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FRACTURE OF A HALF-SPACE COMPRESSED ALONG A PENNY-SHAPED CRACK LOCATED AT A SHORT DISTANCE FROM THE SURFACE

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Non-classical problems of fracture mechanics for a half-space with a crack located at a short distance from the free surface are solved. An axisymmetric problem for a penny-shaped crack is considered. A numerical analysis is performed for materials with harmonic and Bartenev–Khazanovich potentials in the case of unequal roots of the characteristic equation. This case is similar in mathematical structure of the equations to the case of equal roots in the classical theory of isotropic elasticity. The numerical results are tabulated and analyzed

Keywords: near-surface crack, penny-shaped crack, compression, critical stress

Introduction. When the forces acting on a body with a plane crack are parallel to the crack plane, the stress intensity factors predicted by linear fracture mechanics are equal to zero and the Irwin–Griffith failure criteria are inapplicable. When a load acts along the crack plane, use is made of the approach proposed in [3–6] and employed and developed in [2, 6–13, 14–17]. The failure criterion in this case is local loss of stability near the crack within the framework of the three-dimensional linearized theory of elastic stability [3]. According to this approach, fracture is initiated by local loss of stability near cracks, and the critical compressive loads are determined by solving eigenvalue problems within the framework of the three-dimensional linearized theory of elastic stability.

In [2, 3, 6–13, 14–17], the critical compressive strains and stresses were determined for different positions of interacting cracks and for different distances between cracks and between the crack and the free surface. In [2, 7–12], for example, the relationship between β and the critical compressive strain (stress) was established for a near-surface circular crack (Fig. 1: $\beta = ha^{-1}$, where h is the distance between the free surface and the crack plane; a is the radius of the crack) and potentials of different types. It is also important to know the behavior of the critical parameters in asymptotic cases ($\beta \to \infty$, $\beta \to 0$). Such asymptotics as $\beta \to \infty$ were obtained in [6, 9]. The critical compressive stress tends to the level corresponding to a single crack and equal values of these parameters.

The situation is more complicated when $\beta \to 0$. In this case, the available numerical methods fail to determine the critical stresses because the matrix determinants that should be equated to zero to find the critical points appear close to zero and the calculation errors show up. Note that approximate design models (buckling of a thin element isolated by the crack and the free surface) were used to find solutions to the problem. However, it is necessary to choose boundary conditions for this thin plate. Boundary conditions may range from simple support to clamping, which yields values differing by a factor of 4 for a rectangular plate under uniaxial compression and by a factor of 3.5 for a circular plate under uniform biaxial compression. This issue of boundary conditions is of theoretical and practical interest when designing thin interlayers formed after spraying, thermal shock, etc.

Since it is difficult to analytically solve such problems as $\beta \to 0$, we used a Wolfram Mathematica package for symbolic mathematics for arbitrary β. This allowed using the method for solving integral equations proposed in [18] to obtain a solution for short distances between the crack and the free surface in the case of equal roots of the characteristic equation.

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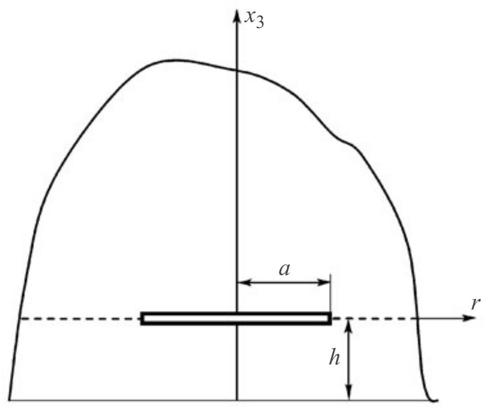

Fig. 1

1. Problem Formulation and General Solution. Let us consider a penny-shaped crack of radius a in the half-space $x_3 \geq -h$. The crack is located in the plane $x_3 = 0$ and its center is on the Ox_3-axis. The prestresses acting along the crack are induced by biaxial tension/compression [6]:

$$S_{33}^0 = 0, \quad S_{11}^0 = S_{22}^0 \neq 0, \quad u_m^0 = \delta_{jm}(\lambda_j - 1)x_j, \quad \lambda_1 = \lambda_2 \neq \lambda_3, \quad \lambda_j = \text{const},$$

where λ_j are the elongations along the axes; x_j are the Lagrange coordinates that coincide with the Cartesian coordinates in undeformed state; S_{ij}^0 are the components of the symmetric stress tensor; u_j^0 are the displacements corresponding to the prestresses S_{ij}^0.

For the axisymmetric linearized problem, we will use the following boundary conditions on the crack faces $x_3 = \pm 0$ and on the free surface $x_3 = -h$ [9]:

$$t_{33} = 0, \quad x_3 = \pm 0, \quad 0 \leq r < a, \qquad t_{3r} = 0, \quad x_3 = \pm 0, \quad 0 \leq r < a,$$

$$t_{33} = 0, \quad x_3 = -h, \quad 0 \leq r < \infty, \qquad t_{3r} = 0, \quad x_3 = -h, \quad 0 \leq r < \infty, \tag{1.1}$$

where t_{ij} is the asymmetrical Kirchhoff stress tensor; (r, θ, x_3) are the cylindrical coordinates corresponding to the Cartesian coordinates x_j.

In the case of equal roots of the characteristic equation, the linearized equations of the axisymmetric problem have the following generalized solutions [6]:

$$u_r = -\frac{\partial \varphi}{\partial r} - z_1 \frac{\partial F}{\partial r},$$

$$u_3 = (m_1^0 - m_2^0 + 1)(n_1^0)^{-1/2} F - m_1^0 (n_1^0)^{-1/2} \Phi - m_1^0 (n_1^0)^{-1/2} \frac{\partial F}{\partial z_1},$$

$$t_{33} = C_{44}^0 \left\{ (d_1^0 l_1^0 - d_2^0 l_2^0) \frac{\partial F}{\partial z_1} - d_1^0 l_1^0 \frac{\partial \Phi}{\partial z_1} - d_1^0 l_1^0 z_1 \frac{\partial^2 F}{\partial z_1^2} \right\},$$

$$t_{3r} = C_{44}^0 \left\{ (n_1^0)^{-1/2} \frac{\partial}{\partial r} \left[(d_1^0 - d_2^0) F - d_1^0 \Phi - d_1^0 z_1 \frac{\partial F}{\partial z_1} \right] \right\}, \tag{1.2}$$

where $\varphi = -(\varphi_1 + \varphi_2)$, $F = -\dfrac{\partial \varphi_2}{\partial z_1}$, $\Phi = \dfrac{\partial \varphi}{\partial z_1}$ are harmonic functions satisfying the Laplace equation; $z_1 = (n_1^0)^{-1/2} x_3$.

2. Fredholm Equations. Following [9], we obtain dual integral equations.

Let us partition the half-space $x_3 \geq -h$ into sections 1 ($x_3 \geq 0$) and 2 ($-h \leq x_3 \leq 0$) on which the potential function is represented as the Hankel transform:

$$\varphi^{(1)}(r, z_1) = -\int_0^\infty B(\lambda) e^{-\lambda z_1} J_0(\lambda r)\frac{d\lambda}{\lambda}, \quad F^{(1)}(r, z_1) = \int_0^\infty A(\lambda) e^{-\lambda z_1} J_0(\lambda r) d\lambda,$$

$$\Phi^{(1)}(r, z_1) = \int_0^\infty B(\lambda) e^{-\lambda z_1} J_0(\lambda r) d\lambda,$$

$$\varphi^{(2)}(r, z_1) = \int_0^\infty [D_1(\lambda)\sinh\lambda(z_1 + h_1) + D_2(\lambda)\sinh\lambda(z_1 + h_1)] J_0(\lambda r)\frac{d\lambda}{\lambda \sinh \lambda h_1},$$

$$F^{(2)}(r, z_2) = \int_0^\infty [C_1(\lambda) + \cosh\lambda(z_1 + h_1) + C_2(\lambda) + \sinh\lambda(z_1 + h_1)] J_0(\lambda r)\frac{d\lambda}{\sinh \lambda h_1},$$

$$\Phi^{(2)}(r, z_1) = \int_0^\infty [D_1(\lambda) + \cosh\lambda(z_1 + h_1) + D_2(\lambda) + \sinh\lambda(z_1 + h_1)] J_0(\lambda r)\frac{d\lambda}{\sinh \lambda h_1}, \tag{2.1}$$

where $h_1 = (n_1^0)^{-1/2} h$; $A(\lambda), B(\lambda), C_i(\lambda), D_i(\lambda)$ $(i = 1, 2)$ are unknown functions.

Following [9], we use (1.1), (1.2), and (2.1), we obtain the following dual integral equations:

$$d_1 l_1^0 \int_0^\infty [C_1(k - \mu_1 \coth\mu_1) - C_2 \mu_1] J_0(\lambda r)\lambda \mathrm{d}\lambda = 0 \quad (0 \leq r < a),$$

$$(n_1^0)^{-1/2} d_1 \int_0^\infty [(\mu_1 C_1 + C_2(k + \mu_1 \coth\mu_1)] J_0(\lambda r) \mathrm{d}\lambda = \tilde{c} \quad (r < a),$$

$$\int_0^\infty x_1 J_0(\lambda r) \mathrm{d}\lambda = 0 \quad (r > a), \quad \int_0^\infty x_2 J_0(\lambda r)\lambda \mathrm{d}\lambda = 0 \quad (r > a), \tag{2.2}$$

where $\tilde{c} = \text{const}$ has units of length, and the functions in (2.1) are expressed in terms of the functions C_1 and C_2 as

$$D_1(\lambda) = \left(1 - \frac{d_1^0}{d_2^0}\right) C_1(\lambda) + \mu_1 C_2(\lambda), \quad D_2(\lambda) = \left(1 - \frac{d_2^0 l_2^0}{d_1^0 l_1^0}\right) C_2(\lambda) + \mu_1 C_1(\lambda),$$

$$A(\lambda) = C_1(\lambda)\left[\frac{\mu_1}{k}(1 + \coth\mu_1) - 1\right] + C_2(\lambda)\left[\frac{\mu_1}{k}(1 + \coth\mu_1) + 1\right], \tag{2.3}$$

$$B(\lambda) = C_1(\lambda)\left[\frac{\mu_1}{k}\left(1 - \frac{d_2^0 l_2^0}{d_1^0 l_1^0}\right)(1 + \coth\mu_1) - \left(1 - \frac{d_2^0}{d_1^0}\right) - \mu_1 \coth\mu_1\right]$$

$$+ C_2(\lambda)\left[\frac{\mu_1}{k}\left(1 - \frac{d_2^0 l_2^0}{d_1^0 l_1^0}\right)(1 + \coth\mu_1) + \left(1 - \frac{d_2^0 l_2^0}{d_1^0 l_1^0}\right) - \mu_1\right],$$

$$x_1 = \left(C_1\left(1 - \frac{\mu_1}{k}\right) - C_2 \frac{\mu_1}{k}\right)(\coth\mu_1 + 1), \quad x_2 = \left(C_1 \frac{\mu_1}{k} + C_2\left(1 + \frac{\mu_1}{k}\right)\right)(\coth\mu_1 + 1),$$

$$\mu_1 = \lambda h_1, \qquad k = \frac{(l_1^0 - l_2^0) d_2^0}{l_1^0 d_1^0}. \tag{2.4}$$

Let the solution of the dual integral equations have the form

$$x_1 = \frac{1}{\lambda} \int_0^a \varphi(t)(\cos \lambda t - \cos \lambda a) dt, \qquad x_2 = h \int_0^a \psi(t) \cos \lambda t \, dt, \tag{2.5}$$

where $\varphi(t)$ and $\psi(t)$ are unknown functions continuous together with their first derivatives on the interval $[0, a]$.

Performing transformations [9] and nondimensionalizing the variables and functions, we obtain a system of two Fredholm equations of the second kind with an additional condition:

$$f(\xi) + \frac{1}{\pi k} \int_0^1 M_1(\xi, \eta) f(\eta) d\eta - \frac{2}{\pi k} \int_0^1 N_1(\xi, \eta) g(\eta) d\eta = 0,$$

$$g(\xi) + \frac{1}{\pi k} \int_0^1 M_2(\xi, \eta) g(\eta) d\eta - \frac{2}{\pi k} \int_0^1 N_2(\xi, \eta) f(\eta) d\eta + \widetilde{C}_1 = 0,$$

$$\int_0^1 g(\xi) d\xi = 0 \quad (0 \le \xi \le 1, 0 \le \eta \le 1), \qquad f(\xi) \equiv \varphi(a\xi), \qquad g(\xi) \equiv \psi(a\xi), \tag{2.6}$$

where $\widetilde{C}_1$ is an unknown constant related to the additional condition.

The kernels of the integral equations are given by

$$M_1(\xi, \eta) = R_1(\eta + \xi) - R_1(1 + \xi) + R_1(\eta - \xi) - R_1(1 - \xi),$$

$$N_1(\xi, \eta) = S_1(\eta + \xi) + S_1(\eta - \xi), \qquad M_2(\xi, \eta) = S_2(\eta + \xi) + S_2(\eta - \xi),$$

$$N_2(\xi, \eta) = R_2(\eta + \xi) - R_2(1 + \xi) + R_2(\eta - \xi) - R_2(1 - \xi),$$

$$R_1(\zeta) = -2\left[\frac{k}{2} L_0(\zeta) + \frac{1}{k} L_2(\zeta) + L_1(\zeta)\right], \qquad R_2(\zeta) = -\frac{1}{k} L_1(\zeta),$$

$$S_1(\zeta) = -\frac{1}{k} L_3(\zeta), \qquad S_2(\zeta) = -2\left[\frac{k}{2} L_0(\zeta) + \frac{1}{k} L_2(\zeta) - L_1(\zeta)\right],$$

$$L_0(\zeta) = \frac{\beta_1}{\beta_1^2 + \zeta^2}, \qquad L_1(\zeta) = \frac{\beta_1}{2} \frac{\beta_1^2 - \zeta^2}{(\beta_1^2 + \zeta^2)^2}, \qquad L_2(\zeta) = \frac{\beta_1^3}{2} \frac{4\beta_1^2 - 3\zeta^2}{(\beta_1^2 + \zeta^2)^3},$$

$$L_2(\zeta) = \frac{\beta_1}{2} \frac{\zeta^4 - 6\beta_1^2 \zeta^2 + \beta_1^4}{(\beta_1^2 + \zeta^2)^4}, \qquad \beta_1 = 2(n_1^0)^{-1/2} \beta, \qquad \beta = h a^{-1}. \tag{2.7}$$

3. Procedure of Analysis. The further analysis employs Wolfram Mathematica 7, which is an efficient package for symbolic computation with guaranteed accuracy.

To determine the critical compressive and tensile strains and stresses from the integral equations (2.6), we use a procedure based on the Bubnov–Galerkin method. Let the coordinate functions be power functions. For N coordinate functions, we have

$$f(x) = \sum_{i=0}^{N} F_i x^i, \qquad g(x) = \sum_{i=0}^{N} G_i x^i. \tag{3.1}$$

Unlike the previous studies [2, 8, 9] where system (2.6) was numerically integrated after substitution of the coordinate functions, we use Mathematica for analytic evaluation of the integrals for the chosen system of coordinate functions. This allows us to increase the accuracy of computation by excluding the error of numerical integration.

To accelerate the evaluation of integrals, we use an algorithm based on recurrence formulas.

Let us introduce the following functions:

$$L(n)=\int_0^1 \frac{x^n}{(a^2+(x+y)^2)^2}\,dx, \qquad V(n)=\int_0^1 \frac{x^n}{(a^2+(x-y)^2)^2}\,dx, \tag{3.2}$$

$$LL(n)=\int_0^1 \frac{x^n}{(a^2+(x+y)^2)^4}\,dx, \qquad VV(n)=\int_0^1 \frac{x^n}{(a^2+(x-y)^2)^4}\,dx\,. \tag{3.3}$$

We use the following recurrence formulas to evaluate integrals (3.2) and (3.3):

$$L(n)=\frac{1}{n-3}\left(\frac{1}{a^2+(1+y)^2}-2y(n-2)V(n-1)-(a^2+y^2)(n-1)V(n-2)\right) \quad (n\neq 3),$$

$$V(n)=\frac{1}{n-3}\left(\frac{1}{a^2+(1-y)^2}+2y(n-2)V(n-1)-(a^2+y^2)(n-1)V(n-2)\right) \quad (n\neq 3),$$

$$LL(n)=\frac{1}{n-5}\left(\frac{1}{(a^2+(x+y)^2)^3}-2y(n-4)LL(n-1)-(a^2+y^2)(n-1)LL(n-2)\right) \quad (n\neq 5),$$

$$VV(n)=\frac{1}{n-5}\left(\frac{1}{(a^2+(1-y)^2)^3}+2y(n-4)VV(n-1)-(a^2+y^2)(n-1)VV(n-2)\right) \quad (n\neq 5). \tag{3.4}$$

Having the values of integrals (3.2) for $n=0,\ 1,\ 3$ and (3.3) for $n=0,\ 1,\ 5$ and using the recurrence formulas (3.4), we can evaluate the following integrals:

$$\int_0^1 \frac{x^n}{a^2+(x+y)^2}\,dx=\frac{1}{n+1}\left(\frac{1}{a^2+(1+y)^2}+2L(n+2)+2yL(n+1)\right),$$

$$\int_0^1 \frac{x^n(a^2-(x+y)^2)^2}{(a^2+(x+y)^2)^2}\,dx=(a^2-y^2)L(n)-2yL(n+1)-L(n+2),$$

$$\int_0^1 \frac{x^n(a^2-3(x+y)^2)^2}{(a^2+(x+y)^2)^3}\,dx=4a^2(LL(n)(a^2+y^2)+LL(n+1)2y+LL(n+2))-3L(n),$$

$$\int_0^1 \frac{x^n(a^4-6a^2(x+y)^2+(x+y)^4)^2}{(a^2+(x+y)^2)^4}\,dx=L(n)-8a^2(y^2LL(n)+2yLL(n+1)+LL(n+2)),$$

$$\int_0^1 \frac{x^n}{a^2+(x-y)^2}\,dx=\frac{1}{n+1}\left(\frac{1}{a^2+(1-y)^2}+2V(n+2)-2yV(n+1)\right),$$

$$\int_0^1 \frac{x^n(a^2-(x-y)^2)^2}{(a^2+(x-y)^2)^2}\,dx=(a^2-y^2)V(n)+2yV(n+1)-V(n+2),$$

TABLE 1

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\varepsilon \cdot 10^2$	0.73	2.63	5.32	8.45	11.65	14.62	17.18	19.29	21.00	22.37

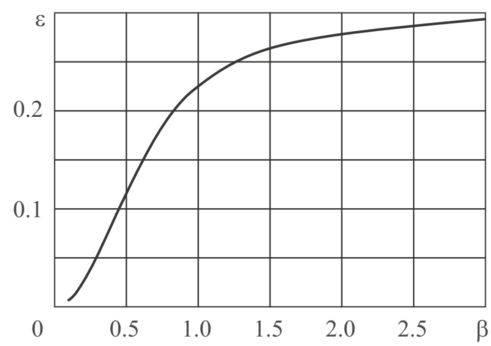

Fig. 2

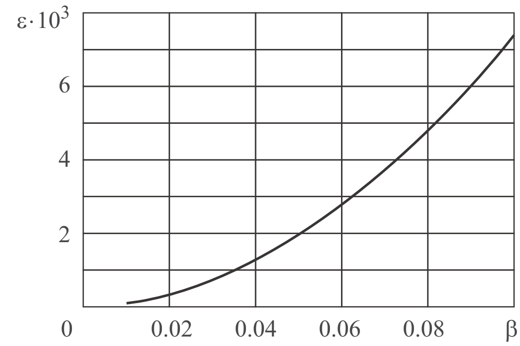

Fig. 3

$$\int_0^1 \frac{x^n (a^2 - 3(x-y)^2)^2}{(a^2 + (x-y)^2)^3} dx = 4a^2 (VV(n)(a^2 + y^2) - VV(n+1)2y + VV(n+2)) - 3V(n),$$

$$\int_0^1 \frac{x^n (a^4 - 6a^2 (x-y)^2 + (x-y)^4)^2}{(a^2 + (x-y)^2)^4} dx = V(n) - 8a^2 (y^2 VV(n) - 2yVV(n+1) + VV(n+2)). \tag{3.5}$$

Substituting (3.1) into the Fredholm equations of the second kind (2.6) and using (3.5) for integrating the kernels of the Fredholm equations (2.7), we obtain a system of $2(N+1)+1$ equations with the same number of unknowns $F_i, G_i, \widetilde{C}_1, i \in [0, N]$. Let us represent it as

$$\sum_{i=0}^{N} F_i F_{1ji} + \sum_{i=0}^{N} G_i G_{1ji} = 0,$$

$$\sum_{i=0}^{N} F_i F_{2ji} + \sum_{i=0}^{N} G_i G_{2ji} + \widetilde{C}_1 = 0,$$

$$\sum_{i=0}^{N} \frac{1}{i+1} G_i = 0, \quad 0 \le j \le N, \tag{3.6}$$

where F_{kji} and G_{kji} are the exact expressions derived using Wolfram Mathematica; they depend on the dimensionless distance β_1 between the crack and the free surface and on k.

4. Numerical Results. Let us consider, as an example, a near-surface penny-shaped crack in materials with Bartenev–Khazanovich potential and with harmonic potential (case of equal roots).

Bartenev–Khazanovich potential:

$$n_1^0 = n_2^0 = \lambda_1^3, \quad k = (3\lambda_1^3 - 1)/(1 + \lambda_1^3), \quad \beta_1 = 2(n_1^0)^{-1/2} \beta. \tag{4.1}$$

TABLE 2

β	ε	A
$9\cdot10^{-2}$	$5.96074\cdot10^{-3}$	–2.24766
$8\cdot10^{-2}$	$4.76082\cdot10^{-3}$	–2.26383
$7\cdot10^{-2}$	$3.68500\cdot10^{-3}$	–2.28126
$6\cdot10^{-2}$	$2.73743\cdot10^{-3}$	–2.30003
$5\cdot10^{-2}$	$1.92240\cdot10^{-3}$	–2.32024
$4\cdot10^{-2}$	$1.24441\cdot10^{-3}$	–2.34200
$3\cdot10^{-2}$	$7.08118\cdot10^{-4}$	–2.36541
$2\cdot10^{-2}$	$3.18445\cdot10^{-4}$	–2.39062
$1\cdot10^{-2}$	$8.05725\cdot10^{-5}$	–2.41776
$1\cdot10^{-3}$	$8.14575\cdot10^{-7}$	–2.44373
$1\cdot10^{-4}$	$8.15063\cdot10^{-9}$	–2.44519
$1\cdot10^{-5}$	$8.14758\cdot10^{-11}$	–2.44427
$1\cdot10^{-6}$	$8.14606\cdot10^{-13}$	–2.44382
$1\cdot10^{-7}$	$8.14575\cdot10^{-15}$	–2.44373
$1\cdot10^{-8}$	$8.14575\cdot10^{-17}$	–2.44373
$1\cdot10^{-9}$	$8.14575\cdot10^{-19}$	–2.44373

Substituting (4.1) into (3.6), we obtain a system of equations with coefficients F_{kji} and G_{kji} dependent on the parameters β and λ_1. Analyzing this system numerically, we can determine the minimum critical strain $\varepsilon = 1-\lambda_1$ and critical stress at which the system loses stability for different values of the dimensionless distance β between the crack and the free surface.

Using 20 coordinate functions, we calculate the function $\varepsilon(\beta)$ (see Table 1 and Fig. 2). These results are in good agreement with the data obtained in [9] which means that the proposed method provides good accuracy.

Using approximate design models for an isolated plane disk-shaped plate under compression, we determine the critical stress $\sigma_{\mathrm{cr}} = K(D/(b^2 a))$, where D is a coefficient depending on the material properties; a and b are the thickness and radius of the plate; K is a coefficient depending on the boundary condition of the plate. For clamped boundary conditions and Bartenev–Khazanovich potential, we have $\sigma_{\mathrm{cr}} = A_{\mathrm{cr}}\beta^2\mu$ ($A_{\mathrm{cr}} = -2.44667$).

Assuming that σ_{11}^0 behaves as $A\beta^2\mu$ as $\beta \to 0$, we numerically analyze the system of equations (3.5) for small values of β. The calculated values of the function $\varepsilon(\beta)$ and coefficient A for $0.01 < \beta < 0.1$ are presented in Table 2 and Fig. 3.

Harmonic potential:

TABLE 3

β	ε				
	ν = 0.1	ν = 0.2	ν = 0.3	ν = 0.4	ν = 0.5
1.0	$2.716 \cdot 10^{-1}$	$2.548 \cdot 10^{-1}$	$2.399 \cdot 10^{-1}$	$2.266 \cdot 10^{-1}$	$2.148 \cdot 10^{-1}$
$9 \cdot 10^{-1}$	$2.505 \cdot 10^{-1}$	$2.345 \cdot 10^{-1}$	$2.204 \cdot 10^{-1}$	$2.079 \cdot 10^{-1}$	$1.968 \cdot 10^{-1}$
$8 \cdot 10^{-1}$	$2.264 \cdot 10^{-1}$	$2.115 \cdot 10^{-1}$	$1.985 \cdot 10^{-1}$	$1.870 \cdot 10^{-1}$	$1.767 \cdot 10^{-1}$
$7 \cdot 10^{-1}$	$1.993 \cdot 10^{-1}$	$1.858 \cdot 10^{-1}$	$1.740 \cdot 10^{-1}$	$1.636 \cdot 10^{-1}$	$1.544 \cdot 10^{-1}$
$6 \cdot 10^{-1}$	$1.692 \cdot 10^{-1}$	$1.573 \cdot 10^{-1}$	$1.470 \cdot 10^{-1}$	$1.379 \cdot 10^{-1}$	$1.299 \cdot 10^{-1}$
$5 \cdot 10^{-1}$	$1.361 \cdot 10^{-1}$	$1.262 \cdot 10^{-1}$	$1.177 \cdot 10^{-1}$	$1.102 \cdot 10^{-1}$	$1.036 \cdot 10^{-1}$
$4 \cdot 10^{-1}$	$1.012 \cdot 10^{-1}$	$9.355 \cdot 10^{-2}$	$8.698 \cdot 10^{-2}$	$8.127 \cdot 10^{-2}$	$7.627 \cdot 10^{-2}$
$3 \cdot 10^{-1}$	$6.609 \cdot 10^{-2}$	$6.092 \cdot 10^{-2}$	$5.650 \cdot 10^{-2}$	$5.268 \cdot 10^{-2}$	$4.934 \cdot 10^{-2}$
$2 \cdot 10^{-1}$	$3.400 \cdot 10^{-2}$	$3.126 \cdot 10^{-2}$	$2.892 \cdot 10^{-2}$	$2.691 \cdot 10^{-2}$	$2.516 \cdot 10^{-2}$
$1 \cdot 10^{-1}$	$9.772 \cdot 10^{-3}$	$8.965 \cdot 10^{-3}$	$8.281 \cdot 10^{-3}$	$7.694 \cdot 10^{-3}$	$7.185 \cdot 10^{-3}$
$9 \cdot 10^{-2}$	$8.023 \cdot 10^{-3}$	$7.359 \cdot 10^{-3}$	$6.797 \cdot 10^{-3}$	$6.314 \cdot 10^{-3}$	$5.896 \cdot 10^{-3}$
$8 \cdot 10^{-2}$	$6.424 \cdot 10^{-3}$	$5.892 \cdot 10^{-3}$	$5.441 \cdot 10^{-3}$	$5.054 \cdot 10^{-3}$	$4.719 \cdot 10^{-3}$
$7 \cdot 10^{-2}$	$4.983 \cdot 10^{-3}$	$4.570 \cdot 10^{-3}$	$4.220 \cdot 10^{-3}$	$3.920 \cdot 10^{-3}$	$3.660 \cdot 10^{-3}$
$6 \cdot 10^{-2}$	$3.710 \cdot 10^{-3}$	$3.492 \cdot 10^{-3}$	$3.141 \cdot 10^{-3}$	$2.917 \cdot 10^{-3}$	$2.723 \cdot 10^{-3}$
$5 \cdot 10^{-2}$	$2.610 \cdot 10^{-3}$	$2.393 \cdot 10^{-3}$	$2.209 \cdot 10^{-3}$	$2.052 \cdot 10^{-3}$	$1.915 \cdot 10^{-3}$
$4 \cdot 10^{-2}$	$1.692 \cdot 10^{-3}$	$1.551 \cdot 10^{-3}$	$1.432 \cdot 10^{-3}$	$1.330 \cdot 10^{-3}$	$1.241 \cdot 10^{-3}$
$3 \cdot 10^{-2}$	$9.640 \cdot 10^{-4}$	$8.838 \cdot 10^{-4}$	$8.158 \cdot 10^{-4}$	$7.576 \cdot 10^{-4}$	$7.071 \cdot 10^{-4}$
$2 \cdot 10^{-2}$	$4.339 \cdot 10^{-4}$	$3.978 \cdot 10^{-4}$	$3.672 \cdot 10^{-4}$	$3.410 \cdot 10^{-4}$	$3.182 \cdot 10^{-4}$
$1 \cdot 10^{-2}$	$1.098 \cdot 10^{-4}$	$1.007 \cdot 10^{-4}$	$9.295 \cdot 10^{-5}$	$8.631 \cdot 10^{-5}$	$8.056 \cdot 10^{-5}$
$1 \cdot 10^{-3}$	$1.111 \cdot 10^{-6}$	$1.018 \cdot 10^{-6}$	$9.399 \cdot 10^{-7}$	$8.727 \cdot 10^{-7}$	$8.146 \cdot 10^{-7}$
$1 \cdot 10^{-4}$	$1.111 \cdot 10^{-8}$	$1.019 \cdot 10^{-8}$	$9.405 \cdot 10^{-9}$	$8.733 \cdot 10^{-9}$	$9.151 \cdot 10^{-9}$
$1 \cdot 10^{-5}$	$1.111 \cdot 10^{-10}$	$1.018 \cdot 10^{-10}$	$9.401 \cdot 10^{-11}$	$8.730 \cdot 10^{-11}$	$8.148 \cdot 10^{-11}$
$1 \cdot 10^{-6}$	$1.111 \cdot 10^{-12}$	$1.018 \cdot 10^{-12}$	$9.399 \cdot 10^{-13}$	$8.728 \cdot 10^{-13}$	$8.146 \cdot 10^{-13}$
$1 \cdot 10^{-7}$	$1.111 \cdot 10^{-14}$	$1.018 \cdot 10^{-14}$	$9.399 \cdot 10^{-15}$	$8.728 \cdot 10^{-15}$	$8.146 \cdot 10^{-15}$
$1 \cdot 10^{-8}$	$1.111 \cdot 10^{-16}$	$1.018 \cdot 10^{-16}$	$9.399 \cdot 10^{-17}$	$8.728 \cdot 10^{-17}$	$8.146 \cdot 10^{-17}$
$1 \cdot 10^{-9}$	$1.111 \cdot 10^{-18}$	$1.018 \cdot 10^{-18}$	$9.399 \cdot 10^{-18}$	$8.728 \cdot 10^{-19}$	$8.146 \cdot 10^{-19}$

TABLE 4

β	A				
	$A_{cr} = -2.7185$ $\nu = 0.1$	$A_{cr} = -3.0583$ $\nu = 0.2$	$A_{cr} = -3.4952$ $\nu = 0.3$	$A_{cr} = -4.0778$ $\nu = 0.4$	$A_{cr} = -4.8933$ $\nu = 0.5$
$9 \cdot 10^{-2}$	–2.44070	–2.74579	–3.13804	–3.66105	–4.39326
$8 \cdot 10^{-2}$	–2.46947	–2.77816	–3.17504	–3.70421	–4.44505
$7 \cdot 10^{-2}$	–2.49872	–2.81106	–3.21264	–3.74808	–4.49770
$6 \cdot 10^{-2}$	–2.52846	–2.84452	–3.25088	–3.79269	–4.55123
$5 \cdot 10^{-2}$	–2.55872	–2.87856	–3.28979	–3.83809	–4.60570
$4 \cdot 10^{-2}$	–2.58953	–2.91322	–3.32939	–3.88429	–4.66115
$3 \cdot 10^{-2}$	–2.62090	–2.94852	–3.36974	–3.93136	–4.71763
$2 \cdot 10^{-2}$	–2.65289	–2.98450	–3.41085	–3.97934	–4.77522
$1 \cdot 10^{-2}$	–2.68547	–3.02118	–3.45277	– 4.02828	–4.83382
$1 \cdot 10^{-3}$	–2.71524	–3.05466	–3.49099	– 4.07280	–4.88746
$1 \cdot 10^{-4}$	–2.71695	–3.05658	–3.49325	– 4.07536	–4.89038
$1 \cdot 10^{-5}$	–2.71583	–3.05530	–3.49178	– 4.07379	–4.88855
$1 \cdot 10^{-6}$	–2.71539	–3.05484	–3.49121	– 4.07308	–4.88763
$1 \cdot 10^{-7}$	–2.71531	–3.05466	–3.49110	– 4.07294	–4.88745
$1 \cdot 10^{-8}$	–2.71524	–3.05466	–3.49110	– 4.07294	–4.88745
$1 \cdot 10^{-9}$	–2.71524	–3.05466	–3.49110	– 4.07294	–4.88745

$$n_1^0 = n_2^0 = 1, \quad \lambda_3 = 1 - 2\nu(\lambda_1 - 1)/(1-\nu), \; k = (\lambda_1(2-\nu) - (1-\nu)\lambda_3)/(\nu\lambda_1 + (1-\nu)\lambda_3). \tag{4.2}$$

Substituting (4.2) into (3.6) and using 20 coordinate functions, as in the case of the Bartenev–Khazanovich potential, we calculate the function $\varepsilon(\beta)$ for different values of Poisson's ratio (see Table 3 and in Figs. 4 ($0.1 < \beta < 3$) and ($0.01 < \beta < 0.1$)).

Table 4 collects the values of the coefficient A obtained on the assumption that $\sigma_{11}^0 \to A\beta^2\mu$ as $\beta \to 0$, and the values of A_{cr} for a clamped disk-shaped plate for different values of Poisson's ratio.

Note that practical convergence is reached by increasing the number of coordinate functions and the number of significant digits to improve the precision of computation.

Tables 2 and 4 show that the difference from the results obtained with approximate design models for a clamped circular plate is less than 5% even for $\beta = 0.05$ for both potentials [1].

The minimum value of β used in the computation is 10^{-9}. The difference between A and A_{cr} does not exceed 0.2%.

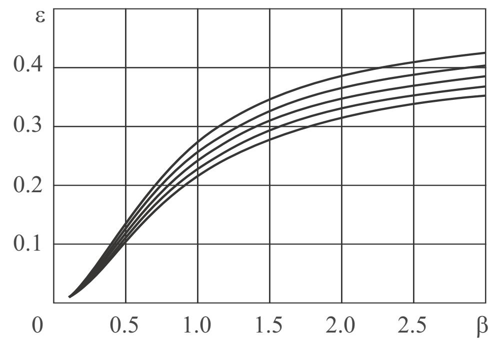

Fig. 4

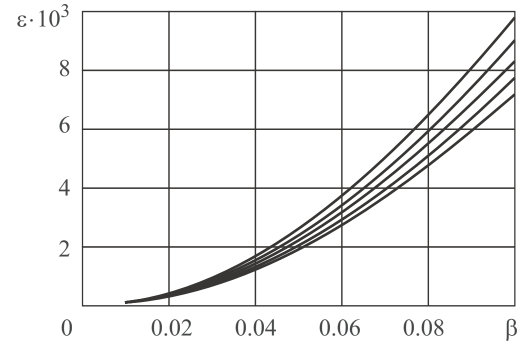

Fig. 5

Conclusions. We have analyzed, for the first time, the critical parameters defining the fracture of a half-space with a near-surface penny-shaped crack under compression for a wide range of distances between the crack and the free surface. Materials with harmonic and Bartenev–Khazanovich potential have been considered as examples. Using the method proposed in [18], we have calculated, for the first time, the relative distance between the crack and the free surface up to $\beta = 10^{-9}$, which is several orders of magnitude less than those obtained earlier. This may be considered an exact solution for a thin element isolated by the crack and the free surface. The data obtained here for the case of equal roots of the characteristic equation and in [18] for the case of unequal roots allow us to establish applicability criteria for approximate design models. For example, the thin plate isolated by a near-surface crack and the free surface should be considered clamped, and approximate design models are applicable if $\beta < 0.05$.

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