ON INERTIAL NAVIGATION SYSTEM ERROR CORRECTION

V. B. Larin[1] and A. A. Tunik[2]

A simple algorithm for the integration of inertial and global navigation systems, magnetometer, and barometric altimeter is considered. The algorithm is capable of compensating the biases of angular-rate sensors. A number of simplifying assumptions are made. This is because the sensors of the system are not very accurate, on the one hand, and, on the other hand, such systems are intended for objects (such as low-cost unmanned aerial vehicles) that move with low speed over relatively short distances. An example is considered to demonstrate the advisability of compensating the biases of angular-rate sensors

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Introduction. Simple inertial navigation systems (INSs) [7, 8, 23] are used, particularly, in low-cost unmanned aerial vehicles (UAVs) (see, e.g., [6]). However, such INSs cannot provide adequate accuracy of navigation over long periods of autonomous operation. It is, therefore, makes sense to integrate these INSs with a global navigation system (GPS) [13], i.e., to regard them as a component of a GPS/INS navigation system [22, 24]. It was pointed out in [4] that similar navigation systems can also be used in wheeled robotic vehicles [20, 21]. However, other systems [19] should be used to solve more complex navigation problems [16, 17]. It is significant that UAVs use not only GPS, but also additional navigation data channels such as magnetometers, barometric altimeters, etc. [11, 12, 15] to correct the solution from the INS.

Here, as in [2], we will discuss a simple algorithm for the integration of GPS, INS, magnetometer, and barometric altimeter. We will make some simplifying assumptions (neglect Coriolis acceleration and use a rectangular coordinate frame). This is because the INS sensors are not very accurate, on the one hand, and, on the other hand, such systems are intended for objects that move with low speed over relatively short distances.

However, in contrast to [2], we will assume that the angular-rate sensors of the INS are affected by a bias that requires the GPS/INS system to employ a much more complicated (compared with [2]) algorithm.

1. Governing Equations. The well-known equations related to the attitude-determination problem for rigid bodies [3, 5, 19, 25] are presented below. Let us describe different ways to determine the attitude.

The Euler angles ψ, ϑ, φ (precession, nutation, and intrinsic rotation) describe the orientation of a body, i.e., the transition of the body from the initial position defined by the axes of $Oxyz$ to the final position defined by the axes of $Ox'y'z'$ (Fig. 1). This transition can be carried out by rotating the body through an angle χ about the axis defined by angles α, β, γ. Therefore, the orientation of a body can be characterized by four Euler–Rodrigues parameters [3]: $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ (Euler's parameters [25]):

$$\lambda_1 = \cos\alpha \sin\frac{\chi}{2}, \quad \lambda_2 = \cos\beta \sin\frac{\chi}{2}, \quad \lambda_3 = \cos\gamma \sin\frac{\chi}{2}, \quad \lambda_0 = \cos\frac{\chi}{2}.$$

The equality $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$ obviously holds. The Euler–Rodrigues parameters are expressed in terms of the Euler angles as follows:

[1]S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterova St., Kyiv, Ukraine 03057, e-mail: model@inmech.kiev.ua. [2]National Aviation University, 1 Komarova Av., Kyiv, Ukraine 03680, e-mail: aatunik@hotmail.com. Translated from Prikladnaya Mekhanika, Vol. 48, No. 2, pp. 114–126, March–April 2012. Original article submitted July 12, 2010.

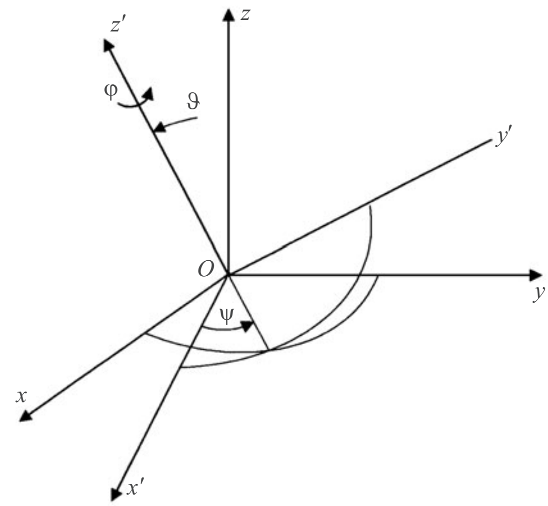

Fig. 1

$$\lambda_0 = \cos\frac{\vartheta}{2}\cos\frac{\varphi+\psi}{2}, \qquad \lambda_1 = \sin\frac{\vartheta}{2}\cos\frac{\psi-\varphi}{2},$$

$$\lambda_2 = \sin\frac{\vartheta}{2}\sin\frac{\psi-\varphi}{2}, \qquad \lambda_3 = \cos\frac{\vartheta}{2}\sin\frac{\varphi+\psi}{2}. \tag{1.1}$$

The orientation of a rigid body relative to a fixed coordinate frame $Oxyz$ can be defined by a coordinate transformation matrix A (direction cosine matrix between the fixed and moving coordinate frames); i.e., if m is some vector in the fixed frame, and its components k are the projections of this vector onto the axes of the moving frame ($Ox'y'z'$), then

$$k = Am. \tag{1.2}$$

This matrix can be represented in terms of the Euler–Rodrigues parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ as follows:

$$A(\lambda) = \begin{bmatrix} \lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2 & 2(\lambda_1\lambda_2+\lambda_0\lambda_3) & 2(\lambda_1\lambda_3-\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & \lambda_0^2-\lambda_1^2+\lambda_2^2-\lambda_3^2 & 2(\lambda_2\lambda_3+\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3+\lambda_0\lambda_2) & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) & \lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2 \end{bmatrix}. \tag{1.3}$$

The inverse formulas hold as well. For example, if $A = [a_{ij}]$, $i, j = \overline{1,3}$, and $1 + a_{11} + a_{22} + a_{33} > 0$, then we have the following [5, 25]:

$$\lambda_0 = \frac{1}{2}\sqrt{1+a_{11}+a_{22}+a_{33}}, \qquad \lambda_1 = \frac{a_{23}-a_{32}}{2\sqrt{1+a_{11}+a_{22}+a_{33}}},$$

$$\lambda_2 = \frac{a_{31}-a_{13}}{2\sqrt{1+a_{11}+a_{22}+a_{33}}}, \qquad \lambda_3 = \frac{a_{12}-a_{21}}{2\sqrt{1+a_{11}+a_{22}+a_{33}}}. \tag{1.4}$$

The projections $\omega_1, \omega_2, \omega_3$ of the angular-velocity vector of the body onto the body-fixed axes are expressed in terms of the Euler angles as

$$\omega_1 = \dot\psi \sin\vartheta \sin\varphi + \dot\vartheta \cos\varphi,$$

$$\omega_2 = \dot\psi \sin\vartheta \cos\varphi - \dot\vartheta \sin\varphi,$$

$$\omega_3 = \dot{\psi}\cos\vartheta + \dot{\varphi}. \tag{1.5}$$

Measuring the projections of the angular-velocity vector $\omega = [\omega_1\ \omega_2\ \omega_3]^T$ onto the body-fixed axes and knowing the initial position of the rigid body, we can find the vector (quaternion) of Euler–Rodrigues parameters $\lambda = [\lambda_0\ \lambda_1\ \lambda_2\ \lambda_3]^T$ by integrating the kinematic equations

$$\dot{\lambda} = 1/2\Omega\lambda, \quad \Omega = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}, \quad ||\lambda||^2 = \lambda^T\lambda = 1, \tag{1.6}$$

where $||\cdot||$ denotes spectral matrix norm; the superscript "T" denotes transposition.

If the frames $Oxyz$ and $Ox'y'z'$ are close (the Euler angles are small), we can use an approximate expression (say (26) in [25]) for the matrix A:

$$A \cong \begin{bmatrix} 1 & \mu_3 & -\mu_2 \\ -\mu_3 & 1 & \mu_1 \\ \mu_2 & -\mu_1 & 1 \end{bmatrix}, \tag{1.7}$$

where μ_1, μ_2, μ_3 are the small angles of rotation of $Oxyz$ about the x-, y-, z-axes, respectively.

We will use (as in [19]) Eq. (30) from [9] that describes the variation in the coordinates of the object and represents the acceleration summation theorem (Coriolis theorem):

$$\frac{dv}{dt} = w - 2\Omega_z \times v - \Omega_z \times \Omega_z \times R, \tag{1.8}$$

where w is absolute acceleration; v is the relative velocity of the object; Ω_z is the Earth's rotation rate; R is the position vector of a point in the geocentric coordinate frame. The accelerometers measure the following quantity:

$$w_{\mathrm{a}} = w + g, \tag{1.9}$$

where g is the acceleration of gravity;

2. INS Algorithm. The above formulas allow us to describe the operation of the INS including angular-rate sensors (ARSs) and accelerometers. The ARS readings can be used to find the matrix A by using (1.3) and integrating (1.6) (given initial conditions). Next, using this matrix to transform the readings (1.9) of the onboard accelerometer to the coordinate frame of (1.8), we can integrate (1.8) to determine the relative velocity v and, then, the coordinates of the object.

Thus, the INS algorithm involves the integration of a system of differential equations. For implementation purposes, it would be appropriate to examine the case of "discretization" where the INS sensors are read not continuously, but also at regular time intervals Δt, i.e., with frequency $f = 1/\Delta t$. Then the required navigation parameters (cosine matrix $A(\lambda)$, velocity v, coordinates r) are calculated at time intervals Δt. Since various discretization procedures can be used to calculate the navigation parameters, we will dwell on each of them. We start with the estimation of the quaternions at times t_i, $t_i - t_{i-1} = \Delta t$, $i = 1, 2, 3, \ldots$ [19]. Let the quasicoordinates (elements of the vector $\nabla\theta_i = \int_{t_{i-1}}^{t_i} \omega dt$) be known on the time interval Δt. After the solution of Eq. (1.6) subject to the initial condition $[1\quad 0\quad 0\quad 0]^T$ on the time interval Δt is expressed in terms of these quasicoordinates $\delta\lambda(t_i)$ (i.e., calculating the quaternion corresponding to a small-angle rotation of the rigid body in time Δt), the orientation of the body can be described by successive multiplication of $\delta\lambda(t_i)$ "elementary" quaternions:

$$\lambda(t_i) = \lambda(t_{i-1})\delta\lambda(t_i),$$

$$\delta\lambda(t_i) = \left[\delta\lambda_0(t_i)\quad \delta\lambda_1(t_i)\quad \delta\lambda_2(t_i)\quad \delta\lambda_3(t_i)\right]^T. \tag{2.1}$$

In matrix form:

$$\lambda(t_i)=\begin{bmatrix}\delta\lambda_0(t_i) & -\delta\lambda_1(t_i) & -\delta\lambda_2(t_i) & -\delta\lambda_3(t_i)\\ \delta\lambda_1(t_i) & \delta\lambda_0(t_i) & \delta\lambda_3(t_i) & -\delta\lambda_2(t_i)\\ \delta\lambda_2(t_i) & -\delta\lambda_3(t_i) & \delta\lambda_0(t_i) & \delta\lambda_1(t_i)\\ \delta\lambda_3(t_i) & \delta\lambda_2(t_i) & -\delta\lambda_1(t_i) & \delta\lambda_0(t_i)\end{bmatrix}\begin{bmatrix}\lambda_0(t_{i-1})\\ \lambda_1(t_{i-1})\\ \lambda_2(t_{i-1})\\ \lambda_3(t_{i-1})\end{bmatrix}. \tag{2.2}$$

Expressions for the quaternions $\delta\lambda(t_i)$ in terms of the quasicoordinate vector $\nabla\theta_i$ that provide certain quality of approximation depending on complexity can be found in [19].

In what follows, we will use the following approximation of the quaternion $\delta\lambda(t_i)$ (formula (2.6) in [19]):

$$\delta\lambda(t_i)=\begin{bmatrix}1-\frac{1}{12}||\nabla\theta_i||^2\\ \frac{1}{2}\nabla\theta_i-\frac{1}{24}\left(\nabla\theta_i\times\nabla\theta_{i-1}\right)\end{bmatrix}. \tag{2.3}$$

As in [19], we can calculate $\nabla\theta_i$ using the quadratic spline-approximation of the angular-velocity vector $\omega(t)$. For example, if $\omega(t_{i-2})$, $\omega(t_{i-1})$, $\omega(t_i)$ are known, then

$$\nabla\theta_i=\frac{\Delta t}{12}(5\omega(t_i)+8\omega(t_{i-1})-\omega(t_{i-2})). \tag{2.4}$$

Thus, having ARS readings, we can use formulas (2.1)–(2.4) and then (1.3) to find the direction cosine matrix. Using this matrix to transform the accelerometer readings, we can determine the acceleration w in (1.8) from (1.9). The next step is to integrate Eq. (1.8) to determine the current coordinates and velocity of the object.

Note that, as in [9], the term $2\Omega_z\times v$ (Coriolis acceleration) is considered as a small correction (neglected in the example below). In calculating (if necessary) the Coriolis acceleration at the ith step, it is possible, as in [19], to use the velocity v at the time t_{i-1}. This assumption allows us to evaluate quadratures instead of the integration of Eq. (1.8).

Thus, the right-hand side $\tilde{w}$ of Eq. (1.8) can be found from the accelerometer readings and direction cosine matrix. Having the values of $\tilde{w}$ at t_{i-2}, t_{i-1}, t_i, we can write formulas for $v(t_i), r(t_i)$ similar to (2.4):

$$v(t_i)=(5\tilde{w}(t_i)+8\tilde{w}(t_{i-1})-\tilde{w}(t_{i-2}))\cdot\frac{\Delta t}{12}+v(t_{i-1}), \tag{2.5}$$

$$r(t_i)=(3\tilde{w}(t_i)+10\tilde{w}(t_{i-1})-\tilde{w}(t_{i-2}))\cdot\frac{\Delta t^2}{24}+\Delta t v(t_{i-1})+r(t_{i-1}). \tag{2.6}$$

Formulas (1.3), (2.1)–(2.6) constitute the INS algorithm, i.e., allow estimating the navigation parameters at times t_i from the ARS and accelerometer readings at t_i. This, in turn, makes it possible to use conventional GPS–INS integration algorithms to correct solutions from the INS.

3. Filter Equations. Denote the error vectors of the INS in the coordinate frame of Eq. (1.8) by μ, δv, δr (μ is the small-angle rotation vector of the attitude error; δv and δr are the velocity and coordinate error vectors); the bias vector of the ARS by δc; the total acceleration vector by $w=[w_1, w_2, w_3]^T$; the direction cosine matrix A is defined by (1.2). The equation of variation in the INS errors has a form similar to (7.149) of [14]:

$$\dot{x}=Fx+n,\quad x=\begin{bmatrix}\mu\\ \delta v\\ \delta r\\ \delta c\end{bmatrix},\quad F=\begin{bmatrix}0 & 0 & 0 & A^T\\ C & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\quad C=\begin{bmatrix}0 & -w_3 & w_2\\ w_3 & 0 & -w_1\\ -w_2 & w_1 & 0\end{bmatrix}, \tag{3.1}$$

n is the white noise vector; 0 is a zero matrix. I is a unit matrix.

We will use the following equation as a discrete analog of (3.1), i.e., errors at small time intervals Δt:

$$x_{k+1} = \Phi_k x_k + n_k,$$

$$\Phi_k = I + F\Delta t + \frac{(\Delta t)^2}{2} F^2 = \begin{bmatrix} I & 0 & 0 & A^T \Delta t \\ C\Delta t & I & 0 & CA^T \dfrac{(\Delta t)^2}{2} \\ C\dfrac{(\Delta t)^2}{2} & I\Delta t & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad (3.2)$$

n_k is the random error vector of the INS. The subscript k corresponds to the time $k\Delta t$. It may be assumed that Δt is the INS cycle, and the original error equation is (3.2). Let the GPS supply measured coordinates and velocity of the object to the INS at the kth cycle, i.e., we have the following observation process:

$$z_k = Hx_k + \xi_k, \qquad H = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}, \quad (3.3)$$

ξ_k is the measurement error.

Thus, with formulas (3.3), the error-correction problem for the INS may be formulated as an optimal filtering problem. The solution of this problem is known (see, e.g., Sec. 12.4 in [10]) to have the form

$$\hat{x}_k = \bar{x}_k + K_k (z_k - H\bar{x}_k), \quad \bar{x}_{k+1} = \Phi_k \hat{x}_k. \quad (3.4)$$

The feedback-gain matrix (K_k) of the filter that generates the optimal estimate vector x_k is defined as follows (filter equations):

$$K_k = M_k H^T (HM_k H^T + R_k)^{-1}, \quad (3.5)$$

$$M_{k+1} = \Phi_k S_k \Phi'_k + Q_k^T, \quad (3.6)$$

$$S_k = M_k - K_k (HM_k H^T + R_k) K_k^T, \quad (3.7)$$

where Q_k and R_k are the covariance matrices of noises n_k and ξ_k that appear in (3.2) and (3.3); M_0 is the given covariance matrix of the initial estimate of the vector x. Note that INS errors are usually corrected every $j > 1$cycles. Between corrections, the variation in the INS errors is described by Eq. (3.2), and the variation in their correlation matrix is described by (3.6) (it may be assumed that $H = 0$ during these cycles). During each cycle that involves correction, the variation in the correlation matrix is described by Eq. (3.7).

Thus, the first nine elements of the vector $\bar{x}_k$ in (3.4) define estimates of the error vectors μ_k, δv_k, δr_k and, hence, estimates of the attitude, velocity, and coordinates of the object at time t_k. The last three elements of the vector $\bar{x}_k$ (vector δc_k) define estimates of the bias of the ARS. It is reasonable to use this estimate to correct the ARS readings. For example, if the vector $\overline{\omega}(t_k)$ is the ARS output at the time t_k, then the following expression for the angular-velocity vector should be substituted into (2.4):

$$\omega(t_k) = \overline{\omega}(t_k) - \delta c_k. \quad (3.8)$$

Note that δc_k in (3.8) is changed only at the time the INS is corrected by the GPS, i.e., when $H \neq 0$.

A significant feature of the problem is that the matrices Φ_k and H form an incompletely observable pair [18]. This requires higher accuracy of computational procedures. In this connection, solving such problems usually involves algorithms to compute the Cholesky factors of covariance matrices. An algorithm [18] based on the QR-decomposition of a matrix will be described below. It is assumed that the matrix R_k is invertible (see [18] for the general case).

4. Computing the Cholesky Factors. Let m_k, p_k, q_k, η_k be the Cholesky factors of the matrices M_k, S_k, Q_k, R_k, respectively:

$$M_k = m_k m_k^T, \quad S_k = p_k p_k^T, \quad Q_k = q_k q_k^T, \quad R_k = \eta_k \eta_k^T.$$

Since the matrix R_k is invertible, formula (3.7) can be rearranged as follows:

$$p_k p_k^T = m_k (I + m_k^T H^T R_k^{-1} H m_k)^{-1} m_k^T. \tag{4.1}$$

Let us represent the bracketed expression as the product of two rectangular matrices:

$$I + m_k^T H^T R_k^{-1} H m_k = N_k N_k^T, \quad N_k = [I \quad m_k^T H^T \eta_k^{-1}].$$

Using an orthogonal matrix U and QR-decomposition algorithm, we transform the matrix N^T as follows:

$$\begin{bmatrix} \Lambda_k \\ 0 \end{bmatrix} = U_k N_k^T, \tag{4.2}$$

where Λ_k is an invertible matrix.

According to (4.1) and (4.2), we get

$$p_k = m_k \Lambda_k^{-1}. \tag{4.3}$$

Likewise, we represent the right-hand side of (3.6) as the product of two rectangular matrices and do the QR-decomposition of these matrices using an orthogonal matrix Z_k:

$$m_{k+1} m_{k+1}^T = T_k T_k^T, \quad T_k = [\Phi_k p_k q_k], \tag{4.4}$$

$$\begin{bmatrix} X_k^T \\ 0 \end{bmatrix} = Z_k T^T, \tag{4.5}$$

$$m_{k+1} = X_k. \tag{4.6}$$

Thus, given m_k and η_k, we calculate the factor p_k by formulas (4.2) and (4.3) and the factor m_{k+1} by formulas (4.4)–(4.6).

5. Using the Signals of the Magnetometer and Altimeter. Section 3 described the process of correcting the INS solution using GPS signals (formulas (3.4)–(3.7)).

Let us generalize the problem formulation assuming that not only GPS, but also magnetometer and altimeter can be used to correct a solution from the INS.

Thus, the input data for the correction algorithm are not only the statistical parameters of signals and measurement noise, but also the residual vector (ε_k) calculated as the difference between the GPS signal (vector z) and the estimates of the current coordinates and velocity of the object (vector $H\bar{x}_k$):

$$\varepsilon_k = z_k - H\bar{x}_k. \tag{5.1}$$

It is natural that the generalization of the problem formulation must be related to the generalization of the procedure of computing the residual vector. For example, when the measuring channels include the altimeter, the problem formulation is generalized by expanding the vector z_k and matrix H in (5.1). Using the readings of the magnetometer, however, requires additional considerations. For the sake of simplicity, we will model the magnetometer data channel as follows. Assume that an onboard instrument measures the vector ($\bar{m}$), which is a unit vector directed along the x-axis ($m = [1 \quad 0 \quad 0]^T$) in the Earth-fixed frame.

Thus, having $\bar{m}$ and $\bar{A}$, we can find the estimate γ of the small-angle rotation vector, which is the error in the attitude of the object. To this end, we can use the following formula (see, e.g., (1.8) in [1]):

$$\bar{A}\bar{m} - m = m \times \gamma. \tag{5.2}$$

Formula (5.2) can be interpreted as a formalization of the fact that the vector γ rotates the vector m until it coincides with the vector $\bar{A}\bar{m}$. Multiplying both sides of (5.2) by $m\times$, we obtain the following expression for γ:

$$-\gamma = m(\bar{A}\ \bar{m} - m). \tag{5.3}$$

Since the vectors m and γ are assumed orthogonal, the first element of the vector γ is zero and, thus, may be excluded from consideration. The other elements of the vector γ can be interpreted as the measurement of the respective two elements of the vector μ appearing in (3.1).

Thus, when the signals not only from the GPS, but also from the magnetometer and altimeter are used, the following (9×1)-vector may be taken for z_k in (3.3):

$$z_k = \begin{bmatrix} -\tilde{\gamma} \\ \tilde{v} \\ \tilde{r} \end{bmatrix}, \tag{5.4}$$

where $\tilde{\gamma}$ is a (2×1)-vector consisting of the last two elements of the vector γ defined by (5.3); $\tilde{v}$ is a (3×1) velocity vector measured by the GPS; $\tilde{r}$ is a (4×1)-vector consisting of the coordinates determined from the GPS and altimeter readings. Naturally, the matrix H in (3.3) should be changed in an appropriate way.

6. General Case. Let us continue considering the problem of using magnetometer signals to correct a solution from the INS. In contrast to Sec. 5, we will not assume here that the magnetic-field vector is determined by a unit vector of the OX-axis, i.e., $m = [1\quad 0\quad 0]^T$. Let us show that in this general case we can use the algorithm of Sec. 5 modified in an appropriate way. Denote by m_τ a unit vector defining the magnetic field, but not coinciding with the unit vector of the OX-axis. Let the orthogonal matrix τ be such that

$$\tau m_\tau = m = [1\quad 0\quad 0]^T. \tag{6.1}$$

In this case, it makes sense to consider the first three elements of the vector x_k that describe the small-angle rotation vector in the coordinate frame defined by the matrix τ appearing in (6.1). In other words, it is necessary to introduce a small-angle rotation vector $\bar{\mu}$ related to the vector μ as follows:

$$\bar{\mu} = \tau\mu. \tag{6.2}$$

In this connection, the matrix Φ_k appearing in (3.2) should be subjected to the following linear transformation:

$$\bar{\Phi}_k = \theta\ \Phi_k \theta^T, \qquad \theta = \mathrm{diag}\{\tau,\quad I,\quad I\}. \tag{6.3}$$

The resulting matrix $\bar{\Phi}_k$ should be used in the formulas of Secs. 3 and 4.

Let us discuss the changes that should be made in the procedure described in Sec. 5. The following formula is an analog of (5.2) in this case:

$$\bar{A}\bar{m}_\tau - m_\tau = m_\tau \times \gamma_\tau, \tag{6.4}$$

where $\bar{m}_\tau$ is the measured magnetic-field vector in the moving coordinate frame; γ_τ is the corresponding small-angle rotation vector. Multiplying (6.4) by τ, we obtain

$$\tau\bar{A}\bar{m}_\tau - m = m \times \bar{\gamma}, \qquad \bar{\gamma} = \tau\gamma_\tau, \tag{6.5}$$

whence follows an analog of (5.3):

$$-\bar{\gamma} = m \times \tau\bar{A}\bar{m}_\tau. \tag{6.6}$$

Since the vector $\bar{\gamma}$ is orthogonal to the vector m, the first element of the vector $\bar{\gamma}$ is zero. In this connection, the vector $\tilde{\gamma}$ appearing in (5.4) has only two elements that coincide with the last two elements of the vector $\bar{\gamma}$. Thus, we have determined the vector z_k in (5.4) in the general case.

The estimate of the vector x_k obtained with (3.4) should be multiplied by θ^T. This is because the first three elements of the vector x_k correspond to the vector $\bar{\mu}$ related by (6.2) to the small-angle rotation vector (μ) in the initial coordinate frame.

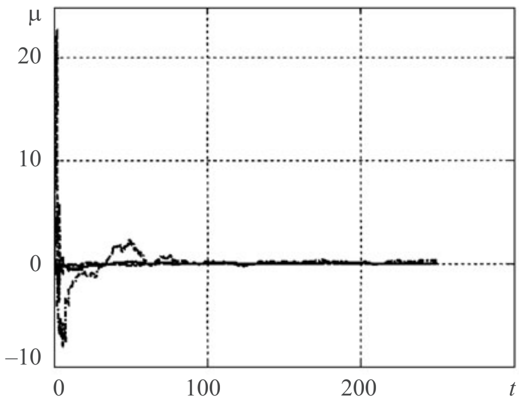

Fig. 2

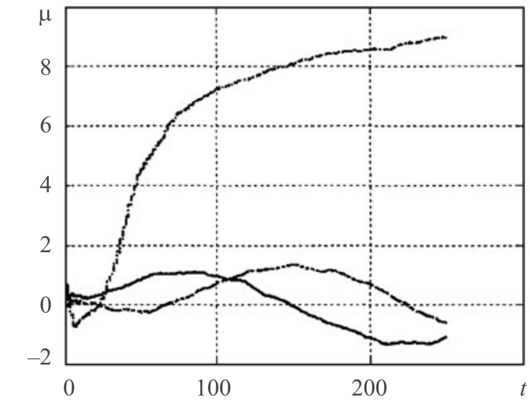

Fig. 3

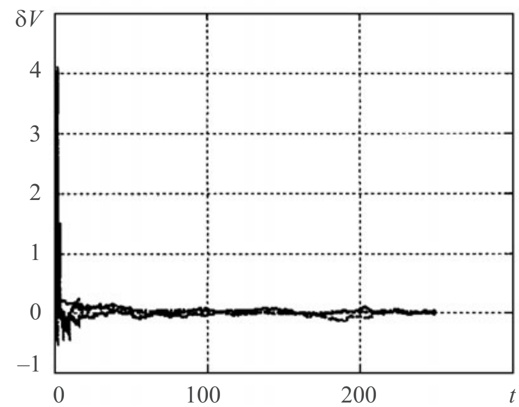

Fig. 4

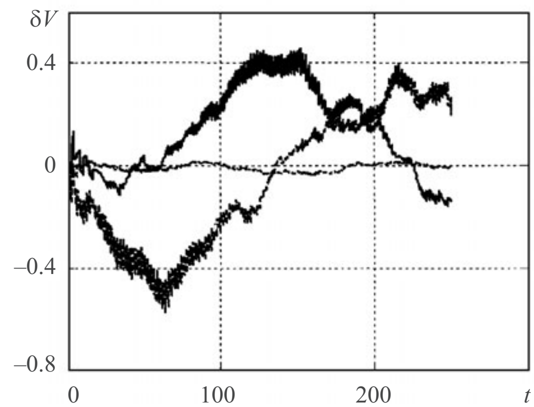

Fig. 5

7. Example. Let us illustrate the algorithm of using GPS, magnetometer, and altimeter signals to correct a solution from the INS. To demonstrate the effect of the bias of the ARS, we will compare the above algorithm and the algorithm [2] that disregards the biases of the ARSs. The value of the ARS bias is assumed to be the same as in the example of [6]. Let us consider an example similar to the example of [2]. Let the coordinate frame *xyz* shown in Fig. 1 be oriented as follows: the *x*-axis is pointed south, the *y*-axis east, and the *z*-axis toward the zenith. The origin (point *O*) of this frame is on the Earth's surface at latitude 45° north. In this frame, the object circles in the plane *xy* with period $T = 300$ sec and velocity $V = 60$ m/sec. During motion, its orientation is described by the following time-dependent Euler angles: $\psi = 2\pi t / T$, $\vartheta = 0$, $\varphi = 0.3\sin(10\psi)$. The projections of the angular velocity onto the axes of the body-fixed frame $(x'y'z')$ (regardless of the Earth's rotation rate) are defined by (1.5). These data are used to model the readings of the onboard ARSs. Namely, the angular-velocity vector $\omega = [\omega_1 \quad \omega_2 \quad \omega_3]^T$ obtained from (1.5) at t_k is summed with the bias vector $n_\text{b} = [\sigma_1 \quad \sigma_2 \quad \sigma_3]^T$ and random-error (3×1)-vector n_ω. The elements of the vector n_ω are random numbers uniformly distributed with zero mean and variance σ_ω. To integrate the kinematic equations (1.6) according to algorithm (2.1), we use (2.3) as an "elementary" quaternion. The necessary vectors of quasicoordinates $\nabla\theta_i$ are calculated using (2.4). The errors in the readings of the accelerometers are assumed to be random numbers uniformly distributed with zero mean and variance σ_a.

To integrate Eq. (1.8), we use formulas (2.5) and (2.6) and neglect the Coriolis acceleration.

The errors in the readings of the magnetometer and altimeter are modeled in a similar manner. The errors of the magnetometer are assumed to have the same variance σ_m in all coordinates. The errors of the altimeter have a variance σ_v.

Let the INS operate at a frequency of 20 Hz as in [2], i.e., the time interval $\Delta t = 5 \cdot 10^{-2}$ sec. The readings of the ARS and accelerometers are noised ($\sigma_\omega = 3$ arcmin/sec, $\sigma_a = 10^{-2}$ m/sec^2). The elements of the vector n_b have the following values, as in

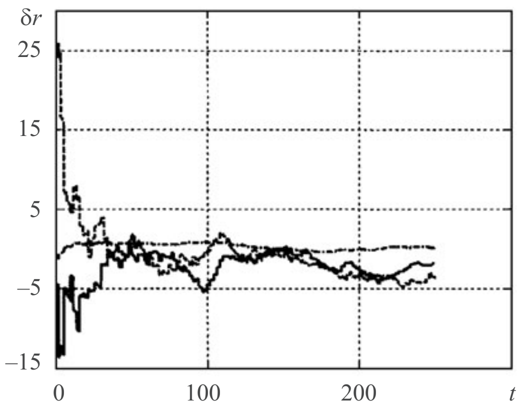

Fig. 6

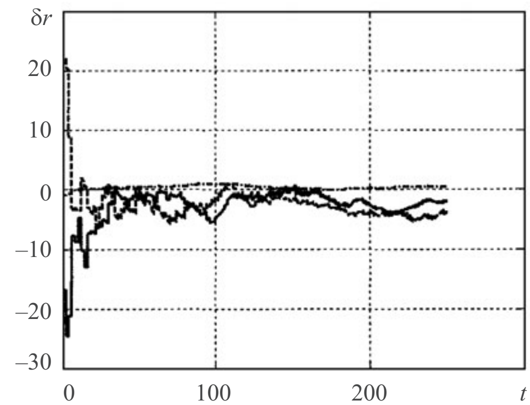

Fig. 7

[6]: $\sigma_1 = 94$ arcmin/sec, $\sigma_2 = -56$ arcmin/sec, $\sigma_3 = 22$ arcmin/sec. Note that the bias is less than σ_ω. The solution from the INS is corrected every two second. As in [2], we assume that the GPS measures the velocity and coordinates of the object with the following errors: 0.1 m/sec and 50 m. The variances of the errors of the magnetometer (coordinates of the vector $\overline{m}$) and altimeter are $\sigma_m = 0.0524$, $\sigma_v = 1$ m. With these data, we can find the values of the Cholesky factors q_k, η_k:

$$\overline{q}_k = 10^{-3} \operatorname{diag}\{\alpha_q I,\ \beta_q I,\ \gamma_q I,\ \delta_q I\},$$

$$\eta_k = \operatorname{diag}\{\alpha_\eta I_2,\ \beta_\eta I,\ \gamma_\eta I,\ 1\}, \tag{7.1}$$

$\alpha_q = 0.0218$, $\beta_k = 0.25$, $\gamma_q = 0.0063$, $\delta_q = 5\cdot 10^{-5}$, $\alpha_\eta = 0.0524$, $\beta_\eta = 0.1$, $\gamma_\eta = 50$, where I is a (3×3)-matrix; I_2 is a (2×2)-matrix. Assume that $g = 9.81$ m/sec^2 and the object is on the y-axis at distance $VT/(2\pi)$ at the initial time ($t = 0$), i.e., the initial position of the object is defined by the vector $r_0 = \begin{bmatrix} 0 & \dfrac{VT}{2\pi} & 0 \end{bmatrix}^T$. The velocity vector $v_0 = [-V \quad 0 \quad 0]^T$. The initial INS alignment involves the following errors: (i) the error of the initial orientation determined by a (non-normalized) quaternion, $\lambda(0) = [1 \quad 0.05 \quad -0.05 \quad 0.05]^T$; (ii) the initial alignment error determined by the relative errors $\varepsilon_r = 0.01$ and $\varepsilon_v = 0.01$, i.e., the following initial coordinates (r) and velocity (v) are set: $r = r_0(1+\varepsilon_r)$, $v = v_0(1+\varepsilon_v)$.

In this connection, the following matrix is used as the Cholesky factor m_0:

$$m_0 = \operatorname{diag}\left\{\alpha_m I,\ \beta_m I,\ \gamma_m I,\ \delta_m I\right\}, \quad \alpha_m = 0.04, \quad \beta_m = 0.6, \quad \gamma_m = 28.65, \quad \delta_m = 0.4,$$

where I are (3×3)-matrices, as in (7.1). Note also that the observation vector z_k is formed according to (5.4), and the (9×12)-matrix H has the following structure: $H = [H_1 \quad O_3]$, where $H_1 = \begin{bmatrix} O & I \\ O^T & 1 \end{bmatrix}$; O_3 is a zero (9×3)-matrix; O is a zero (8×1)-matrix; I is a unit (8×8)-matrix.

With such input data, the operation of the INS was simulated on a time interval of 250 sec. The results of the simulation (time-dependence the first nine elements of the vector x in (3.1)) are presented in Figs. 2, 4, and 6.

Figure 2 shows the time-dependence of the elements (μ_x, μ_y, μ_z measured in degrees of arc) of the vector μ, and Figs. 4 and 6 show the time-dependence of the elements of the vectors δv and δr (measured in meters per second and meters, respectively). To assess the effect of the ARS bias, Figs. 3, 5, and 7 present similar results obtained with the algorithm [2] that neglects the bias. Figures 2–7 employ the following notation for the coordinates of the vectors $\mu, \delta v, \delta r$: the solid line corresponds to the x-axis, the dashed line to the y-axis, and the dash-and-dot line to the z-axis.

The effectiveness of compensating the ARS bias may be judged from Fig. 8, which shows the time-dependence of $\alpha = \log(\|n_b - \delta c\| / \|n_b\|)$, where δc is the current estimate of the ARS bias, and $\|\cdot\|$ is the norm of a vector.

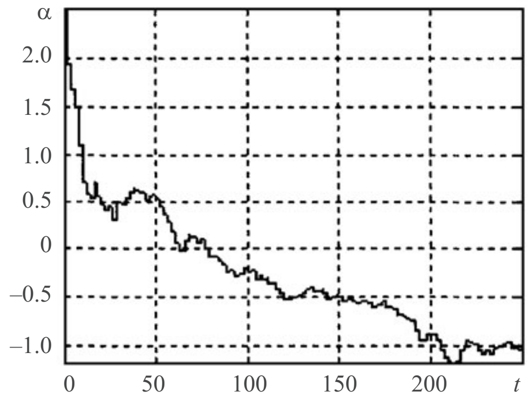

Fig. 8

It follows from Figs. 2, 4, and 6 that the proposed correction algorithm is also effective in the presence of ARS biases. For example, after a relatively short (2 to 3 cycles of correction) transient, which is due to a rough estimate of the ARS bias, the system provides high-accuracy estimates of the orientation, velocity, and coordinates of the object.

Comparing between Figs. 2, 4, 6 and Figs. 3, 5, 7 indicates the effectiveness of the algorithm for compensating the ARS bias.

Conclusions. A simple algorithm for the integration of inertial navigation system, global navigation system, magnetometer, and barometric altimeter has been expounded. The algorithm compensates the biases of the angular-rate sensors. A number of simplifying assumptions have been made. This is because the INS sensors are not very accurate, on the one hand, and, on the other hand, such systems are intended for objects (such as low-cost unmanned aerial vehicles) that move with low speed over relatively short distances. An example has been considered to demonstrate the advisability of compensating the biases of angular-rate sensors.

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