stress–strain curve is determined by the physical nonlinearity of the material and the increase in the porosity during physically nonlinear deformation.

Let us analyze, as an example, the coupled processes of nonlinear deformation and microdamage of a material with bulk strains being linear and shear strains described by a linear-hardening diagram:

$$\langle\sigma_{rr}\rangle = K\langle\varepsilon_{rr}\rangle, \quad \langle\sigma_{ij}\rangle' = 2\mu(J)\langle\varepsilon_{ij}\rangle', \quad (3.40)$$

where the bulk modulus K does not depend on the strains, and the shear modulus $\mu(J)$ is described by

$$\mu(J) = \begin{cases} \mu_0, & T \le T_0, \\ \mu' + \left(1 - \dfrac{\mu'}{\mu_0}\right)\dfrac{T_0}{2J}, & T \ge T_0, \end{cases} \quad (3.41)$$

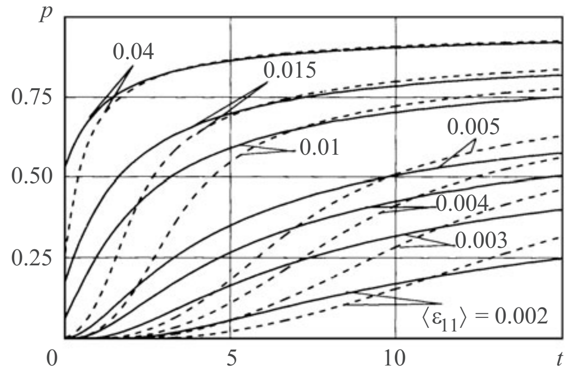

Fig. 3.1

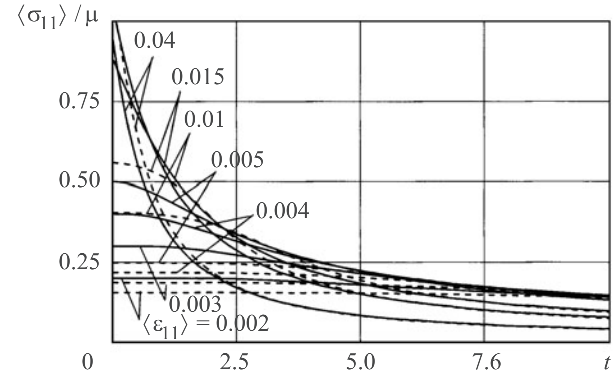

Fig. 3.2

$$\Big(J = (\langle \varepsilon_{ij} \rangle' \langle \varepsilon_{ij} \rangle')^{1/2}, \; T = (\langle \sigma_{ij} \rangle' \langle \sigma_{ij} \rangle')^{1/2}, \; T_0 = \sigma_0 \sqrt{2/3}, \tag{3.42}$$

where $\langle \varepsilon_{ij} \rangle'$ and $\langle \sigma_{ij} \rangle'$ are the strain and stress deviators; σ_0 is the tensile proportional limit assumed to be independent of the coordinates; μ_0 and μ' are material constants.

Formulas (3.16)–(3.18), (3.35), (3.26), (3.27), (3.34) (or (3.39)) and the secant method [4] can be used to develop an iterative algorithm for the determination of the stress–strain state of the material and the volume fraction of microdamages in it. For the Weibull distribution and functions $\psi(\bar{t})$ defined by (3.34) and (3.39), we studied the coupled processes of nonlinear deformation and microdamage of a homogeneous material described by the linear-hardening diagram (3.40)–(3.42) with the following constants [11, 22]:

$$K = 3.33 \text{ GPa}, \quad \mu_0 = 1.11 \text{ GPa}, \quad \mu' = 0.331 \text{ GPa} \tag{3.43}$$

and the following proportional limits and minimum tensile microstrength ($\sigma_p = k_0 \sqrt{3/2}$):

$$\sigma_0 = 0.003 \text{ GPa}, \quad \sigma_p = 0.011 \text{ GPa}. \tag{3.44}$$

If

$$\langle \varepsilon_{11} \rangle \neq 0, \quad \langle \sigma_{22} \rangle = \langle \sigma_{33} \rangle = 0, \tag{3.45}$$

then, according to (3.1), the macrostress $\langle \sigma_{11} \rangle$ is related to the macrostrain $\langle \varepsilon_{11} \rangle$ by

$$\langle \sigma_{11} \rangle = \frac{3K^* \mu^*}{K^* + 1/3\mu^*} \langle \varepsilon_{11} \rangle. \tag{3.46}$$

In the porosity balance equation (3.30), (3.36), we use

$$I_{\langle \varepsilon \rangle} = \sqrt{\frac{2}{3}} \frac{3K^* \langle \varepsilon_{11} \rangle}{2(K^* + 1/3\mu^*)}, \tag{3.47}$$

which is equivalent to (3.45).

Figure 3.1 shows (by solid lines) the porosity p of the linear-hardening material with $\psi(\bar{t})$ defined by (3.34) as a function of time $\bar{t}$ for different values of $\langle \varepsilon_{11} \rangle$. For comparison, the figure shows (by dashed lines) p versus $\bar{t}$ for a linear elastic material. The same notation is used in Figs. 3.2–3.4. As is seen, physical nonlinearity has a significant effect on microdamage. The microdamage of the linear-hardening material occurs later (at greater values of $\bar{t}$) and more intensively than in the linear elastic material, i.e., at great values of $\bar{t}$, the porosity of the linear-hardening material is higher than that of the linear material.

Figure 3.2 shows the macrostress $\langle \sigma_{11} \rangle / \mu$ as a function of time $\bar{t}$ for different values of macrostrain $\langle \varepsilon_{11} \rangle$ in the linear-hardening material (solid lines) and the linear elastic material (dashed lines) for the fractional power durability function

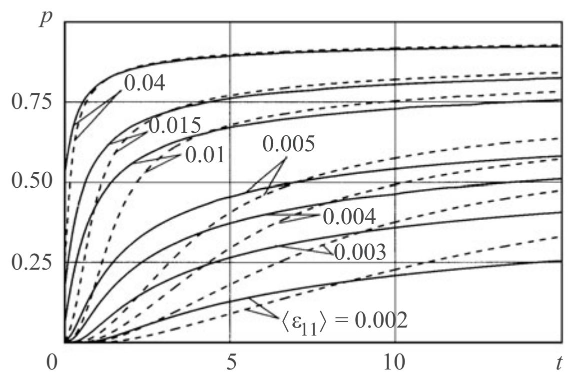

Fig. 3.3

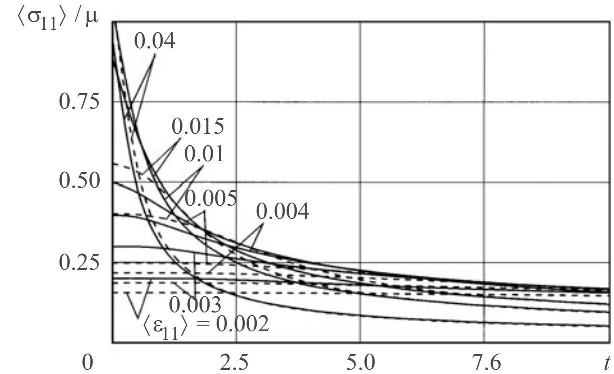

Fig. 3.4

$\psi(\bar{t})$ defined by (3.34). As is seen, at small values of $\bar{t}$, the physical nonlinearity of the material has a significant effect on its stress state as well. At great values of $\bar{t}$, the effect of nonlinearity on the stress state is weak.

Figures 3.3 and 3.4 show the porosity p and macrostress $\langle\sigma_{11}\rangle/\mu$, respectively, of linear-hardening and linear elastic materials with $\psi(\bar{t})$ defined by (3.39) as a function of time $\bar{t}$ for different values of $\langle\varepsilon_{11}\rangle$. As is seen, the curves are qualitatively similar to those for the function $\psi(\bar{t})$ defined by (3.34).

3.2. Particulate Composite Material. The physically nonlinear deformation of a particulate composite is described as the dependence of the bulk (K_ν) and shear (μ_ν, $\nu = 1, 2$) moduli on strains. The microdamage of the composite components caused by loading is modeled by randomly arranged quasispherical micropores occurring in those microvolumes where the stresses exceed the ultimate microstrength. The macrostresses $\langle\sigma_{ij}\rangle$ and macrostrains $\langle\varepsilon_{ij}\rangle$ in an elementary macrovolume are related by

$$\langle\sigma_{ij}\rangle = (K^* - 2/3\mu^*)\langle\varepsilon_{rr}\rangle\delta_{jk} + 2\mu^*\langle\varepsilon_{ij}\rangle, \tag{3.48}$$

where K^*, μ^* are the effective moduli dependent on the macrostrains $\langle\varepsilon_{ij}\rangle$ due to physical nonlinearity and microdamage.

The porosity of the inclusions and matrix is denoted by p_1 and p_2, respectively. Denote the bulk and shear moduli of the porous inclusions and matrix by K_1, μ_1 and K_2, μ_2, respectively, and their volume fractions by c_1 and c_2, respectively. The effective moduli of a physically nonlinear particulate composite with porous components can be determined using the following iterative algorithm. The effective bulk (K^*) and shear (μ^*) moduli are expressed [11, 36, 37, 39] in terms of those of the inclusions (K_{1p}, μ_{1p}) and matrix (K_{2p}, μ_{2p}) as

$$K^* = c_1 K_{1p} + c_2 K_{2p} - c_1 c_2 \frac{(K_{1p} - K_{2p})^2}{c_1 K_{2p} + c_2 K_{1p} + n_c},$$

$$\mu^* = c_1 \mu_{1p} + c_2 \mu_{2p} - c_1 c_2 \frac{(\mu_{1p} - \mu_{2p})^2}{c_1 \mu_{2p} + c_2 \mu_{1p} + m_c} \tag{3.49}$$

$$\left(n_c = \frac{4}{3}\mu_c, \ \ m_c = \frac{\mu_c (9K_c + 8\mu_c)}{6(K_c + 2\mu_c)}, \right. \tag{3.50}$$

and

$$K_c = c_1 K_{1p} + c_2 K_{2p}, \quad \mu_c = c_1 \mu_{1p} + c_2 \mu_{2p} \tag{3.51}$$

if the porous matrix is stiffer than the porous inclusions and

$$K_c = \frac{K_{1p}K_{2p}}{c_1K_{2p} + c_2K_{1p}}, \quad \mu_c = \frac{\mu_{1p}\mu_{2p}}{c_1\mu_{2p} + c_2\mu_{1p}} \tag{3.52}$$

otherwise. The effective moduli of porous inclusions, K_{1p}, μ_{1p}, and porous matrix, K_{2p}, μ_{2p}, are defined by the following formulas [11, 35]:

$$K_{\nu p} = K_{\nu p}(\langle \varepsilon_{lm}^{1\nu} \rangle) = \frac{4K_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle)\mu_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle)(1-p_\nu)^2}{3K_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle)p_\nu + 4\mu_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle)(1-p_\nu)},$$

$$\mu_{\nu p} = \mu_{\nu p}(\langle \varepsilon_{lm}^{1\nu} \rangle) = \frac{[9K_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle) + 8\mu_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle)]\mu_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle)(1-p_\nu)^2}{3K_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle)(3-p_\nu) + 4\mu_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle)(2+p_\nu)}$$

$$(\nu = 1, 2), \tag{3.53}$$

where $\langle \varepsilon_{lm}^{1\nu} \rangle$ are the average strains in the undamaged portion of the inclusions and matrix. Since they are expressed in terms of the elastic moduli K_ν, μ_ν $(\nu = 1, 2)$ of the components, which, in turn, are functions of the average strains in the undamaged portion of the ν component, they can be determined using the following iterative algorithm. The $(n+1)$th approximation is related to the nth approximation $\langle \varepsilon_{ij}^{\nu} \rangle^{(n)}$ of the average strains in the components by

$$\langle \varepsilon_{ij}^{1\nu} \rangle^{(n+1)} = \frac{1}{(1-p_\nu)}$$

$$\times \left\{ \frac{\mu_{\nu p}^{(n)}}{\mu_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle^{(n)})} \langle \varepsilon_{ij}^{\nu} \rangle^{(n)} + \frac{1}{3}\left[\frac{K_{\nu p}^{(n)}}{K_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle^{(n)})} - \frac{\mu_{\nu p}^{(n)}}{\mu_\nu(\langle \varepsilon_{lm}^{1\nu} \rangle^{(n)})} \right] \langle \varepsilon_{rr}^{\nu} \rangle^{(n)} \delta_{ij} \right\}$$

$$(\nu = 1, 2). \tag{3.54}$$

The average strains $\langle \varepsilon_{ij}^{\nu} \rangle^{(n)}$ are determined in terms of the macrostrains $\langle \varepsilon_{ij} \rangle$ by the formulas

$$\langle \varepsilon_{ij}^{\nu} \rangle^{(n)} = (-1)^{\nu+1}$$

$$\times \left\{ \frac{2\mu^{*(n)}(\mu_{1p}^{(n)} - \mu_{2p}^{(n)})(K^{*(n)} - K_{\chi p}^{(n)}) - 3K^{*(n)}(K_{1p}^{(n)} - K_{2p}^{(n)})(\mu^{*(n)} - \mu_{\chi p}^{(n)})}{6c_\nu\mu^{*(n)}(\mu_{1p}^{(n)} - \mu_{2p}^{(n)})(K_{1p}^{(n)} - K_{2p}^{(n)})} \langle \varepsilon_{rr} \rangle \delta_{ij} + \frac{\mu^{*(n)} - \mu_{\chi p}^{(n)}}{c_k(\mu_{1p}^{(n)} - \mu_{2p}^{(n)})} \langle \varepsilon_{ij} \rangle \right\}$$

$$(\chi = 3 - k). \tag{3.55}$$

The zero-order approximation represents physically linear components.

We will use the Huber–Mises criterion [15] as a condition for the formation of a microdamage in a microvolume of the undamaged portion of the components:

$$I_{\langle\sigma\rangle}^{1\nu} = k_\nu \quad (\nu = 1, 2), \tag{3.56}$$

where $I_{\langle\sigma\rangle}^{1\nu} = (\langle \sigma_{ij}^{1\nu} \rangle' \langle \sigma_{ij}^{1\nu} \rangle')^{1/2}$ is the second invariant of the deviatoric average-stress tensor $\langle \sigma_{ij}^{1\nu} \rangle'$ in the undamaged portion of the νth component; k_ν is the ultimate microstrength, which is a random function of coordinates. Since the average stresses $\langle \sigma_{ij}^{1\nu} \rangle$ in the undamaged portion of the νth component are related to the average stresses $\langle \sigma_{ij}^{\nu} \rangle$ in the component as follows [35, 39]:

$$\langle \sigma_{ij}^{1\nu} \rangle = \frac{1}{1-p_\nu} \langle \sigma_{ij}^{\nu} \rangle, \tag{3.57}$$

the invariant $I_{\langle\sigma\rangle}^{1\nu}$ is related to the invariant of the deviatoric average-stress tensor $I_{\langle\sigma\rangle}^{\nu} = (\langle \sigma_{ij}^{\nu} \rangle' \langle \sigma_{ij}^{\nu} \rangle')^{1/2}$ in the component by the formula

$$I_{\langle\sigma\rangle}^{1\nu} = \frac{1}{1-p_\nu} I_{\langle\sigma\rangle}^{\nu}, \tag{3.58}$$

and is related to the invariant of the deviatoric average-strain tensor $I_{\langle\varepsilon\rangle}^{\nu} = (\langle \varepsilon_{ij}^{\nu} \rangle' \langle \varepsilon_{ij}^{\nu} \rangle')^{1/2}$ in the components by the formula

$$I_{\langle\sigma\rangle}^{1\nu} = \frac{2\mu_{\nu p}}{1-p_\nu} I_{\langle\varepsilon\rangle}^{\nu}, \tag{3.59}$$

where $I_{\langle\varepsilon\rangle}^{\nu}$ is related to the invariant $I_{\langle\varepsilon\rangle} = (\langle \varepsilon_{ij} \rangle' \langle \varepsilon_{ij} \rangle')^{1/2}$ for the whole composite by

$$I_{\langle\varepsilon\rangle}^{\nu} = (-1)^{\nu+1} \frac{\mu^* - \mu_{(3-\nu)p}}{c_\nu (\mu_{1p} - \mu_{2p})} I_{\langle\varepsilon\rangle}. \tag{3.60}$$

With (3.59) and (3.60), the failure criterion (3.56) takes the following form in the macrostrain space:

$$(-1)^{\nu+1} \frac{2\mu_{\nu p} (\mu^* - \mu_{(3-\nu)p})}{c_\nu (1-p_\nu)(\mu_{1p} - \mu_{2p})} I_{\langle\varepsilon\rangle} = k_\nu \quad (\nu = 1, 2). \tag{3.61}$$

If the invariant $I_{\langle\sigma\rangle}^{1\nu}$ does not reach the limiting value k_ν in some microvolume of the νth component, then, according to the stress-rupture criterion, failure will occur in some time τ_k^ν, which depends on the difference between $I_{\langle\sigma\rangle}^{1\nu}$ and k_ν. In the general case, this dependence can be represented as some function:

$$\tau_k^\nu = \varphi_\nu (I_{\langle\sigma\rangle}^{1\nu}, k_\nu), \tag{3.62}$$

where $\varphi_\nu (k_\nu, k_\nu) = 0$ and $\varphi_\nu (0, k_\nu) = \infty$ according to (3.56).

The one-point distribution function $F_\nu (k_\nu)$ for some microvolume in the undamaged portion of a component can be approximated by a power function on some interval

$$F_\nu (k_\nu) = \begin{cases} 0, & k_\nu < k_{\nu 0}, \\ \left(\dfrac{k_\nu - k_{\nu 0}}{k_{1\nu} - k_{\nu 0}} \right)^{n_\nu}, & k_{\nu 0} \le k_\nu \le k_{\nu 1}, \\ 1, & k_\nu > k_{\nu 1}, \end{cases} \tag{3.63}$$

or by the Weibull function

$$F_\nu (k_\nu) = \begin{cases} 0, & k_\nu < k_{\nu 0}, \\ 1 - \exp[-m_\nu (k_\nu - k_{\nu 0})^{n_\nu}], & k_\nu \ge k_{\nu 0}, \end{cases} \tag{3.64}$$

where $k_{\nu 0}$ is the minimum value of ultimate microstrength in a component; $k_{\nu 1}, m_\nu, n_\nu$ are deterministic constants describing the behavior of the distribution function and determined by fitting experimental microstrength scatter or stress–strain curves.

Assume that the random field of ultimate microstrength k_ν is statistically homogeneous in real materials, and its correlation scale and the size of single microdamages and the distances between them are negligible compared with the macrovolume. Then the random field k_ν and the distribution of macrostresses in the component under uniform loading are ergodic, and the distribution function $F_\nu(k_\nu)$ defines the fraction of the undamaged portion of the component in which the ultimate microstrength is less than k_ν. Therefore, if the stresses $\langle\sigma_{ij}^{1\nu}\rangle$ are nonzero, the function $F_\nu(I_{\langle\sigma\rangle}^{1\nu})$ defines, according to (3.56), (3.63), and (3.64), the content of instantaneously damaged microvolumes of the skeleton of the component. Since the damaged microvolumes are modeled by pores, we can write a porosity balance equation [55]:

$$p_\nu = p_{\nu 0} + (1 - p_{\nu 0})F_\nu(I_{\langle\sigma\rangle}^{1\nu}). \tag{3.65}$$

With (3.59) and (3.60), the porosity balance equation (3.65) takes the following form in the macrostrain space:

$$p_\nu = p_{\nu 0} + (1 - p_{\nu 0})F_\nu\left[(-1)^{\nu+1}\frac{2\mu_{\nu p}(\mu^* - \mu_{(3-\nu)p})}{c_\nu(1-p_\nu)(\mu_{1p} - \mu_{2p})}I_{\langle\varepsilon\rangle}\right], \tag{3.66}$$

where μ^* and μ_{1p}, μ_{2p} are defined by (3.49)–(3.53).

If the stresses $\langle\sigma_{ij}^{\nu}\rangle$ act for some time t, then, according to the stress-rupture criterion (3.62), those microvolumes of the νth component are damaged that have k_ν such that

$$t \ge \tau_k^\nu = \varphi_\nu(I_{\langle\sigma\rangle}^{1\nu}, k_\nu), \tag{3.67}$$

where $I_{\langle\sigma\rangle}^{1\nu}$ is defined by (3.58) or (3.59).

The time to brittle failure τ_k^ν for the νth component of real materials at low temperatures is finite beginning only from some value of $I_{\langle\sigma\rangle}^{1\nu} > 0$. In this case, the durability function $\phi_\nu(I_{\langle\sigma\rangle}^{1\nu}, k_\nu)$ can be represented as follows [57]:

$$\phi_\nu(I_{\langle\sigma\rangle}^{1\nu}, k_\nu) = \tau_{0\nu}\left(\frac{k_\nu - I_{\langle\sigma\rangle}^{1\nu}}{I_{\langle\sigma\rangle}^{1\nu} - \gamma_\nu k_\nu}\right)^{n_{1\nu}} \quad (\gamma_\nu k_\nu \le I_{\langle\sigma\rangle}^{1\nu} \le k_\nu,\ \gamma_\nu < 1), \tag{3.68}$$

where some typical time $\tau_{0\nu}$, exponent $n_{1\nu}$, and coefficient γ_ν are determined from the fit of experimental durability curves for the νth component.

Substituting (3.68) into (3.67), we arrive at the inequality

$$k_\nu \le I_{\langle\sigma\rangle}^{1\nu}\frac{1 + \bar{t}_\nu^{-1/n_{1\nu}}}{1 + \gamma_\nu \bar{t}_\nu^{-1/n_{1\nu}}} \quad \left(\bar{t}_\nu = \frac{t}{\tau_{0\nu}}\right). \tag{3.69}$$

Considering the definition of the distribution function $F_\nu(k_\nu)$, we conclude that the function $F_\nu[I_{\langle\sigma\rangle}^{1\nu}\psi_\nu(\bar{t}_\nu)]$, where

$$\psi_\nu(\bar{t}_\nu) = \frac{1 + \bar{t}_\nu^{-1/n_{1\nu}}}{1 + \gamma_\nu \bar{t}_\nu^{-1/n_{1\nu}}}, \tag{3.70}$$

defines the relative content of the destroyed microvolumes in the undamaged portion of the νth component at the time $\bar{t}_\nu$. Then, in view of (3.57), the porosity balance equation for the νth component subject to long-term damage can be represented as

$$p_\nu = p_{0\nu} + (1 - p_{0\nu})F_\nu\left[\frac{I_{\langle\sigma\rangle}^{\nu}}{1 - p_\nu}\psi_\nu(\bar{t}_\nu)\right], \tag{3.71}$$

where p_ν is a function of dimensionless time $\bar{t}_\nu$, and $I_{\langle\sigma\rangle}^{\nu}$ is defined by (3.58)–(3.60).

If the time τ_k^ν is finite for arbitrary values of $I_{\langle\sigma\rangle}^{1\nu}$, which may be observed at high temperatures, then the durability function can be approximated by an exponential power function [57]:

$$\varphi_\nu(I_{\langle\sigma\rangle}^\nu, k_\nu) = \tau_{0\nu}\{\exp m_{1\nu}[(k_\nu / I_{\langle\sigma\rangle}^\nu)^{n_{1\nu}} - 1] - 1\}^{n_{2\nu}}, \tag{3.72}$$

which has enough constants $\tau_{0\nu}$, $m_{1\nu}, n_{1\nu}, n_{2\nu}$ to fit experimental curves. Substituting (3.72) into (3.67), we arrive at the inequality

$$k_\nu \le I_{\langle\sigma\rangle}^\nu \left[1 + \frac{1}{m_{1\nu}} \ln(1 + \bar{t}_\nu^{\,1/n_{2\nu}})\right]^{1/n_{1\nu}} \qquad \left(\bar{t}_\nu = \frac{t}{\tau_{0\nu}}\right). \tag{3.73}$$

Considering the definition of the distribution function $F_\nu(k_\nu)$, we conclude that the function $F_\nu[I_{\langle\sigma\rangle}^{1\nu}\,\psi_\nu(\bar{t}_\nu)]$, where

$$\psi_\nu(\bar{t}_\nu) = \left[1 + \frac{1}{m_{1\nu}} \ln(1 + \bar{t}_\nu^{\,1/n_{2\nu}})\right]^{1/n_{1\nu}}, \tag{3.74}$$

defines the relative content of the destroyed microvolumes in the undamaged portion of the νth component at the time $\bar{t}_\nu$. Then, in view of (3.57), the porosity balance equation for the νth component subject to long-term damage can be represented in the form (3.71), where p_ν is a function of dimensionless time $\bar{t}_\nu$, and $I_{\langle\sigma\rangle}^\nu$ is defined by (3.58)–(3.60).

At $\bar{t}_\nu = 0$, the porosity balance equation (3.71) with (3.58)–(3.60), (3.70) (or (3.74)) defines the short-term (instantaneous) damage of the νth component. As time elapses, Eq. (3.71) with (3.58)–(3.60), (3.70) (or (3.74)) defines its long-term damage, which consists of short-term damage and additional time-dependent damage.

Equations (3.48), (3.49)–(3.55), (3.71), (3.58)–(3.60), (3.70) (or (3.74)) form a closed-form system describing the coupled processes of statistically homogeneous physically nonlinear deformation and long-term damage of a particulate composite. The physical nonlinearity of its components affects the way pores form during deformation, and the porosity of the components has an effect on its stress–strain curve. This is why the nonlinearity of the stress–strain curve of the particulate composite is determined by the physical nonlinearity of its components and the increase in the porosity during physically nonlinear deformation.

Let us analyze, as an example, the coupled processes of nonlinear deformation and long-term microdamage of a particulate composite with linear elastic inclusions and nonlinear elastic matrix with bulk strains being linear and shear strains described by a linear-hardening diagram

$$\langle\sigma_{rr}^2\rangle = K_2\langle\varepsilon_{rr}^2\rangle, \quad \langle\sigma_{ij}^2\rangle' = 2\mu_2(J_2)\langle\varepsilon_{ij}^2\rangle', \tag{3.75}$$

where the bulk modulus K_2 does not depend on the strains, and the shear modulus $\mu_2(J_2)$ is described by

$$\mu_2(J_2) = \begin{cases} \mu_{20}, & T_2 \le T_{20}, \\ \mu_2' + \left(1 - \dfrac{\mu_2'}{\mu_{20}}\right)\dfrac{T_{20}}{2J_2}, & T_2 \ge T_{20}, \end{cases} \tag{3.76}$$

$$(J_2 = (\langle\varepsilon_{ij}^2\rangle'\langle\varepsilon_{ij}^2\rangle')^{1/2}, \quad T_2 = (\langle\sigma_{ij}^2\rangle'\langle\sigma_{ij}^2\rangle')^{1/2}, \quad T_{20} = \sigma_{20}\sqrt{2/3}, \tag{3.77}$$

where $\langle\varepsilon_{ij}^2\rangle'$ and $\langle\sigma_{ij}^2\rangle'$ are the strain and stress deviators in the matrix; σ_{20} is the tensile proportional limit assumed to be independent of the coordinates; μ_{20}, μ_2' are the material constants of the matrix.

To describe the coupled processes of physically nonlinear deformation and long-term damage of a particulate composite, it is necessary to find the macrostrain-dependent effective elastic moduli by the iterative algorithm (3.49)–(3.55) and to determine the porosity from Eq. (3.71), (3.58)–(3.60), (3.70) (or 3.74)) also by an iterative method based on the secant method [4]. We analyzed the coupled processed of nonlinear deformation and long-term damage of a particulate composite for Weibull

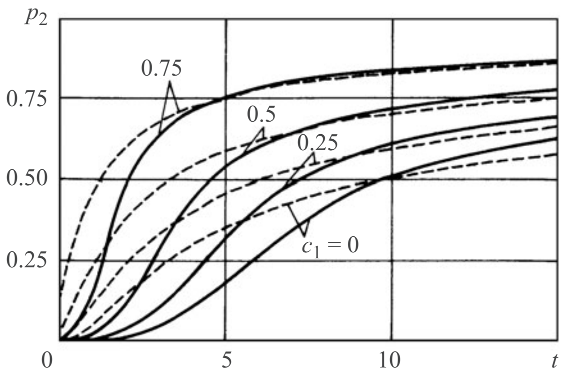

Fig. 3.5

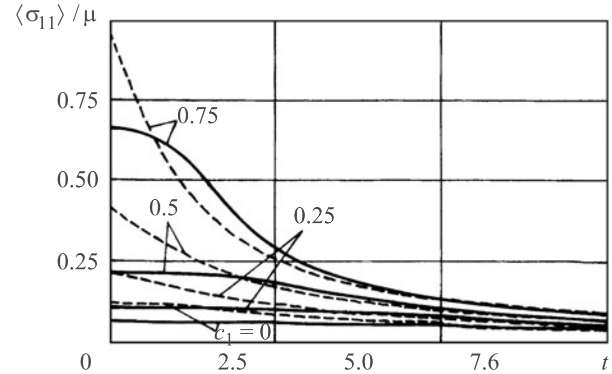

Fig. 3.6

distribution (3.64) and for both fractional power durability function $\psi_2(\bar{t}_2)$ defined by (3.70) and exponential power durability function $\psi_2(\bar{t}_2)$ defined by (3.74). The inclusions are linear elastic particles with the following characteristics [22] and volume fraction:

$$K_1 = 38.89 \text{ GPa}, \quad \mu_1 = 29{,}17 \text{ GPa}, \quad c_1 = 0, 0.25, 0.5, 0.75, 1.0 \tag{3.78}$$

and the matrix is described by the linear-hardening diagram (3.75)–(3.77) with the following constants [11, 22]:

$$K_2 = 3.33 \text{ GPa}, \quad \mu_{20} = 1.11 \text{ GPa}, \quad \mu'_2 = 0.331 \text{ GPa} \tag{3.79}$$

and the following proportional limits and minimum tensile microstrength ($\sigma_{2p} = \sqrt{3/2}\, k_{20}$):

$$\sigma_{20} = 0.003 \text{ GPa}, \quad \sigma_{2p} = 0.011 \text{ GPa} \tag{3.80}$$

$$(p_{02} = 0,\ k_{02}/\mu_2 = 0.01,\ m_2 = 1000, \quad \alpha_2 = 2, \quad \gamma_2 = 0.05, \quad n_{12} = 1). \tag{3.81}$$

If

$$\langle\varepsilon_{11}\rangle \neq 0, \quad \langle\sigma_{22}\rangle = \langle\sigma_{33}\rangle = 0, \tag{3.82}$$

then, according to (3.48), the macrostress $\langle\sigma_{11}\rangle$ is related to the macrostrain $\langle\varepsilon_{11}\rangle$ by

$$\langle\sigma_{11}\rangle = \frac{3K^*\mu^*}{K^* + 1/3\mu^*}\langle\varepsilon_{11}\rangle. \tag{3.83}$$

In the porosity balance equation (3.66), we use

$$I_{\langle\varepsilon\rangle} = \sqrt{\frac{2}{3}}\frac{3K^*\langle\varepsilon_{11}\rangle}{2(K^* + 1/3\mu^*)}, \tag{3.84}$$

which is equivalent to (3.82).

Figure 3.5 shows (solid lines) the porosity p_2 of the linear-hardening matrix as a function of time $\bar{t}_2$ for fractional-power function $\psi_2(\bar{t}_2)$ defined by (3.70) and for different values of c_1. For comparison, the figure shows (dashed lines) p_2 versus $\bar{t}_2$ for the linear elastic matrix. The same notation is used in Figs. 3.6–3.8. As is seen, the physical nonlinearity of the matrix has a significant effect on the microdamage of the particulate composite. The microdamage of the composite with linear-hardening matrix sets in at greater values of $\bar{t}_2$ and occurs more intensively than in the composite with linear elastic matrix, i.e., at great values of $\bar{t}_2$, the porosity of the composite with linear-hardening matrix is higher than in the composite with linear elastic matrix.

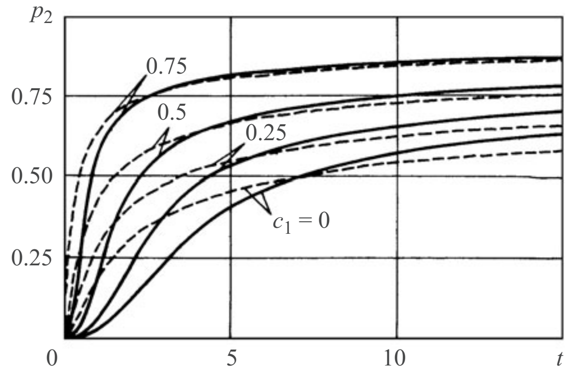

Fig. 3.7

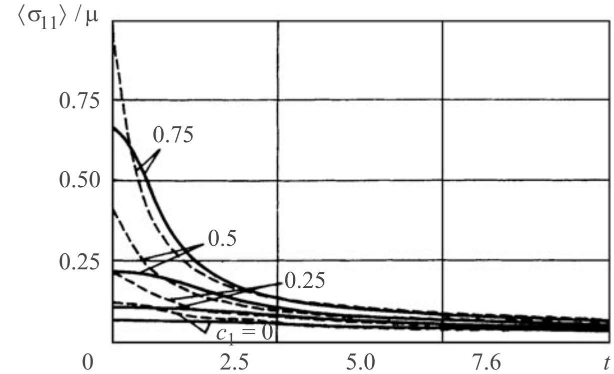

Fig. 3.8

Figure 3.6 shows the macrostress $\langle\sigma_{11}\rangle/\mu_2$ in composites with linear-hardening and linear elastic matrices as a function of time $\bar{t}_2$ for fractional-power function $\psi_2(\bar{t}_2)$ defined by (3.70) and for different values of c_1. As is seen, at small values of $\bar{t}_2$, the physical nonlinearity of the matrix has a significant effect on the stress state of the particulate composite as well. At great values of $\bar{t}_2$, the effect of nonlinearity on the stress state is weak.

Figures 3.7 and 3.8 show the porosity p_2 of the matrix of and the macrostress $\langle\sigma_{11}\rangle/\mu_2$ in a particulate composite with linear-hardening and linear elastic matrices as functions of time $\bar{t}_2$ for exponential-power function $\psi_2(\bar{t}_2)$ defined by (3.74) and for different values of c_1. As is seen, the curves are qualitatively similar to those for the function $\psi_2(\bar{t}_2)$ defined by (3.70).

3.3. Laminated Composite Material. The physically nonlinear deformation of a laminated composite with N isotropic components is described as the dependence of the bulk (K_ν) and shear (μ_ν, $\nu = 1, 2, \dots, N$) moduli on strains. The damage of a component of the composite is modeled by randomly arranged quasispherical micropores occurring in those microvolumes where the stresses exceed the ultimate microstrength. The macrostresses $\langle\sigma_{ij}\rangle$ and macrostrains $\langle\varepsilon_{ij}\rangle$ in an elementary macrovolume are related by

$$\langle\sigma_{ij}\rangle = (\lambda_{11}^* - \lambda_{12}^*)\langle\varepsilon_{ij}\rangle + (\lambda_{12}^*\langle\varepsilon_{rr}\rangle + \lambda_{13}^*\langle\varepsilon_{33}\rangle)\delta_{ij}$$

$$\langle\sigma_{33}\rangle = \lambda_{13}^*\langle\varepsilon_{rr}\rangle + \lambda_{33}^*\langle\varepsilon_{33}\rangle, \qquad \langle\sigma_{i3}\rangle = 2\lambda_{44}^*\langle\varepsilon_{i3}\rangle$$

$$(i, j, r = 1, 2), \tag{3.85}$$

where $\lambda_{11}^*, \lambda_{12}^*, \lambda_{13}^*, \lambda_{33}^*, \lambda_{44}^*$ are the effective elastic moduli dependent on the macrostrains $\langle\varepsilon_{ij}\rangle$ due to physical nonlinearity and microdamage.

Denote the bulk and shear moduli of the skeleton of the νth component by K_ν, μ_ν, its porosity by p_ν, and the volume fraction of the porous νth component by c_ν $(\nu = 1, \dots, N)$. The effective moduli of a physically nonlinear laminated composite with porous components can be determined using the following iterative algorithm. The effective moduli $\lambda_{11}^*, \lambda_{12}^*, \lambda_{13}^*, \lambda_{33}^*, \lambda_{44}^*$ of the composite are expressed [11, 36, 37, 39] in terms of the moduli $\lambda_{\nu p}, \mu_{\nu p}$ $(\nu = 1, 2, \dots, N)$ of its components as

$$\lambda_{11}^* = \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle^{-1} \left\langle \frac{\lambda_p}{\lambda_p + 2\mu_p} \right\rangle^2 + 4\left\langle \frac{\mu_p(\lambda_p + \mu_p)}{\lambda_p + 2\mu_p} \right\rangle,$$

$$\lambda_{12}^* = \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle^{-1} \left\langle \frac{\lambda_p}{\lambda_p + 2\mu_p} \right\rangle^2 + 2\left\langle \frac{\lambda_p \mu_p}{\lambda_p + 2\mu_p} \right\rangle, \qquad \lambda_{44}^* = \left\langle \frac{1}{\mu_p} \right\rangle^{-1},$$

$$\lambda_{13}^* = \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle^{-1} \left\langle \frac{\lambda_p}{\lambda_p + 2\mu_p} \right\rangle, \qquad \lambda_{33}^* = \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle^{-1}, \tag{3.86}$$

where f is an arbitrary function,

$$\langle f_p \rangle = \sum_{\nu=1}^{N} c_\nu f_{\nu p}. \tag{3.87}$$

The effective moduli $K_{\nu p}, \lambda_{\nu p}, \mu_{\nu p}$ $(\lambda_{\nu p} = K_{\nu p} - 2\mu_{\nu p} / 3)$ $(\nu = 1, \ldots, N)$ of the porous νth component are defined by formulas (3.53), according to [11, 35]. The average strains $\langle \varepsilon_{ij}^{1\nu} \rangle$ appearing in these formulas are determined using the following iterative algorithm. The $(n+1)$th approximation is related to the nth approximation by (3.54) $(\nu = 1, \ldots, N)$. The average strains $\langle \varepsilon_{ij}^{\nu} \rangle$ are determined in terms of the macrostrains $\langle \varepsilon_{ij} \rangle$ by the formulas

$$\langle \varepsilon_{ij}^{1} \rangle = \ldots = \langle \varepsilon_{ij}^{N} \rangle = \langle \varepsilon_{ij} \rangle, \quad \langle \varepsilon_{i3}^{\nu} \rangle = \frac{1}{\mu_{\nu p}} \left\langle \frac{1}{\mu_p} \right\rangle^{-1} \langle \varepsilon_{i3} \rangle,$$

$$\langle \varepsilon_{33}^{\nu} \rangle = \frac{1}{\lambda_{\nu p} + 2\mu_{\nu p}} \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle^{-1} \left[\left\langle \frac{\lambda_p}{\lambda_p + 2\mu_p} \right\rangle - \lambda_{\nu p} \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle \right] \langle \varepsilon_{rr} \rangle + \langle \varepsilon_{33} \rangle$$

$$(i, j = 1, 2,\ \nu = 1, \ldots, N). \tag{3.88}$$

The zero-order approximation represents physically linear components.

We will use the Huber–Mises criterion (3.56) $(\nu = 1, \ldots, N)$ as a condition for the formation of a microdamage in a microvolume of the undamaged portion of the components. If the invariant $I_{\langle \sigma \rangle}^{1\nu}$ does not reach the limiting value k_ν in some microvolume of the νth component, then, according to the stress-rupture criterion, failure will occur in some time τ_k^{ν}, which depends on the difference between $I_{\langle \sigma \rangle}^{1\nu}$ and k_ν. In the general case, this dependence can be represented as some function (3.62). The one-point distribution function $F_\nu(k_\nu)$ for some microvolume in the undamaged portion of the νth component can be approximated by a power function on some interval (3.63) or by Weibull function (3.64).

Assume that the random field of ultimate microstrength k_ν is statistically homogeneous in real materials, and its correlation scale and the size of single microdamages and the distances between them are negligible compared with the macrovolume. Then the random field k_ν and the distribution of macrostresses in the component under uniform loading are ergodic, and the distribution function $F_\nu(k_\nu)$ defines the fraction of the undamaged portion of the component in which the ultimate microstrength is less than k_ν. Therefore, if the stresses $\langle \sigma_{ij}^{1\nu} \rangle$ are nonzero, the function $F_\nu(I_{\langle \sigma \rangle}^{1\nu})$ defines, according to (3.56), (3.63), and (3.64), the content of instantaneously damaged microvolumes of the skeleton of the component. Since the damaged microvolumes are modeled by pores, we can write a porosity balance equation in the form (3.65), $(\nu = 1, \ldots, N)$, where $p_{\nu 0}$ is the initial porosity. Given macrostrains $\langle \varepsilon_{ij} \rangle$, the average stresses $\langle \sigma_{ij}^{\nu} \rangle$ are related to the macrostrains $\langle \varepsilon_{ij} \rangle$ as follows [11, 36, 37, 39]:

$$\langle \sigma_{ij}^{1} \rangle = \ldots = \langle \sigma_{ij}^{N} \rangle = 2\mu_{\nu p} \langle \varepsilon_{ij} \rangle + \frac{\lambda_{\nu p}}{\lambda_{\nu p} + 2\mu_{\nu p}} \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle^{-1}$$

$$\times \left[\left(\left\langle \frac{\lambda_p}{\lambda_p + 2\mu_p} \right\rangle + 2\mu_{\nu p} \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle \right) \langle \varepsilon_{rr} \rangle + \langle \varepsilon_{33} \rangle \right] \delta_{ij}$$

$$\langle \sigma_{33}^{1} \rangle = \ldots = \langle \sigma_{33}^{N} \rangle = \left\langle \frac{1}{\lambda_p + 2\mu_p} \right\rangle^{-1} \left(\left\langle \frac{\lambda_p}{\lambda_p + 2\mu_p} \right\rangle \langle \varepsilon_{rr} \rangle + \langle \varepsilon_{33} \rangle \right),$$

$$\langle \sigma_{i3}^{1} \rangle = \ldots = \langle \sigma_{i3}^{N} \rangle = 2\left\langle \frac{1}{\mu_p} \right\rangle^{-1} \langle \varepsilon_{i3} \rangle \qquad (i, j, r = 1, 2). \tag{3.89}$$

If the stresses $\langle \sigma_{ij}^{\nu} \rangle$ act for some time t, then, according to the stress-rupture criterion (3.62), those microvolumes of the νth component are damaged that have k_ν satisfying inequality (3.67), where $I_{\langle\sigma\rangle}^{1\nu}$ is defined by (3.89).

The time to brittle failure τ_k^{ν} for the νth component of real materials at low temperatures is finite beginning only from some value of $I_{\langle\sigma\rangle}^{1\nu} > 0$. In this case, the durability function $\phi_\nu(I_{\langle\sigma\rangle}^{1\nu}, k_\nu)$ can be represented in the form (3.68). If the time τ_k^{ν} is finite for arbitrary values of $I_{\langle\sigma\rangle}^{1\nu}$, which may be observed at high temperatures, then the durability function can be represented as (3.72). Substituting (3.68) or (3.72) into (3.67) and considering the definition of the distribution function $F_\nu(k_\nu)$, we conclude that the function $F_\nu[I_{\langle\sigma\rangle}^{1\nu}\psi_\nu(\bar{t}_\nu)]$, where $\psi_\nu(\bar{t}_\nu)$ is defined by (3.70) or (3.74), respectively, defines the relative content of the destroyed microvolumes in the undamaged portion of the νth component at the time $\bar{t}_\nu$. Then, in view of (3.57), the porosity balance equation for the νth component subject to long-term damage can be represented in the form (3.71), ($\nu = 1, \ldots, N$), where the porosity p_ν of the νth component is a function of dimensionless time $\bar{t}_\nu$, and the average stresses $\langle \sigma_{jk}^{\nu} \rangle$ in the ith component are defined by (3.89).

At $\bar{t}_\nu = 0$, the porosity balance equation (3.71) with (3.89), (3.70) (or (3.74)) defines the short-term (instantaneous) damage of the νth component. As time elapses, Eq. (3.71) with (3.89), (3.70) (or (3.74)) defines its long-term damage, which consists of short-term damage and additional time-dependent damage.

Equations (3.48)–(3.54), (3.71), (3.89), (3.70) (or (3.74)) form a closed-form system describing the coupled processes of statistically homogeneous physically nonlinear deformation and long-term damage of a laminated material. The physical nonlinearity of its components affects the way pores form during deformation, and the porosity of the components has an effect on its stress–strain curve. This is why the nonlinearity of the stress–strain curve of the laminated composite is determined by the physical nonlinearity of its components and the increase in the porosity during physically nonlinear deformation.

Let us analyze, as an example, the coupled processes of nonlinear deformation and long-term microdamage of a two-component laminated composite with linear elastic reinforcement and microdamaged nonlinear elastic matrix with bulk strains being linear and shear strains described by linear-hardening diagram (3.75)–(3.77). Formulas (3.48)–(3.54), (3.71), (3.89), (3.70) (or (3.74)) and the secant method [4] can be used to develop an iterative algorithm for the determination of the stress–strain state of the nonlinear laminated composite and the volume fraction of microdamages in its components (layers). We conducted calculations to plot macrodeformation curves for a two-layer composite with microdamaged matrix for Weibull distribution (3.64) and for fractional power durability function $\psi_2(\bar{t}_2)$ defined by (3.70). The reinforcement is an elastic layer with characteristics and volume fractions specified in (3.78). The matrix is described by linear-hardening diagram (3.75)–(3.77) with constants (3.79) and proportional limits and minimum tensile microstrength (3.80), (3.81).

If

$$\langle \varepsilon_{33} \rangle \neq 0, \qquad \langle \sigma_{11} \rangle = \langle \sigma_{22} \rangle = 0, \tag{3.90}$$

then, according to (3.85), the macrostress $\langle \sigma_{33} \rangle$ is related to the macrostrain $\langle \varepsilon_{33} \rangle$ by

$$\langle \sigma_{33} \rangle = \frac{1}{\lambda_{11}^{*} + \lambda_{12}^{*}} [(\lambda_{11}^{*} + \lambda_{12}^{*})\lambda_{33}^{*} - 2(\lambda_{13}^{*})^2] \langle \varepsilon_{33} \rangle. \tag{3.91}$$

In the porosity balance equation (3.71), (3.89), (3.70), we use

$$\langle \varepsilon_{11} \rangle = \langle \varepsilon_{22} \rangle = -\frac{\lambda_{13}^{*}}{\lambda_{11}^{*} + \lambda_{12}^{*}} \langle \varepsilon_{33} \rangle, \tag{3.92}$$

which is equivalent to (3.90).

The analysis demonstrates that the physical nonlinearity of the matrix has a significant effect on the microdamage of the laminated composite. The microdamage of the composite with linear-hardening matrix sets in at greater values of $\bar{t}_2$ and occurs more intensively than in the composite with linear elastic matrix, i.e., at great values of $\bar{t}_2$, the porosity of the composite with linear-hardening matrix is higher than in the composite with linear elastic matrix.

At small values of $\bar{t}_2$, the physical nonlinearity of the matrix has a significant effect on the stress state of the laminated composite as well. At great values of $\bar{t}_2$, the effect of nonlinearity on the stress state is weak.

3.4. Fibrous Composite Material. Let us consider the nonlinear deformation of a unidirectional fibrous composite with transversely isotropic fibers and isotropic matrix. The nonlinearity is due to the physical nonlinearity of the matrix and the accumulation of microdamages in it. The physically nonlinear deformation of the fibrous composite is described as the dependence of the bulk (K_2) and shear (μ_2) moduli of its matrix on strains. The microdamage of the matrix caused by loading is modeled by randomly arranged quasispherical micropores occurring in those microvolumes where the stresses exceed the ultimate microstrength. The macrostresses $\langle\sigma_{ij}\rangle$ and macrostrains $\langle\varepsilon_{ij}\rangle$ in an elementary macrovolume of a fibrous material are related by (3.85), where $\lambda^*_{11}, \lambda^*_{12}, \lambda^*_{13}, \lambda^*_{33}, \lambda^*_{44}$ are the effective elastic moduli of the composite, which are functions of the macrostrains $\langle\varepsilon_{ij}\rangle$ because of the physical nonlinearity and microdamage of the matrix.

Let the fibers be transversely isotropic and normal to the isotropy plane x_1x_2. Denote the elastic moduli of the fibers by $\lambda^1_{11}, \lambda^1_{12}, \lambda^1_{13}, \lambda^1_{33}, \lambda^1_{44}$, the bulk and shear moduli of the skeleton of the matrix by K_2, μ_2, its porosity by p_2, and the volume fractions of fibers and porous matrix by c_1 and c_2, respectively. The effective moduli of the physically nonlinear laminated composite with porous matrix can be determined using the following iterative algorithm. The effective moduli $\lambda^*_{11}, \lambda^*_{12}, \lambda^*_{13}$, $\lambda^*_{33}, \lambda^*_{44}$ of the composite are expressed [11, 36, 37, 39] in terms of the respective moduli of the fibers ($\lambda^1_{11}, \lambda^1_{12}, \lambda^1_{13}, \lambda^1_{33}, \lambda^1_{44}$) and the matrix ($\lambda_{2p}, \mu_{2p}, \lambda_{2p} = K_{2p} - 2\mu_{2p}/3$) as

$$\lambda^*_{11} + \lambda^*_{12} = c_1(\lambda^1_{11} + \lambda^1_{12}) + 2c_2(\lambda_{2p} + \mu_{2p}) - \frac{c_1c_2(\lambda^1_{11} + \lambda^1_{12} - 2\lambda_{2p} - 2\mu_{2p})^2}{2c_1(\lambda_{2p} + \mu_{2p}) + c_2(\lambda^1_{11} + \lambda^1_{12}) + 2m},$$

$$\lambda^*_{11} - \lambda^*_{12} = c_1(\lambda^1_{11} - \lambda^1_{12}) + 2c_2\mu_{2p} - \frac{c_1c_2(\lambda^1_{11} - \lambda^1_{12} - 2\mu_{2p})^2}{2c_1\mu_{2p} + c_2(\lambda^1_{11} - \lambda^1_{12}) + \dfrac{2mn}{n+2m}},$$

$$\lambda^{*(n)}_{13} = c_1\lambda^1_{13} + c_2\lambda^{(n)}_{2p} - \frac{c_1c_2(\lambda^1_{11} + \lambda^1_{12} - 2\lambda^{(n)}_{2p} - 2\mu^{(n)}_{2p})(\lambda^1_{13} - \lambda^{(n)}_{2p})}{2c_1(\lambda^{(n)}_{2p} + \mu^{(n)}_{2p}) + c_2(\lambda^1_{11} + \lambda^1_{12}) + 2m^{(n)}}, \tag{3.93}$$

$$\lambda^*_{33} = c_1\lambda^1_{33} + c_2(\lambda_{2p} + 2\mu_{2p}) - \frac{2c_1c_2(\lambda^1_{13} - \lambda_{2p})^2}{2c_1(\lambda_{2p} + \mu_{2p}) + c_2(\lambda^1_{11} + \lambda^1_{12}) + 2m},$$

$$\lambda^*_{44} = c_1\lambda^1_{44} + c_2\mu_{2p} - \frac{c_1c_2(\lambda^1_{44} - \mu_{2p})^2}{c_1\mu_{2p} + c_2\lambda^1_{44} + s}$$

$$(2m = c_1(\lambda^1_{11} - \lambda^1_{12}) + 2c_2\mu_{2p},\ 2n = c_1(\lambda^1_{11} + \lambda^1_{12}) + 2c_2(\lambda_{2p} + \mu_{2p}),\ s = c_1\lambda^1_{44} + c_2\mu_{2p} \tag{3.94}$$

if the matrix is stiffer than the fibers and

$$2m = \left(\frac{c_1}{\lambda^1_{11} - \lambda^1_{12}} + \frac{c_2}{2\mu_{2p}}\right)^{-1},\quad 2n = \left(\frac{c_1}{\lambda^1_{11} + \lambda^1_{12}} + \frac{c_2}{2(\lambda_{2p} + \mu_{2p})}\right)^{-1},\quad s = \left(\frac{c_1}{\lambda^1_{44}} + \frac{c_2}{\mu_{2p}}\right)^{-1} \tag{3.95}$$

if the fibers are stiffer than the matrix.

The effective moduli $K_{2p}, \lambda_{2p}, \mu_{2p}$ $(\lambda_{2p} = K_{2p} - 2\mu_{2p}/3)$ of the porous matrix are defined by formulas (3.53), $\nu = 2$, according to [35]. The average strains $\langle \varepsilon_{ij}^{12} \rangle$ appearing in these formulas are determined using the following iterative algorithm. The $(n+1)$th approximation is related to the nth approximation by (3.54), $\nu = 2$. The average strains $\langle \varepsilon_{ij}^{2} \rangle$ are determined in terms of the macrostrains $\langle \varepsilon_{ij} \rangle$ by the formulas

$$\langle \varepsilon_{ij}^{2} \rangle = \frac{\lambda_{11}^{*} - \lambda_{12}^{*} - \lambda_{11}^{1} + \lambda_{12}^{1}}{c_2 (2\mu_{2p} - \lambda_{11}^{1} + \lambda_{12}^{1})} \langle \varepsilon_{ij} \rangle$$

$$- \frac{1}{\Delta_2} \{[(\lambda_{11}^{*} - \lambda_{11}^{1}) a_1 - (\lambda_{12}^{*} - \lambda_{12}^{1}) a_2 - (\lambda_{13}^{*} - \lambda_{13}^{1}) a_3] \langle \varepsilon_{rr} \rangle$$

$$+ [(\lambda_{13}^{*} - \lambda_{13}^{1})(a_1 - a_2) - (\lambda_{33}^{*} - \lambda_{33}^{1}) a_3] \langle \varepsilon_{33} \rangle\} \delta_{ij}$$

$$\langle \varepsilon_{33}^{2} \rangle = -\frac{1}{\Delta_2^{(n)}} \{[(\lambda_{13}^{*} - \lambda_{13}^{1}) a_4 - (\lambda_{11}^{*} + \lambda_{12}^{*} - \lambda_{11}^{1} - \lambda_{12}^{1}) a_3] \langle \varepsilon_{rr} \rangle$$

$$+ [(\lambda_{33}^{*} - \lambda_{33}^{1}) a_4 - 2(\lambda_{13}^{*} - \lambda_{13}^{1}) a_3] \langle \varepsilon_{rr} \rangle\}, \tag{3.96}$$

$$\langle \varepsilon_{i3}^{2} \rangle = \frac{\lambda_{44}^{*(n)} - \lambda_{44}^{1}}{c_2 (\mu_{2p} - \lambda_{44}^{1})} \langle \varepsilon_{i3} \rangle \quad (i, j, r = 1, 2),$$

$$(\Delta_2 = c_2 (\lambda_{11}^{1} - \lambda_{12}^{1} - 2\mu_{2p}) [(\lambda_{11}^{1} + \lambda_{12}^{1} - 2\lambda_{2p} - 2\mu_{2p})(\lambda_{33}^{1} - \lambda_{2p} - 2\mu_{2p}) - 2(\lambda_{13}^{1} - \lambda_{2p})^2],$$

$$a_1 = (\lambda_{13}^{1} - \lambda_{2p})^2 - (\lambda_{12}^{1} - \lambda_{2p})(\lambda_{33}^{1} - \lambda_{2p} - 2\mu_{2p}),$$

$$a_2 = (\lambda_{13}^{1} - \lambda_{2p})^2 - (\lambda_{11}^{1} - \lambda_{2p} - 2\mu_{2p})(\lambda_{33}^{1} - \lambda_{2p} - 2\mu_{2p}),$$

$$a_3 = (\lambda_{13}^{1} - \lambda_{2p})(\lambda_{11}^{1} - \lambda_{12}^{1} - 2\mu_{2p}),$$

$$a_4 = (\lambda_{11}^{1} + \lambda_{12}^{1} - 2\lambda_{2p} - 2\mu_{2p})(\lambda_{11}^{1} - \lambda_{12}^{1} - 2\mu_{2p}), \tag{3.97}$$

and the effective moduli $K_{2p}, \lambda_{2p}, \mu_{2p}$ are defined by (3.53).

The zero-order approximation represents a physically linear matrix.

We will use the Huber–Mises criterion (3.56), $\nu = 2$, as a condition for the formation of a microdamage in a microvolume of the undamaged portion of the matrix. If the invariant $I_{\langle \sigma \rangle}^{12}$ does not reach the limiting value k_2 in some microvolume of the matrix, then, according to the stress-rupture criterion, failure will occur in some time τ_k^2 dependent on the difference between $I_{\langle \sigma \rangle}^{12}$ and k_2. In the general case, this dependence can be represented as some function (3.62). The one-point distribution function $F_2(k_2)$ for some microvolume in the undamaged portion of the matrix can be approximated by a power function on some interval (3.63) or by Weibull function (3.64).

Assume that the random field of ultimate microstrength k_2 is statistically homogeneous in real materials, and its correlation scale and the size of single microdamages and the distances between them are negligible compared with the macrovolume. Then the random field k_2 and the distribution of macrostresses in the matrix under uniform loading are ergodic, and the distribution function $F_2(k_2)$ defines the fraction of the undamaged portion of the matrix in which the ultimate microstrength is less than k_2. Therefore, if the stresses $\langle \sigma_{ij}^{12} \rangle$ are nonzero, the function $F_2(I_{\langle \sigma \rangle}^{12})$ defines, according to (3.56), (3.63), and (3.64), the content of instantaneously damaged microvolumes of the skeleton of the matrix. Since the damaged microvolumes are modeled by pores, we can write a porosity balance equation in the form (3.65), $\nu = 2$, where p_{20} is the initial porosity. Given macrostrains $\langle \varepsilon_{ij} \rangle$, the average stresses $\langle \sigma_{ij}^{12} \rangle$ in the undamaged portion of the matrix are related to the average macrostrains $\langle \sigma_{ij}^{2} \rangle$ in the matrix by (3.57), $\nu = 2$. The average stresses $\langle \sigma_{ij}^{2} \rangle$ are related to the average strains $\langle \varepsilon_{ij}^{2} \rangle$ by

$$\langle \sigma_{ij}^{2} \rangle = \lambda_{2p} \langle \varepsilon_{rr}^{2} \rangle \delta_{ij} + 2\mu_{2p} \langle \varepsilon_{ij}^{2} \rangle, \tag{3.98}$$

and the average strains $\langle \varepsilon_{ij}^{2} \rangle$ in the matrix are related to the macrostrains $\langle \varepsilon_{ij} \rangle$ by (3.96), (3.97), the effective moduli λ_{2p}, μ_{2p} of the porous matrix being given by (3.53), $\nu = 2$.

If the stresses $\langle \sigma_{ij}^{2} \rangle$ act for some time t, then, according to the stress-rupture criterion (3.62), those microvolumes of the matrix are damaged that have k_2 satisfying inequality (3.67), where $I_{\langle\sigma\rangle}^{12}$ is defined by (3.96), (3.97).

The time to brittle failure τ_k^2 for real materials at low temperatures is finite beginning only from some value of $I_{\langle\sigma\rangle}^{12} > 0$. In this case, the durability function $\varphi_2(I_{\langle\sigma\rangle}^{12}, k_2)$ can be represented as (3.68). If the time τ_k^2 is finite for arbitrary values of $I_{\langle\sigma\rangle}^{12}$, which may be observed at high temperatures, then the durability function can be represented as (3.72). Substituting (3.68) or (3.72) into (3.67) and considering the definition of the distribution function $F_2(k_2)$, we conclude that the function $F_2[I_{\langle\sigma\rangle}^{12} \psi_2(\bar{t}_2)]$, where $\psi_2(\bar{t}_2)$ is defined by (3.70) or (3.74), respectively, defines the relative content of the destroyed microvolumes in the undamaged portion of the matrix at the time $\bar{t}_2$. Then, in view of (3.57), the porosity balance equation for the matrix subject to long-term damage can be represented in the form (3.71), $\nu = 2$, where p_2 is a function of dimensionless time $\bar{t}_2$, and the average stresses $\langle \sigma_{jk}^{2} \rangle$ are defined by (3.96), (3.97).

At $\bar{t}_2 = 0$, the porosity balance equation (3.71) with (3.96), (3.97), (3.70) (or (3.74)) defines the short-term (instantaneous) damage of the matrix. As time elapses, Eq. (3.71) with (3.96), (3.97), (3.70) (or (3.74)) defines its long-term damage, which consists of short-term damage and additional time-dependent damage.

Equations (3.48), (3.93)–(3.95), (3.71), (3.96)–(3.97), (3.70) (or (3.74)) form a closed-form system describing the coupled processes of statistically homogeneous physically nonlinear deformation and long-term damage of a fibrous composite. The physical nonlinearity of the matrix affects the way pores form during deformation, and the porosity of the matrix has an effect on the stress–strain curve of the composite. This is why the nonlinearity of the stress–strain curve of the fibrous composite is determined by the physical nonlinearity of its matrix and the increase in the porosity during physically nonlinear deformation.

Let us analyze, as an example, the coupled processes of nonlinear deformation and long-term microdamage of a fibrous composite with linear elastic fibers and microdamaged nonlinear elastic matrix with bulk strains being linear and shear strains described by a linear-hardening diagram (3.75)–(3.77). Formulas (3.48), (3.93)–(3.95), (3.71), (3.96), (3.97), (3.70) (or (3.74)) and the secant method [4] can be used to develop an iterative algorithm for the determination of the stress–strain state of a physically nonlinear fibrous composite and the volume fraction of microdamages in its matrix. We conducted calculations to plot macrodeformation curves for a fibrous composite with microdamaged matrix for Weibull distribution (3.64) and for fractional power durability function $\psi_2(\bar{t}_2)$ defined by (3.70). Let the composite consist of an epoxy matrix described by linear-hardening diagram (3.75)–(3.77) with constants (3.79) and proportional limits and minimum tensile microstrength (3.80), (3.81) and high-modulus carbon fibers with the following characteristics [22]:

$$E_1^1 = 8\,\text{GPa}, \quad \nu_{12}^1 = 0.2, \quad \nu_{13}^1 = 0.3, \quad G_{12}^1 = 60\,\text{GPa}, \tag{3.99}$$

where E_1^1 and E_3^1, ν_{12}^1 and ν_{13}^1, G_{12}^1 and G_{13}^1 are, respectively, the transverse and longitudinal Young's moduli, Poisson's ratios, shear moduli of the fibers, which are related to the elastic moduli $\lambda_{11}^1, \lambda_{12}^1, \lambda_{13}^1, \lambda_{33}^1, \lambda_{44}^1$ by

$$\lambda_{11}^1 + \lambda_{12}^1 = E_1^1 E_3^1 \left[E_3^1 \left(2 - \frac{E_1^1}{2G_{12}^1} \right) - 2E_1^1 (\nu_{13}^1)^2 \right]^{-1}, \quad \lambda_{11}^1 - \lambda_{12}^1 = 2G_{12}^1,$$

$$\lambda_{13}^1 = \nu_{13}^1 (\lambda_{11}^1 + \lambda_{12}^1), \quad \lambda_{33}^1 = (\lambda_{11}^1 + \lambda_{12}^1) \frac{E_3^1}{E_1^1} \left(2 - \frac{E_1^1}{2G_{12}^1} \right), \quad \lambda_{44}^1 = G_{13}^1. \tag{3.100}$$

If

$$\langle \varepsilon_{11} \rangle \neq 0, \quad \langle \sigma_{22} \rangle = \langle \sigma_{33} \rangle = 0, \tag{3.101}$$

then, according to (3.85), the macrostress $\langle \sigma_{11} \rangle$ is related to the macrostrain $\langle \varepsilon_{11} \rangle$ by

$$\langle \sigma_{11} \rangle = \frac{\lambda_{11}^* - \lambda_{12}^*}{\lambda_{11}^* \lambda_{33}^* - (\lambda_{13}^*)^2} [(\lambda_{11}^* + \lambda_{12}^*)\lambda_{33}^* - 2(\lambda_{13}^*)^2] \langle \varepsilon_{11} \rangle. \tag{3.102}$$

In the porosity balance equation (3.70), (3.71), (3.96), (3.97), we use

$$\langle \varepsilon_{22} \rangle = \frac{(\lambda_{13}^*)^2 - \lambda_{12}^* \lambda_{33}^*}{\lambda_{11}^* \lambda_{33}^* - (\lambda_{13}^*)^2} \langle \varepsilon_{11} \rangle, \quad \langle \varepsilon_{33} \rangle = \frac{(\lambda_{12}^* - \lambda_{11}^*)\lambda_{13}^*}{\lambda_{11}^* \lambda_{33}^* - (\lambda_{13}^*)^2} \langle \varepsilon_{11} \rangle, \tag{3.103}$$

which is equivalent to (3,101).

The analysis demonstrates that the physical nonlinearity of the matrix has a significant effect on the microdamage of the fibrous composite. The microdamage of the composite with linear-hardening matrix sets in at greater values of $\bar{t}_2$ and occurs more intensively than in the composite with linear elastic matrix, i.e., at great values of $\bar{t}_2$, the porosity of the composite with linear-hardening matrix is higher than in the composite with linear elastic matrix.

At small values of $\bar{t}_2$, the physical nonlinearity of the matrix has a significant effect on the stress state of the fibrous composite as well. At great values of $\bar{t}_2$, the effect of nonlinearity on the stress state is weak.

Conclusions. The mathematical theory of coupled processes of deformation and damage of physically nonlinear materials has been developed based on the stochastic equations of elasticity of porous materials whose skeleton is physically nonlinear. The damage of a material has been modeled by dispersed microvolumes destroyed to become randomly arranged micropores. A microdamage of a single microvolume has been characterized by its ultimate strength according to the Huber–Mises failure criterion or by its stress-rupture strength described by a fractional or exponential power function, which is determined by the dependence of the time to brittle fracture on the difference between the equivalent stress and its limit (ultimate strength according to the Huber–Mises criterion). The ultimate microstrength has been assumed to be a random function of coordinates whose one-point distribution is described by a power function on some interval or by the Weibull function. The effective elastic properties and the stress–strain state of a physically nonlinear material with randomly arranged microdamages have been determined from the stochastic equations of elasticity of physically nonlinear porous materials. The equation of damage (porosity) balance at an arbitrary time has been derived from the properties of the distribution functions and ergodicity of the random field of ultimate microstrength and the dependence of the time to brittle failure for a microvolume on its stress state and ultimate microstrength. The macrostress–macrostrain relations for a physically nonlinear porous material and the porosity balance equation form a closed-loop system describing the coupled processes of physically nonlinear deformation and microdamage. An iteration method has been used to develop algorithms for calculating the macrostresses and microdamage as functions of macrostrains and time and to plot the respective curves. The influence of nonlinearity on the deformation and microdamage of materials has been analyzed.

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