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MODELING THE FRACTURE PROCESS ZONE NEAR A CRACK TIP IN A NONLINEAR ELASTIC BODY

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The model of a fracture process zone near the tip of a mode I crack in a nonlinear elastic body is proposed. Using the numerical solution of an appropriate boundary-value problem, the effect of the fracture process zone on a crack opening displacement is examined

Keywords: nonlinear elastic body, crack, fracture process zone

1. Introduction. As experiments [1, 3, 8, 12] show, fracture occurs in a local area near a crack tip (fracture process zone) where extremely high stresses give rise to submicrocracks that grow and coalesce. The shape and structure of this zone depend on the material properties and loading conditions.

The strains in the fracture process zone are very high (more than 50% in an elastoplastic body [4]).

This is why many attempts to describe the deformation process in this zone using the small-strain deformation theory of plasticity led to physically incorrect results that disagree with experimental data [1].

A great many theoretical studies of the stress–strain behavior near a crack tip in nonlinear bodies disregarding the fracture process zone were published in the 1960–70s [1, 7, 8]. Researchers presumed that by solving nonlinear boundary-value problems, it was possible to eliminate the singularity of the stresses and strains at the crack tip, which would be consistent with reality.

However, the singularity was still there, but of different type. Thus, stresses and strains at the crack tip remained infinitely high.

The lack of information on fracture mechanisms is compensated for by modeling the fracture process zone [1, 3, 10-13] based on experimental data [1, 8] on the localization of nonlinear deformation in narrow zones (slipbands) at the crack tip. This phenomenon is especially typical for the early stage of loading. Since the fracture process zone is usually located on the continuation of the crack and is much shorter than the crack, it is modeled, expanding upon the Leonov–Panasyuk model [6], by a slit with faces subject to certain stresses.

The experiments [4] indicate that the fracture process zone occurs inside the zone of nonlinear deformation near the crack tip. We will study the effect of the fracture process zone on the opening displacement of a mode I crack. Use will be made of the model proposed in [3, 11, 12]. In this model, the length of the fracture process zone and, hence, the length of the slit remain constant with increase in the external load. What do change are the stresses applied to the faces of the slit. They are determined by solving a boundary-value problem and satisfying the continuity and limitation conditions for the stresses at the slit tip.

We will analyze the role of the fracture process zone near a crack tip in a nonlinear elastic body in the case of a generalized plane stress state assuming small strains. The boundary-value problem will be formulated in terms of the displacement components. By solving the problem, we will ascertain how the fracture process zone influences the crack opening displacement.

1. Preliminaries. To formulate the boundary-value problem, we need the nonlinear constitutive equations relating the components of the stress tensor $\boldsymbol{S}$ and strain tensor $\boldsymbol{D}$.

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1.1. Constitutive Equations. Let us make use of the constitutive equations for an anisotropic body [5]:

$$S^{\alpha\beta} = G^{\alpha\beta\gamma\delta} D_{\gamma\delta} - \tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\left(G^{\alpha\beta\gamma\delta} D_{\gamma\delta} - \frac{E}{Z} g^{\alpha\beta}\right) \tag{1.1}$$

$$(Z = F_{\alpha\beta\gamma\delta} g^{\alpha\beta} g^{\gamma\delta}, \; E = g^{\alpha\beta} D_{\alpha\beta}, \; \Xi = G^{\alpha\beta\gamma\delta} D_{\alpha\beta} D_{\gamma\delta}). \tag{1.2}$$

Since the tensors $\boldsymbol{F}$ and $\boldsymbol{G}$ are highly symmetric, we may interchange both the indices within a pair of indices and pairs themselves.

Let us restrict the consideration of an isotropic body for which the components of the tensor $\boldsymbol{F}$ can be expressed in terms of two constants (ρ and σ):

$$F_{\alpha\beta\gamma\delta} = \rho g_{\alpha\beta} g_{\gamma\delta} + \sigma g_{\alpha\gamma} g_{\beta\delta} (\gamma, \delta). \tag{1.3}$$

The components of the tensor $\boldsymbol{G}$ are as follows [16]:

$$G^{\alpha\beta\gamma\delta} = \frac{1}{\sigma}\left(g^{\alpha\gamma} g^{\beta\delta} - \frac{\rho}{3\rho + \sigma} g^{\alpha\beta} g^{\gamma\delta}\right)(\gamma, \delta). \tag{1.4}$$

Note that the constants ρ and σ are related to Lame's constants (λ and μ) as

$$-\rho = \frac{\lambda}{2\mu(3\lambda + 2\mu)}, \quad \sigma = \frac{1}{2\mu}.$$

Using formulas (4) and the second formula in (1.2), we obtain

$$G^{\alpha\beta\gamma\delta} D_{\gamma\delta} = \frac{1}{\sigma}\left(g^{\alpha\gamma} g^{\beta\delta} D_{\gamma\delta} - \frac{\rho E}{3\rho + \sigma} g^{\alpha\beta}\right). \tag{1.5}$$

According to (1.2)–(1.4), we have

$$Z = 3(3\rho + \sigma), \quad \Xi = \frac{1}{\sigma}\left(Y - \frac{\rho E^2}{3\rho + \sigma}\right) \tag{1.6}$$

$$(Y = g^{\alpha\gamma} g^{\beta\delta} D_{\alpha\beta} D_{\gamma\delta}). \tag{1.7}$$

With the notation

$$\sqrt{\frac{1}{\sigma}\left(Y - \frac{E^2}{3}\right)} \equiv \Omega \tag{1.8}$$

and formulas (1.5) and (1.6), Eqs. (1.1) become

$$S^{\alpha\beta} = \frac{1}{\sigma}\left[g^{\alpha\gamma} g^{\beta\delta} D_{\gamma\delta} - \frac{\rho E}{3\rho + \sigma} g^{\alpha\beta} - \tilde{\varphi}(\Omega)\left(g^{\alpha\gamma} g^{\beta\delta} D_{\gamma\delta} - \frac{E}{3} g^{\alpha\beta}\right)\right]. \tag{1.9}$$

Note that Eqs. (1.9) are identical with Kauderer's equations [15]. However, they have isolated linear terms and, therefore, are preferable for formulating the boundary-value problem.

If $\tilde{\varphi}(\Omega) = 0$, then Eqs. (1.9) degenerate into Hooke's equations [9]:

$$S^{\alpha\beta} = \frac{1}{\sigma}\left(g^{\alpha\gamma} g^{\beta\delta} D_{\gamma\delta} - \frac{\rho E}{3\rho + \sigma} g^{\alpha\beta}\right).$$

1.2. Scalar Function. Following [14], we introduce a constant $\upsilon > 0$ and assume that the function $\widetilde{\varphi}(\Omega)$ is such that

$$\widetilde{\varphi}(\Omega)|_{\Omega\le\upsilon} = 0, \tag{1.10}$$

$$\widetilde{\varphi}(\Omega)|_{\Omega>\upsilon} = \frac{\Omega - \upsilon + p - \sqrt[3]{\sqrt{q^3 + r^2} - r} + \sqrt[3]{\sqrt{q^3 + r^2} + r}}{\Omega}, \tag{1.11}$$

$$p = \frac{b}{3c}, \quad q = p^2 + \frac{1}{3c}, \quad r = p^3 - \frac{1}{2c}(\Omega - \upsilon + p). \tag{1.12}$$

The constant υ and the coefficients b and c will be particularized below.

2. Generalities. We choose Cartesian coordinates x^1, x^2, x^3 to describe the body. Then the components of the metric tensor $\boldsymbol{g}$ are expressed as

$$g^{\varepsilon\varsigma} = \begin{cases} 1, & \varepsilon = \varsigma, \\ 0, & \varepsilon \ne \varsigma. \end{cases} \tag{2.1}$$

2.1. Governing Equations. Let us derive the governing equations for the components of the displacement vector $\boldsymbol{u}$. The components of the strain tensor $\boldsymbol{D}$ and the components of the displacement vector $\boldsymbol{u}$ are related as follows [9]:

$$D_{\varepsilon\varsigma} = \frac{\partial u_\varepsilon}{\partial x^\varsigma}(\varepsilon, \varsigma). \tag{2.2}$$

Using the second formula in (1.2) and formulas (2.1), (2.2), we get

$$\mathrm{E} = \sum_{\beta=1}^{3} \frac{\partial u_\beta}{\partial x^\beta}. \tag{2.3}$$

Formulas (1.7), (2.1), and (2.2) yield

$$\mathrm{Y} = \frac{1}{4}\sum_{\gamma=1}^{3}\sum_{\delta=1}^{3}\left(\frac{\partial u_\gamma}{\partial x^\delta} + \frac{\partial u_\delta}{\partial x^\gamma}\right)\left(\frac{\partial u_\gamma}{\partial x^\delta} + \frac{\partial u_\delta}{\partial x^\gamma}\right). \tag{2.4}$$

With (2.2), Eqs. (1.9) can be represented as

$$S^{\alpha\beta} = \frac{1}{\sigma}\left\{\frac{1}{2}g^{\alpha\gamma}g^{\beta\delta}\left(\frac{\partial u_\gamma}{\partial x^\delta} + \frac{\partial u_\delta}{\partial x^\gamma}\right) - \frac{\rho \mathrm{E}}{3\rho + \sigma}g^{\alpha\beta}\right.$$

$$\left. - \widetilde{\varphi}(\Omega)\left[\frac{1}{2}g^{\alpha\gamma}g^{\beta\delta}\left(\frac{\partial u_\gamma}{\partial x^\delta} + \frac{\partial u_\delta}{\partial x^\gamma}\right) - \frac{\mathrm{E}}{3}g^{\alpha\beta}\right]\right\}. \tag{2.5}$$

In the special case of generalized plane stress state, the components of the stress tensor $\boldsymbol{S}$ are given by

$$S^{\alpha\beta} = S^{\alpha\beta}(x^1, x^2) \quad (\alpha = 1, 2, \ \beta = 1, 2), \tag{2.6}$$

$$S^{\alpha\beta} = 0 \quad (\alpha = 1, 2, \beta = 3, \ \alpha = 3, \ \beta = 1, 2, \ \alpha = 3, \beta = 3). \tag{2.7}$$

Using formulas (2.1), (2.3), and (2.5), we get

$$S^{11} = \frac{1}{\sigma}\left\{\frac{1}{3\rho + \sigma}\left[(2\rho + \sigma)\frac{\partial u_1}{\partial x^1} - \rho\left(\frac{\partial u_2}{\partial x^2} + \frac{\partial u_3}{\partial x^3}\right)\right] - \frac{1}{3}\widetilde{\varphi}(\Omega)\left(2\frac{\partial u_1}{\partial x^1} - \frac{\partial u_2}{\partial x^2} - \frac{\partial u_3}{\partial x^3}\right)\right\},$$

$$S^{22} = \frac{1}{\sigma}\left\{\frac{1}{3\rho+\sigma}\left[(2\rho+\sigma)\frac{\partial u_2}{\partial x^2} - \rho\left(\frac{\partial u_1}{\partial x^1} + \frac{\partial u_3}{\partial x^3}\right)\right] - \frac{1}{3}\tilde{\varphi}(\Omega)\left(2\frac{\partial u_2}{\partial x^2} - \frac{\partial u_1}{\partial x^1} - \frac{\partial u_3}{\partial x^3}\right)\right\},$$

$$S^{33} = \frac{1}{\sigma}\left\{\frac{1}{3\rho+\sigma}\left[(2\rho+\sigma)\frac{\partial u_3}{\partial x^3} - \rho\left(\frac{\partial u_1}{\partial x^1} + \frac{\partial u_2}{\partial x^2}\right)\right] - \frac{1}{3}\tilde{\varphi}(\Omega)\left(2\frac{\partial u_3}{\partial x^3} - \frac{\partial u_1}{\partial x^1} - \frac{\partial u_2}{\partial x^2}\right)\right\}. \quad (2.8)$$

Equalities (2.7) and the third equation in (2.8) give

$$\frac{\partial u_3}{\partial x^3} = \frac{3\rho+\sigma}{2\rho+\sigma}\left[\frac{\rho}{3\rho+\sigma}\left(\frac{\partial u_1}{\partial x^1} - \frac{\partial u_2}{\partial x^2}\right) + \frac{1}{3}\tilde{\varphi}(\Omega)\left(2\frac{\partial u_3}{\partial x^3} - \frac{\partial u_1}{\partial x^1} - \frac{\partial u_2}{\partial x^2}\right)\right]. \quad (2.9)$$

Substituting the first two equations in (2.8) into (2.9) yields

$$S^{11} = \frac{1}{\sigma(2\rho+\sigma)}\left\{(\rho+\sigma)\frac{\partial u_1}{\partial x^1} - \rho\frac{\partial u_2}{\partial x^2} - \frac{1}{3}\tilde{\varphi}(\Omega)\left[(3\rho+2\sigma)\frac{\partial u_1}{\partial x^1} - (3\rho+\sigma)\frac{\partial u_2}{\partial x^2} - \sigma\frac{\partial u_3}{\partial x^3}\right]\right\},$$

$$S^{22} = \frac{1}{\sigma(2\rho+\sigma)}\left\{(\rho+\sigma)\frac{\partial u_2}{\partial x^2} - \rho\frac{\partial u_1}{\partial x^1} - \frac{1}{3}\tilde{\varphi}(\Omega)\left[(3\rho+2\sigma)\frac{\partial u_2}{\partial x^2} - (3\rho+\sigma)\frac{\partial u_1}{\partial x^1} - \sigma\frac{\partial u_3}{\partial x^3}\right]\right\}. \quad (2.10)$$

Using formulas (2.1) and (2.5), we get

$$S^{12} = \frac{1}{2\sigma}\left[\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1} - \tilde{\varphi}(\Omega)\left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1}\right)\right],$$

$$S^{21} = \frac{1}{2\sigma}\left[\frac{\partial u_2}{\partial x^1} + \frac{\partial u_1}{\partial x^2} - \tilde{\varphi}(\Omega)\left(\frac{\partial u_2}{\partial x^1} + \frac{\partial u_1}{\partial x^2}\right)\right]. \quad (2.11)$$

Since $\tilde{\varphi}(\Omega) \neq 1$, Eqs. (2.5) with (2.1) and (2.7) lead to

$$\frac{\partial u_\gamma}{\partial x^\delta} + \frac{\partial u_\delta}{\partial x^\gamma} = 0 \qquad (\gamma = 1, 2,\ \delta = 3,\ \gamma = 3,\ \delta = 1, 2). \quad (2.12)$$

With (2.12), formula (2.4) becomes:

$$\Upsilon = \frac{1}{4}\sum_{\gamma=1}^{2}\sum_{\delta=1}^{2}\left(\frac{\partial u_\gamma}{\partial x^\delta} + \frac{\partial u_\delta}{\partial x^\gamma}\right)\left(\frac{\partial u_\gamma}{\partial x^\delta} + \frac{\partial u_\delta}{\partial x^\gamma}\right) + \frac{\partial u_3}{\partial x^3}\frac{\partial u_3}{\partial x^3}. \quad (2.13)$$

Let us rearrange Eqs. (2.10) and (2.11) into

$$S^{11} = \frac{1}{\sigma(2\rho+\sigma)}\left[(\rho+\sigma)\frac{\partial u_1}{\partial x^1} - \rho\frac{\partial u_2}{\partial x^2}\right] - T^{11},$$

$$S^{12} = \frac{1}{2\sigma}\left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1}\right) - T^{12}, \qquad S^{21} = \frac{1}{2\sigma}\left(\frac{\partial u_2}{\partial x^1} + \frac{\partial u_1}{\partial x^2}\right) - T^{21},$$

$$S^{22} = \frac{1}{\sigma(2\rho+\sigma)}\left[(\rho+\sigma)\frac{\partial u_2}{\partial x^2} - \rho\frac{\partial u_1}{\partial x^1}\right] - T^{22}, \quad (2.14)$$

where

$$T^{11} = \frac{1}{3\sigma(2\rho+\sigma)} \tilde{\varphi}(\Omega) \left[(3\rho+2\sigma)\frac{\partial u_1}{\partial x^1} - (3\rho+\sigma)\frac{\partial u_2}{\partial x^2} - \sigma\frac{\partial u_3}{\partial x^3} \right],$$

$$T^{12} = \frac{1}{2\sigma} \tilde{\varphi}(\Omega) \left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1} \right), \qquad T^{21} = \frac{1}{2\sigma} \tilde{\varphi}(\Omega) \left(\frac{\partial u_2}{\partial x^1} + \frac{\partial u_1}{\partial x^2} \right),$$

$$T^{22} = \frac{1}{3\sigma(2\rho+\sigma)} \tilde{\varphi}(\Omega) \left[(3\rho+2\sigma)\frac{\partial u_2}{\partial x^2} - (3\rho+\sigma)\frac{\partial u_1}{\partial x^1} - \sigma\frac{\partial u_3}{\partial x^3} \right]. \tag{2.15}$$

The equilibrium equations can be written for the components of the stress tensor $\boldsymbol{S}$ [9]:

$$\frac{\partial S^{\alpha\beta}}{\partial x^{\beta}} = 0. \tag{2.16}$$

With (2.6) and (2.7), Eqs. (2.16) become

$$\frac{\partial S^{11}}{\partial x^1} + \frac{\partial S^{12}}{\partial x^2} = 0,$$

$$\frac{\partial S^{21}}{\partial x^1} + \frac{\partial S^{22}}{\partial x^2} = 0. \tag{2.17}$$

Let the constants ρ and σ be independent of the coordinates x^1 and x^2. Denote

$$2\sigma(2\rho+\sigma)T^{\alpha\beta} \equiv \underline{T}^{\alpha\beta} \qquad (\alpha = 1, 2, \ \beta = 1, 2). \tag{2.18}$$

Then, substituting (2.14) into (2.17) and considering (2.18), we arrive at the following second-order partial differential equations for u_1 and u_2 with respect to the coordinates x^1 and x^2:

$$2(\rho+\sigma)\frac{\partial^2 u_1}{\partial x^1 \partial x^1} + \sigma\frac{\partial^2 u_2}{\partial x^1 \partial x^2} + (2\rho+\sigma)\frac{\partial^2 u_1}{\partial x^2 \partial x^2} = \underline{Q}^1,$$

$$(2\rho+\sigma)\frac{\partial^2 u_2}{\partial x^1 \partial x^1} + \sigma\frac{\partial^2 u_1}{\partial x^1 \partial x^2} + 2(\rho+\sigma)\frac{\partial^2 u_2}{\partial x^2 \partial x^2} = \underline{Q}^2 \tag{2.19}$$

$$\left(\underline{Q}^1 = \frac{\partial \underline{T}^{11}}{\partial x^1} + \frac{\partial \underline{T}^{12}}{\partial x^2}, \underline{Q}^2 = \frac{\partial \underline{T}^{21}}{\partial x^1} + \frac{\partial \underline{T}^{22}}{\partial x^2} \right). \tag{2.20}$$

The boundary conditions for the components of the stress vector $\boldsymbol{P}$ are the following [9]:

$$S^{\alpha\beta} n_{\beta} = P^{\alpha}, \tag{2.21}$$

where n_{β} are the components of the outward unit normal $\boldsymbol{n}$.

With (2.7), conditions (2.21) become

$$S^{11} n_1 + S^{12} n_2 = P^1,$$

$$S^{21} n_1 + S^{22} n_2 = P^2. \tag{2.22}$$

Let

$$2\sigma(2\rho+\sigma)P^{\alpha} \equiv \underline{P}^{\alpha} \qquad (\alpha = 1, 2). \tag{2.23}$$

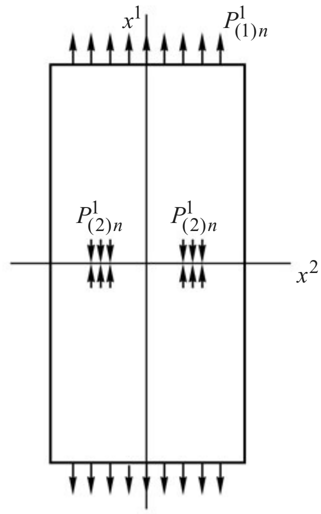


Fig. 1

From (2.14), (2.18), (2.22), (2.23), we derive the following first-order partial differential equations for u_1 and u_2 with respect to x^1 and x^2:

$$2\left[(\rho+\sigma)\frac{\partial u_1}{\partial x^1}-\rho\frac{\partial u_2}{\partial x^2}\right]n_1+(2\rho+\sigma)\left(\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}\right)n_2=\underline{P}^1+\underline{R}^1,$$

$$(2\rho+\sigma)\left(\frac{\partial u_2}{\partial x^1}+\frac{\partial u_1}{\partial x^2}\right)n_1+2\left[(\rho+\sigma)\frac{\partial u_2}{\partial x^2}-\rho\frac{\partial u_1}{\partial x^1}\right]n_2=\underline{P}^2+\underline{R}^2 \tag{2.24}$$

$$(\underline{R}^1=\underline{T}^{11}n_1+\underline{T}^{12}n_2,\ \underline{R}^2=\underline{T}^{21}n_1+\underline{T}^{22}n_2). \tag{2.25}$$

Equations (2.19) and (2.24) can be integrated with Ilyushin's method [2]. In the first approximation, the values of $\underline{Q}^1, \underline{Q}^2$ and $\underline{R}^1, \underline{R}^2$ are equated to zero. In each subsequent approximation, they are found from the previous one.

If $\tilde{\varphi}(\Omega)=0$, then $\underline{Q}^1, \underline{Q}^2$ and $\underline{R}^1, \underline{R}^2$ are equal to zero, according to (2.20), (2.15), (2.18), (2.25). Therefore, integrating Eqs. (2.19) and (2.24) in the first approximation means solving the boundary-value problem for a linear elastic body.

2.2. Boundary-Value Problem Formulation. Consider a thin rectangular body with a central crack (Fig. 1) and symmetry axes aligned with the x^1- and x^2-axes.

If the body is stretched along the x^1-axis, fracture process zones occur near both crack tips. They are modeled by slits. Uniformly distributed stresses are applied to their faces. These stresses should be determined by solving a boundary-value problem.

Let P^1 and P^2 be given on the surfaces of the body, crack, and slits. Since there is symmetry about the x^1- and x^2-axes, we can examine just a quarter (for example, the one occupying the first quadrant) of the body.

On the upper surface of the body, we have

$$n_1=1,\quad n_2=0. \tag{2.26}$$

With (2.26), Eqs. (2.24) become:

$$2\left[(\rho+\sigma)\frac{\partial u_1}{\partial x^1}-\rho\frac{\partial u_2}{\partial x^2}\right]=\underline{P}^1+\underline{R}^1,$$

$$(2\rho+\sigma)\left(\frac{\partial u_2}{\partial x^1}+\frac{\partial u_1}{\partial x^2}\right)=\underline{P}^2+\underline{R}^2. \tag{2.27}$$

Using (2.25) and (2.26), we get

$$\underline{R}^1=\underline{T}^{11}, \quad \underline{R}^2=\underline{T}^{21}. \tag{2.28}$$

On the lateral surface of the body, we have

$$n_1=0, \quad n_2=1. \tag{2.29}$$

With (2.29), Eqs. (2.24) become:

$$(2\rho+\sigma)\left(\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}\right)=\underline{P}^1+\underline{R}^1,$$

$$2\left[(\rho+\sigma)\frac{\partial u_2}{\partial x^2}-\rho\frac{\partial u_1}{\partial x^1}\right]=\underline{P}^2+\underline{R}^2. \tag{2.30}$$

Using (2.25) and (2.29), we get

$$\underline{R}^1=\underline{T}^{12}, \quad \underline{R}^2=\underline{T}^{22}. \tag{2.31}$$

On the upper surfaces of the crack and slit, we have

$$-n_1=1, \quad n_2=0. \tag{2.32}$$

With (2.32), Eqs. (2.24) become:

$$-2\left[(\rho+\sigma)\frac{\partial u_1}{\partial x^1}-\rho\frac{\partial u_2}{\partial x^2}\right]=\underline{P}^1+\underline{R}^1,$$

$$-(2\rho+\sigma)\left(\frac{\partial u_2}{\partial x^1}+\frac{\partial u_1}{\partial x^2}\right)=\underline{P}^2+\underline{R}^2. \tag{2.33}$$

Using (2.25) and (2.32), we get

$$-\underline{R}^1=\underline{T}^{11}, \quad -\underline{R}^2=\underline{T}^{21}. \tag{2.34}$$

For reasons of symmetry about the x^1- and x^2-axes,

$$u_1(x^1,-x^2)-u_1(x^1,+x^2)=0, \quad u_2(x^1,-x^2)+u_2(x^1,+x^2)=0,$$

$$u_1(-x^1,x^2)+u_1(+x^1,x^2)=0, \quad u_2(-x^1,x^2)-u_2(+x^1,x^2)=0. \tag{2.35}$$

Due to symmetry about the x^2-axis, we have

$$u_1=0 \tag{2.36}$$

at the slit tip.

To derive the equation for the component u_2, we choose an arbitrary point (a^1, a^2) near the slit tip and assume that all the partial derivatives of the component u_2 with respect to x^1 and x^2 exist at this point. Then the coordinates of the slit tip are $a^1+\varepsilon^1, a^2+\varepsilon^2$.

Let us expand u_2 into a multiple Taylor series in powers of ε^1 and ε^2:

$$u_2 = u_2(a^1, a^2) + \sum_{\beta=1}^{2} \left.\frac{\partial u_2}{\partial x^\beta}\right|_{(a^1, a^2)} \varepsilon^\beta + \frac{1}{2}\sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} \left.\frac{\partial^2 u_2}{\partial x^\beta \partial x^\gamma}\right|_{(a^1, a^2)} \varepsilon^\beta \varepsilon^\gamma . \qquad (2.37)$$

Rearranging this formula, we arrive at the equation

$$-u_2 + u_2(a^1, a^2) + \left.\frac{\partial u_2}{\partial x^1}\right|_{(a^1, a^2)} \varepsilon^1 + \left.\frac{\partial u_2}{\partial x^2}\right|_{(a^1, a^2)} \varepsilon^2$$

$$+\frac{1}{2}\left(\left.\frac{\partial^2 u_2}{\partial x^1 \partial x^1}\right|_{(a^1, a^2)} \varepsilon^1\varepsilon^1 + 2\left.\frac{\partial^2 u_2}{\partial x^1 \partial x^2}\right|_{(a^1, a^2)} \varepsilon^1\varepsilon^2 + \left.\frac{\partial^2 u_2}{\partial x^2 \partial x^2}\right|_{(a^1, a^2)} \varepsilon^2\varepsilon^2 \right) = 0. \qquad (2.38)$$

2.3. Discretization of Variables. Introducing a step h, we form sets of values of the coordinates:

$$\underset{i}{x}^1 = (i-2)h \quad (i = 1, 2, \ldots, d), \qquad \underset{j}{x}^2 = (j-2)h \quad (j = 1, 2, \ldots, e).$$

Let the crack and slit tips be at a point A with coordinates $\underset{2}{x}^1, \underset{f}{x}^2$ and a point B with coordinates $\underset{2}{x}^1, \underset{g}{x}^2$, respectively.

Expressing the partial derivatives of u_1 and u_2 with respect to x^1 and x^2 in terms of finite differences, using Eqs. (2.19), (2.27), (2.30), (2.33), (2.35), (2.36), (2.38), setting $-\varepsilon^1 = \varepsilon^2 = h$, and taking into account formulas (2.20), (2.28), (2.31), (2.34), notation (2.18), formulas (1.10)–(1.12), (2.15), notation (1.8), formulas (2.3), (2.9), (2.13), we obtain a system of linear algebraic equations for $u_1(\underset{i}{x}^1, \underset{j}{x}^2)$ and $u_2(\underset{i}{x}^1, \underset{j}{x}^2)$. To solve the system, we will use the method [14].

3. Numerical Example. Let us analyze the influence of the fracture process zone on the crack opening displacement in a nonlinear elastic body. To this end, it is necessary to analyze the deformation of a linear elastic body and a nonlinear elastic body with a crack. Let the lengths of the crack and fracture process zone be equal in both bodies.

3.1. Solving the Boundary-Value Problem. Symmetrizing (1.3) and using (2.1), we get

$$F_{\alpha\beta\gamma\delta} = \begin{cases} \rho + \sigma, & \alpha = \beta = \gamma = \delta, \\ \rho, & \alpha = \beta \neq \gamma = \delta. \end{cases} \qquad (3.1)$$

Equating the components $F_{\alpha\beta\gamma\delta}$ $(\alpha = \beta = \gamma = \delta,\ \alpha = \beta \neq \gamma = \delta)$ to the arithmetic means of the values given in [14], we calculate the constants ρ and σ with formulas (3.1): $-\rho = 0.046 \cdot 10^{-10}$ Pa^{-1}, $\sigma = 0.222 \cdot 10^{-10}$ Pa^{-1}.

As in [14], we set $\upsilon = 3.25 \cdot 10^2$ Pa$^{1/2}$, $b = 0.1964347 \cdot 10^{-2}$ Pa$^{-1/2}$, $c = 0.5632820 \cdot 10^{-4}$ Pa^{-1}. Also $h = 0.02 \cdot 10^{-2}$ m, $d = 302$, $e = 152$, $f = 62$, $g = 72$). We examined the cases of absence and presence of a fracture process zone in both bodies.

The crack and slit lengths are $1.20 \cdot 10^{-2}$ m and $0.20 \cdot 10^{-2}$ m, respectively.

Only the component P^1 was nonzero (on the upper surface of the body and on the upper surface of the slit):

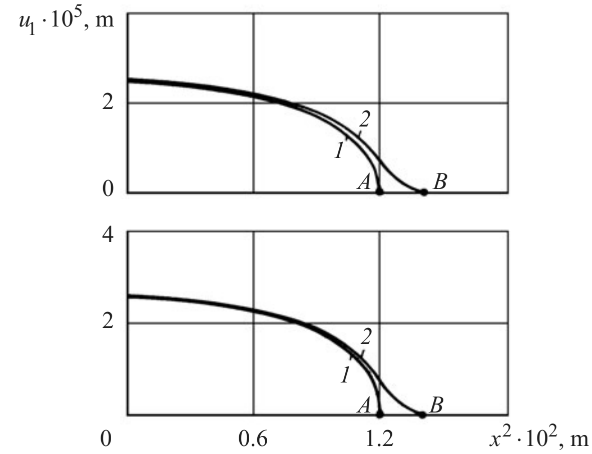
$$P^1(\underset{d}{x}^1, \underset{j}{x}^2) = P^1_{(1)} \qquad (j = 2, \ldots, e-1),$$

$$P^1(\underset{2}{x}^1, \underset{j}{x}^2) = P^1_{(2)} \qquad (j = f, \ldots, g-1).$$

In the presence of fracture process zone, the boundary-value problem is solved assuming that at the slit tip, the component S^{11} has the same value as at all other points of the upper surface of the slit.

The unknowns $u_1(\underset{i}{x}^1, \underset{j}{x}^2)$ and $u_2(\underset{i}{x}^1, \underset{j}{x}^2)$ were calculated to the 15th approximation.

3.2. Analysis of the Results. Consider the case of presence of a fracture process zone. For example, if $P^1_{(1)} = 5.00 \cdot 10^7$ Pa, then $P^1_{(2)} = -15.90 \cdot 10^7$ Pa for the linear elastic body and $P^1_{(2)} = -14.14 \cdot 10^7$ Pa for the nonlinear elastic body. Note that $P^1_{(2)}$ is



b

Fig. 2

considerably greater for the nonlinear elastic body. The difference between the values of $P^1_{(2)}$ for both bodies decreases with $P^1_{(1)}$. For example, if $P^1_{(1)} = 4.00 \cdot 10^7$ Pa, then $P^1_{(2)} = -12.72 \cdot 10^7$ Pa for the linear elastic body and $P^1_{(2)} = -12.31 \cdot 10^7$ Pa for the nonlinear elastic body. This is due to the reduction of the nonlinear zone.

Let us analyze the opening displacements of the crack and slit. For simplicity, by the opening of the crack and slit, we will mean the displacement of particles of their upper surface along the x^1-axis.

The values of $u_1(\underset{i}{x^1}, \underset{j}{x^2})$ found by solving the boundary-value problem are plotted versus the coordinate x^2 for $x^1 = 0$ in Figs. 2*a* and 2*b* for linear elastic and nonlinear elastic bodies, respectively (curves *1* and *2* represent the cases of absence and presence of fracture process zone, respectively).

It can be seen that the fracture process zone causes a minor opening at the middle of the crack in both linear and nonlinear elastic bodies. For example, at $x^2 = 0$, it has increased from $2.445 \cdot 10^{-5}$ to $2.482 \cdot 10^{-5}$ m in the linear elastic body and from $2.556 \cdot 10^{-5}$ to $2.557 \cdot 10^{-5}$ m in the nonlinear elastic body. More substantial changes in the crack opening displacement are observed near the crack tip. For example, at $x^2 = 1.18 \cdot 10^{-2}$ m, it has increased from $0.417 \cdot 10^{-5}$ to $0.787 \cdot 10^{-5}$ m in the linear elastic body and from $0.586 \cdot 10^{-5}$ to $0.896 \cdot 10^{-5}$ m in the nonlinear elastic body. The opening of the crack at its tip in the nonlinear elastic body is noticeably greater than in the linear elastic body. For example, at $x^2 = 1.20 \cdot 10^{-2}$ m, it is equal to $0.597 \cdot 10^{-5}$ m in the linear elastic body and to $0.690 \cdot 10^{-5}$ m inr the nonlinear elastic body. Globally, the effect of the fracture process zone on the crack opening displacement is stronger in the linear elastic body.

The opening of the slit decreases, at first abruptly and then gradually, with distance from the crack tip. For example, at $x^2 = 1.22 \cdot 10^{-2}$ m, it is equal to $0.477 \cdot 10^{-5}$ m in the linear elastic body and to $0.552 \cdot 10^{-5}$ m inr the nonlinear elastic body. At $x^2 = 1.38 \cdot 10^{-2}$ m, it is equal to $0.016 \cdot 10^{-5}$ m in the linear elastic body and to $0.020 \cdot 10^{-5}$ m in the nonlinear elastic body.

Denote the slope of the tangent to the boundaries of the crack and slit by *M*. Since the strains are small, *M* can be calculated with the formula

$$M = \left.\frac{\partial u_1}{\partial x^2}\right|_{x^1 = 0} .$$

At $x^2 = 1.20 \cdot 10^{-2}$ m, the value of *M* is extreme for both bodies, i.e., the boundaries of the crack and slit have inflection points.

At $x^2 = 1.40 \cdot 10^{-2}$ m, the value of *M* is close to zero, i.e., the slit smoothly closes at the tip.

4. Conclusions. We have analyzed a nonlinear elastic body with a mode I crack. The fracture process zone near the crack tip has been modeled by a slit with some stresses being applied to its faces. By solving the problem, we have ascertained how the fracture process zone influences the crack opening displacement. It has been established that the shape of crack faces is in qualitative agreement with the experimental data [3, 12] and that the fracture process zone considerably increases the opening displacement of the crack at its tip. This supports the use of the critical crack opening displacement. It has been shown that the nonlinearity of the body has a strong effect on the crack opening displacement at the crack tip.

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