International Applied Mechanics, Vol. 47, No. 5, November, 2011 (Russian Original Vol. 47, No. 5, September, 2011)

THEORY OF LONG-TERM MICRODAMAGE OF PHYSICALLY NONLINEAR HOMOGENEOUS MATERIALS

L. P. Khoroshun and E. N. Shikula

A theory of long-term damage of physically nonlinear homogeneous materials is proposed. Damage is modeled by randomly dispersed micropores. The failure criterion for a microvolume is characterized by its stress-rupture strength. It is determined by the dependence of the time to brittle fracture on the difference between the equivalent stress and its limit, which is the ultimate strength, according to the Huber–Mises criterion, and assumed to be a random function of coordinates. An equation of damage (porosity) balance in a physically nonlinear material at an arbitrary time is formulated. Algorithms of calculating the time dependence of microdamage and macrostresses are developed and the corresponding curves are plotted. The effect of the nonlinearity of the material on its macrodeformation and damage is analyzed

Keywords: physical nonlinearity, stochastic structure, long-term damage, effective characteristics, porosity balance equation

Introduction. A possible cause of failure of materials and structural members under loading is the occurrence and development of dispersed microdamages, which commonly lead to the formation of main cracks. Physically, the damage of a material may be considered as dispersed defects such as microcracks, microvoids, or destroyed microvolumes. They reduce the effective or bearing portion of the material that resists loads.

Experimental data on and experience of using structural members and structures suggest that damage can be either short-term (occurring instantaneously after the application of stresses or strains) or long-term (building up with time after the application of load). A structural theory of short-term microdamage of homogeneous and composite materials was proposed in [7, 9, 10]. It employs the mechanics of microinhomogeneous bodies of structure stochastic and models dispersed microdamages by quasispherical micropores [5]. Long-term damage is usually considered as accumulation of dispersed microdamages such as micropores and microcracks. At the microscopic level, the strength of a material is statistically inhomogeneous, i.e., the ultimate strength and stress-rupture curves for a microvolume are random functions of coordinates with certain distribution density or cumulative distribution. When a macrospecimen is subject to constant tensile stress, some microvolumes whose ultimate strength is less than the applied stress are damaged, i.e., microcracks or micropores form in their place. Microvolumes where the stress is less than, yet close to the ultimate strength are damaged after a lapse of time, which depends on the difference between the applied stress and the ultimate microstrength. The theory of long-term damage of homogeneous and fibrous materials was developed in [8, 11, 12] based on models and methods of the mechanics of stochastically inhomogeneous materials.

Under high loads, many materials exhibit physical nonlinearity. This type of nonlinearity is typical of metals and polymeric materials at high temperatures. Therefore, it is important to generalize the theory of long-term damage of homogeneous materials [8] based on models and methods of the mechanics of stochastically inhomogeneous materials to physically nonlinear materials. The stochastic equations of elasticity of porous materials whose skeleton is physically nonlinear underlie the structural theory of long-term damage of physically nonlinear homogeneous materials. The damage of a material is

S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterov St., Kyiv, Ukraine 03057, e-mail: stochac@inmech.kiev.ua. Translated from Prikladnaya Mekhanika, Vol. 47, No. 5, pp. 68–78, September 2011. Original article submitted Ocrober 2, 2009.

modeled by dispersed microvolumes destroyed to become randomly arranged micropores. The failure criterion for a single microvolume is determined by its stress-rupture strength described by a fractional or exponential power function, which is, in turn, determined by the dependence of the time to brittle failure on the difference between the equivalent stress and its limit, which characterizes the ultimate strength according to the Huber–Mises criterion. The ultimate strength is assumed to be a random function of coordinates whose one-point distribution is described by a power function on some interval or by the Weibull function. The effective elastic properties and the stress–strain state of a physically nonlinear material with randomly arranged microdamages are determined from the stochastic equations of elasticity of physically nonlinear porous materials. We will derive the equation of damage (porosity) balance at an arbitrary time from the properties of the distribution functions and ergodicity of the random field of ultimate microstrength and the dependence of the time to brittle failure for a microvolume on its stress state and ultimate microstrength. The macrostress–macrostrain relations for a physically nonlinear porous material and the porosity balance equation form a closed-loop system describing the joint processes of physically nonlinear deformation and microdamage. We will use an iteration method to develop algorithms for calculating the microdamage and macrostresses as functions of time and to plot the respective curves.

1. Let us consider the physically nonlinear deformation of an isotropic material described by the dependence of the bulk (K) and shear (μ) moduli on strains and accompanied by microdamage. The microdamage of the material is modeled by randomly arranged quasispherical micropores occurring in those microvolumes where the stresses exceed the ultimate microstrength.

The stresses and strains at an arbitrary point of a physically nonlinear porous material are related by

$$\sigma_{ij} = \lambda(\varepsilon_{\alpha\beta})\varepsilon_{rr}\delta_{ij} + 2\mu(\varepsilon_{\alpha\beta})\varepsilon_{ij} \qquad (\lambda = K - 2/3\mu), \tag{1.1}$$

where the bulk (K) and shear (μ) moduli deterministically depending on the strains $\varepsilon_{\alpha\beta}$ are random functions of coordinates that have the values $K(\varepsilon^1_{\alpha\beta})$, $\mu(\varepsilon^1_{\alpha\beta})$ in the skeleton and $K = \mu = 0$ in pores, the index 1 referring to the skeleton.

If a macrovolume (which is a volume much less than the pores and distances between them) is subject to homogeneous macrostresses and macrostrains, the microstresses σ_{ij} and microstrains ε_{ij} are ergodic statistically homogeneous random functions of coordinates. Their expectations $\langle\sigma_{ij}\rangle$ and $\langle\varepsilon_{ij}\rangle$ at an arbitrary point are equal to the macrostresses and macrostrains, respectively. Substituting (1.1) into the equilibrium equation

$$\sigma_{ij,j} = 0 \tag{1.2}$$

and using the kinematic equations

$$\varepsilon_{ij} = u_{(i,j)} \equiv \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.3}$$

we obtain an equation for the fluctuations of displacements

$$\mu_c u^0_{i,rr} + (\lambda_c + \mu_c)u^0_{r,ri} = \{[\lambda(\varepsilon_{\alpha\beta}) - \lambda_c]\varepsilon_{rr}\delta_{ij} + 2[\mu(\varepsilon_{\alpha\beta}) - \mu_c]\varepsilon_{ij}\}_{,j}, \tag{1.4}$$

where λ_c and μ_c are the constant elastic moduli of a comparison body; the fluctuations of displacements are given by

$$u_i = \langle\varepsilon_{ij}\rangle x_j + u^0_i. \tag{1.5}$$

The boundary condition on a boundary s at infinity follows from (1.5):

$$u^0_i\Big|_s = 0. \tag{1.6}$$

Using Green's function satisfying the equation

$$\mu_c G_{in,rr}(x^{(1)}_r - x^{(2)}_r) + (\lambda_c + \mu_c)G_{rn,ri}(x^{(1)}_r - x^{(2)}_r) + \delta(x^{(1)}_r - x^{(2)}_r)\delta_{in} = 0, \tag{1.7}$$

we reduce the boundary-value problem (1.4), (1.6) to an integral equation for the strain tensor:

$$\varepsilon_{ij}^{(1)} = \langle \varepsilon_{ij} \rangle + K_{ijpq}(x_r^{(1)} - x_r^{(2)})\{[\lambda^{(2)}(\varepsilon_{\alpha\beta}^{(2)}) - \lambda_c]\varepsilon_{rr}^{(2)}\delta_{pq} + 2[\mu^{(2)}(\varepsilon_{\alpha\beta}^{(2)}) - \mu_c]\varepsilon_{pq}^{(2)}\}, \tag{1.8}$$

where the integral operator K_{ijpq} is defined by

$$K_{ijpq}(x_r^{(1)} - x_r^{(2)})\varphi^{(2)} = \int_{V^{(2)}} G_{(ip,j)q}(x_r^{(1)} - x_r^{(2)})(\varphi^{(2)} - \langle\varphi\rangle)dV^{(2)}, \tag{1.9}$$

where the superscript in parentheses denotes a point in space.

The stresses σ_{ij}^1 and strains ε_{ij}^1 in the skeleton (undamaged portion of the material) can be represented as the sum

$$\sigma_{ij}^1 = \langle \sigma_{ij}^1 \rangle + \sigma_{ij}^{10}, \qquad \varepsilon_{ij}^1 = \langle \varepsilon_{ij}^1 \rangle + \varepsilon_{ij}^{10}, \tag{1.10}$$

where $\langle \sigma_{ij}^1 \rangle$, $\langle \varepsilon_{ij}^1 \rangle$ are the average stresses and strains over the skeleton; σ_{ij}^{10}, ε_{ij}^{10} are the respective fluctuations within the skeleton. If these fluctuations are neglected, the nonlinear relation (1.1) becomes

$$\langle \sigma_{ij}^1 \rangle = \lambda(\langle \varepsilon_{\alpha\beta}^1 \rangle)\langle \varepsilon_{rr}^1 \rangle\delta_{ij} + 2\mu(\langle \varepsilon_{\alpha\beta}^1 \rangle)\langle \varepsilon_{ij}^1 \rangle, \tag{1.11}$$

whence follows an expression for the macrostresses:

$$\langle \sigma_{ij} \rangle = (1-p)\langle \sigma_{ij}^1 \rangle = (1-p)[\lambda(\langle \varepsilon_{\alpha\beta}^1 \rangle)\langle \varepsilon_{rr}^1 \rangle\delta_{ij} + 2\mu(\langle \varepsilon_{\alpha\beta}^1 \rangle)\langle \varepsilon_{ij}^1 \rangle]. \tag{1.12}$$

To establish the relationship between the macrostresses and macrostrains, it is necessary to calculate $\langle \varepsilon_{ij}^1 \rangle$ as a function of $\langle \varepsilon_{ij} \rangle$ and substitute it into (1.12). To this end, we will average Eq. (1.8) using conditional density $f(\varepsilon_{ij}^{(1)}, \varepsilon_{ij}^{(2)}|_1^{(1)})$ (distribution density of strains at points $x_r^{(1)}$ and $x_r^{(2)}$ provided that these points are in the skeleton). Then, neglecting the fluctuations of strains within the skeleton, we obtain a system of nonlinear algebraic equations for the average strains in the skeleton [6]:

$$\langle \varepsilon_{ij}^1 \rangle = \langle \varepsilon_{ij} \rangle + K_{ijpq}^{11}\{[\lambda(\langle \varepsilon_{\alpha\beta}^1 \rangle) - \lambda_c]\langle \varepsilon_{rr}^1 \rangle\delta_{pq} + 2[\mu(\langle \varepsilon_{\alpha\beta}^1 \rangle) - \mu_c]\langle \varepsilon_{pq}^1 \rangle\}, \tag{1.13}$$

where the matrix operator K_{ijpq}^{11} is defined by

$$K_{ijpq}^{11} = K_{ijpq}(x_r^{(1)} - x_r^{(2)})p_{11}(x_r^{(1)} - x_r^{(2)}), \tag{1.14}$$

where $p_{11}(x_r^{(1)} - x_r^{(2)}) = f({}_1^{(2)}|_1^{(1)})$ is the probability of transition from the point $x_r^{(1)}$, $x_r^{(2)}$ to the point $x_r^{(2)}$ within the skeleton. If the pores are quasispherical and dispersed statistically isotropically, the transition probability is defined by

$$p_{11}(r) = 1 - p[1 - \varphi(r)], \qquad r^2 = (x_i^{(2)} - x_i^{(1)})(x_i^{(2)} - x_i^{(1)}), \tag{1.15}$$

where $\varphi(r)$ is a correlation coefficient such that $\varphi(0) = 1$, $\varphi(\infty) = 0$.

The macrostress–macrostrain relationship follows from (1.12)–(1.15):

$$\langle \sigma_{ij} \rangle = \left(K^* - \frac{2}{3}\mu^*\right)\langle \varepsilon_{rr} \rangle\delta_{ij} + 2\mu^*\langle \varepsilon_{ij} \rangle, \tag{1.16}$$

where the effective moduli K^* and μ^* are functions of p and $\langle \varepsilon_{ij} \rangle$. The effective moduli of a porous physically nonlinear material can be determined using the following iterative algorithm [2, 6]:

$$K^{*(n)} = \frac{4K(\langle \varepsilon_{ij}^1 \rangle^{(n)})\mu(\langle \varepsilon_{ij}^1 \rangle^{(n)})(1-p)^2}{3K(\langle \varepsilon_{ij}^1 \rangle^{(n)})p + 4\mu(\langle \varepsilon_{ij}^1 \rangle^{(n)})(1-p)},$$

$$\mu^{*(n)} = \frac{[9K(\langle\varepsilon_{ij}^1\rangle^{(n)}) + 8\mu(\langle\varepsilon_{ij}^1\rangle^{(n)})]\mu(\langle\varepsilon_{ij}^1\rangle^{(n)})(1-p)^2}{3K(\langle\varepsilon_{ij}^1\rangle^{(n)})(3-p) + 4\mu(\langle\varepsilon_{ij}^1\rangle^{(n)})(2+p)},$$

$$\langle\varepsilon_{ij}^1\rangle^{(n)} = \frac{1}{(1-p)}\left[\frac{K^{*(n-1)}}{K(\langle\varepsilon_{ij}^1\rangle^{(n)})}V_{ij\alpha\beta} + \frac{\mu^{*(n-1)}}{\mu(\langle\varepsilon_{ij}^1\rangle^{(n)})}D_{ij\alpha\beta}\right]\langle\varepsilon_{\alpha\beta}\rangle, \tag{1.17}$$

where $V_{ij\alpha\beta}$ and $D_{ij\alpha\beta}$ are the volumetric and deviatoric components of the unit tensor $I_{ij\alpha\beta}$,

$$I_{ij\alpha\beta} = V_{ij\alpha\beta} + D_{ij\alpha\beta}, \quad V_{ij\alpha\beta} = 1/3\delta_{ij}\delta_{\alpha\beta},$$
$$D_{ij\alpha\beta} = 1/2(\delta_{\alpha j}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha} - 2/3\delta_{ij}\delta_{\alpha\beta}). \tag{1.18}$$

Given macrostrains $\langle\varepsilon_{ij}\rangle$, the effective moduli are determined as the limits of the iterative process

$$K^* = \lim_{n\to\infty} K^{*(n)}, \quad \mu^* = \lim_{n\to\infty} \mu^{*(n)}. \tag{1.19}$$

We will use the Huber–Mises criterion [3] as a condition for the formation of a microdamage in a microvolume of the undamaged portion of the material:

$$I_{\langle\sigma\rangle}^1 = k, \tag{1.20}$$

$I_{\langle\sigma\rangle}^1 = (\langle\sigma_{ij}^1\rangle'\langle\sigma_{ij}^1\rangle')^{1/2}$ is the second invariant of the deviatoric average-stress tensor $\langle\sigma_{ij}^1\rangle'$ in the undamaged portion of the material; k is the ultimate microstrength, which is a random function of coordinates. Since the average stresses $\langle\sigma_{ij}^1\rangle$ in the undamaged portion are related to the macrostresses $\langle\sigma_{ij}\rangle$ as follows [6, 7]:

$$\langle\sigma_{ij}^1\rangle = \frac{1}{1-p}\langle\sigma_{ij}\rangle, \tag{1.21}$$

the invariant of the average-stress deviator $I_{\langle\sigma\rangle}^1$ is related to the invariant of the macrostress deviator $I_{\langle\sigma\rangle} = (\langle\sigma_{ij}\rangle'\langle\sigma_{ij}\rangle')^{1/2}$ and the invariant of the macrostrain deviator $I_{\langle\varepsilon\rangle} = (\langle\varepsilon_{ij}\rangle'\langle\varepsilon_{ij}\rangle')^{1/2}$ as

$$I_{\langle\sigma\rangle}^1 = \frac{1}{1-p}I_{\langle\sigma\rangle}, \tag{1.22}$$

$$I_{\langle\sigma\rangle}^1 = \frac{2\mu^*}{1-p}I_{\langle\varepsilon\rangle}. \tag{1.23}$$

A failure criterion in the macrostress space follows from (1.20), (1.22):

$$\frac{1}{1-p}I_{\langle\sigma\rangle} = k, \tag{1.24}$$

and a failure criterion in the macrostrain space follows from (1.20), (1.23):

$$\frac{2\mu^*(p,\langle\varepsilon_{ij}\rangle)}{1-p}I_{\langle\varepsilon\rangle} = k. \tag{1.25}$$

If the invariant $I_{\langle\sigma\rangle}^1$ does not reach the limiting value k in some microvolume of the material, then, according to the stress-rupture criterion, failure will occur in some time τ_k, which depends on the difference between $I_{\langle\sigma\rangle}^1$ and k. In the general case, this dependence can be represented as some function:

$$\tau_k = \varphi(I^1_{\langle\sigma\rangle}, k), \tag{1.26}$$

where $\varphi(k, k) = 0$ and $\varphi(0, k) = \infty$ according to (1.3).

The one-point distribution function $F(k)$ for some microvolume in the undamaged portion of the material can be approximated by a power function on some interval

$$F(k) = \begin{cases} 0, & k < k_0, \\ \left(\dfrac{k - k_0}{k_1 - k_0}\right)^\beta, & k_0 \le k \le k_1, \\ 1, & k > k_1 \end{cases} \tag{1.27}$$

or by the Weibull function

$$F(k) = \begin{cases} 0, & k < k_0, \\ 1 - \exp\left[-m(k - k_0)^\beta\right], & k \ge k_0, \end{cases} \tag{1.28}$$

where k_0 is the minimum value of k from which failure begins in some volumes of the material; k_1, m, β are constants found from strength scatter fitting in the material.

Assume that the random field of the ultimate microstrength k is statistically homogeneous, which is typical of real materials, and the microdamages and the distances between them are negligible compared with the inclusions and the distances between them. Then the distribution function $F(k)$ is ergodic because it defines the content of the undamaged portion of the material in which the ultimate microstrength is less than k. Therefore, if the stresses $\langle\sigma^1_{ij}\rangle$ are nonzero, the function $F(I^1_{\langle\sigma\rangle})$ defines, according to (1.20), (1.27), and (1.28), the content of instantaneously destroyed microvolumes. Since the damaged microvolumes are modeled by pores, we can write a balance equation for destroyed microvolumes or porosity of the material subject to short-term damage [7]:

$$p = p_0 + (1 - p_0) F(I^1_{\langle\sigma\rangle}), \tag{1.29}$$

where p_0 is the initial porosity.

If the homogeneous macrostresses $\langle\sigma_{ij}\rangle$ are given, then, according to (1.22), the porosity balance equation (1.29) becomes

$$p = p_0 + (1 - p_0) F\left(\frac{1}{1 - p} I_{\langle\sigma\rangle}\right). \tag{1.30}$$

If the macrostrains $\langle\varepsilon_{ij}\rangle$ are given, then, according to (1.23), we have

$$p = p_0 + (1 - p_0) F\left(\frac{2\mu^*(p, \langle\varepsilon_{ij}\rangle)}{1 - p} I_{\langle\varepsilon\rangle}\right). \tag{1.31}$$

If the stresses $\langle\sigma^1_{ij}\rangle$ act for some time t, then, according to the stress-rupture criterion (1.26), those microvolumes are destroyed that have k such that

$$t \ge \tau_k = \varphi(I^1_{\langle\sigma\rangle}, k), \tag{1.32}$$

where $I^1_{\langle\sigma\rangle}$ is defined by (1.22), (1.23).

At low temperatures, the time to brittle failure τ_k for real materials is finite beginning only from some value of $I^1_{\langle\sigma\rangle} > 0$. In this case, the durability function $\varphi(I^1_{\langle\sigma\rangle}, k)$ can be represented as follows [8]:

$$\varphi(I^1_{\langle\sigma\rangle}, k) = \tau_0 \left(\frac{k - I^1_{\langle\sigma\rangle}}{I^1_{\langle\sigma\rangle} - \gamma k} \right)^{n_1} \quad (\gamma k \le I^1_{\langle\sigma\rangle} \le k, \gamma < 1), \tag{1.33}$$

where some typical time τ_0, exponent n_1, and coefficient γ are determined from the fit of experimental durability curves.

Substituting (1.33) into (1.32), we arrive at the inequality

$$k \le I^1_{\langle\sigma\rangle} \frac{1 + \bar{t}^{-1/n_1}}{1 + \gamma \bar{t}^{-1/n_1}} \quad \left(\bar{t} = \frac{t}{\tau_0} \right). \tag{1.34}$$

Considering the definition of the distribution function $F(k)$, we conclude that the function $F[(I^1_{\langle\sigma\rangle})\psi(\bar{t})]$, where

$$\psi(\bar{t}) = \frac{1 + \bar{t}^{-1/n_1}}{1 + \gamma \bar{t}^{-1/n_1}}, \tag{1.35}$$

defines the relative content of the destroyed microvolumes in the undamaged portion of the material at the time $\bar{t}$. Then, in view of (1.29), the porosity balance equation for a material subject to long-term damage can be represented in the form

$$p = p_0 + (1 - p_0) F[(I^1_{\langle\sigma\rangle})\psi(\bar{t})] \tag{1.36}$$

or, in view of (1.22):

$$p = p_0 + (1 - p_0) F\left[\frac{I_{\langle\sigma\rangle}}{1 - p} \psi(\bar{t}) \right], \tag{1.37}$$

where p is a function of dimensionless time $\bar{t}$, and $I_{\langle\sigma\rangle}$ is defined by (1.23).

If the time τ_k is finite for arbitrary values of $I^1_{\langle\sigma\rangle}$, which may be observed at high temperatures, then the durability function can be approximated by an exponential power function [8]:

$$\varphi(I^1_{\langle\sigma\rangle}, k) = \tau_0 \{\exp m_1 \left[(k / I^1_{\langle\sigma\rangle})^{n_1} - 1 \right] - 1\}^{n_2}, \tag{1.38}$$

which has enough constants τ_0, m_1, n_1, n_2 to fit experimental curves. Substituting (1.38) into (1.32), we arrive at the inequality

$$k \le I^1_{\langle\sigma\rangle} \left[1 + \frac{1}{m_1} \ln(1 + \bar{t}^{1/n_2}) \right]^{1/n_1} \quad \left(\bar{t} = \frac{t}{\tau_0} \right). \tag{1.39}$$

Considering the definition of the distribution function $F(k)$, we conclude that the function $F[(I^1_{\langle\sigma\rangle})\psi(\bar{t})]$, where

$$\psi(\bar{t}) = \left[1 + \frac{1}{m_1} \ln(1 + \bar{t}^{1/n_2}) \right]^{1/n_1}, \tag{1.40}$$

defines the relative content of destroyed microvolumes in the undamaged portion of the material at the time $\bar{t}$. Then, in view of (1.1), the porosity balance equation for a material subject to long-term damage (1.26) can be represented in the form (1.36).

At $\bar{t} = 0$, the porosity balance equation (1.36) with (1.27), (1.28) defines the short-term (instantaneous) damage of the material. As time elapses, Eq. (1.27) with (1.28), (1.35), and (1.36) (or (1.40)) defines its long-term damage, which consists of short-term damage and additional time-dependent damage.

Equations (1.16), (1.27), (1.28), (1.35), (1.36) (or (1.40)) form a closed-loop system describing the joint processes of statistically homogeneous physically nonlinear deformation and long-term damage. Physical nonlinearity affects the way pores form during deformation, and the porosity of the material has an effect on its stress–strain curve. This is why the nonlinearity of the stress–strain curve is determined by the physical nonlinearity of the material and the increase in the porosity during physically nonlinear deformation.

To describe the coupled processes of physically nonlinear deformation and damage, it is necessary to find the macrostrain-dependent effective elastic moduli of the porous material by the iterative algorithm (1.17) and to determine the porosity from Eq. (1.36) also by an iterative method. At the nth step of the iterative process (1.17), Eq. (1.23), (1.36) is represented as

$$f^{(n)}(p) \equiv p - p_0 - (1-p_0)F\left(\frac{2\mu^{*(n)}(p,\langle\varepsilon_{ij}\rangle)}{1-p} I_{\langle\varepsilon\rangle}\right) = 0. \tag{1.41}$$

Then the root p of Eq. (1.29) at the mth step of some iterative process can be expressed as

$$p^{(m,n)} = Af^{(n)}(p^{(m-1)}), \tag{1.42}$$

where A is an operator on the function $f^{(n)}(p)$. The root is found as follows:

$$p = \lim_{\substack{m\to\infty \\ n\to\infty}} p^{(m,n)}. \tag{1.43}$$

At $\bar{t} = 0$, the porosity balance equation (1.29) with (1.22), (1.23) defines the short-term (instantaneous) damage of the material. As time elapses, Eq. (1.22) with (1.23), (1.35), (1.36) (or (1.40)) defines its long-term damage, which consists of short-term damage and additional time-dependent damage.

2. Let us analyze, as an example, the coupled processes of nonlinear deformation and microdamage of a material with linear bulk strains and shear strains described by a linear-hardening diagram

$$\langle\sigma_{rr}\rangle = K\langle\varepsilon_{rr}\rangle,$$

$$\langle\sigma_{ij}\rangle' = 2\mu(S)\langle\varepsilon_{ij}\rangle', \tag{2.1}$$

where the bulk modulus K does not depend on the strains, and the shear modulus $\mu(S)$ is described by

$$\mu(S) = \begin{cases} \mu_0, & T \le T_0, \\ \mu' + \left(1 - \dfrac{\mu'}{\mu_0}\right)\dfrac{T_0}{2S}, & T \ge T_0, \end{cases} \tag{2.2}$$

and

$$S = (\langle\varepsilon_{ij}\rangle'\langle\varepsilon_{ij}\rangle')^{1/2},$$

$$T = (\langle\sigma_{ij}\rangle'\langle\sigma_{ij}\rangle')^{1/2},$$

$$T_0 = \sqrt{2/3}\,\sigma_0, \tag{2.3}$$

where $\langle\varepsilon_{ij}\rangle'$ and $\langle\sigma_{ij}\rangle'$ are the strain and stress deviators; σ_0 is the tensile proportional limit assumed to be independent of the coordinates; μ_0 and μ' are material constants.

The root p of Eq. (2.14) can be found by the secant method [1]. Since the root p falls within $[p_0, 1]$, which follows from

$$f^{(n)}(p_0) \le 0,$$

$$f^{(n)}(1) \ge 0, \tag{2.4}$$

the zero approximation $p^{(0,n)}$ is defined by

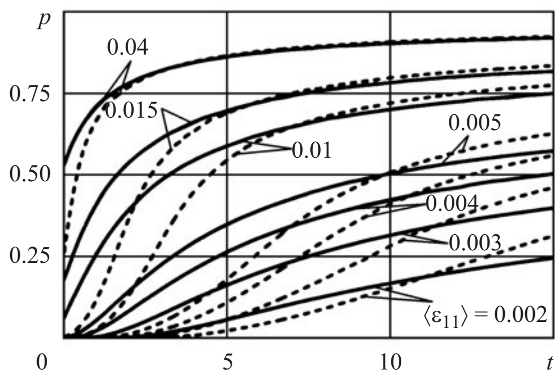

Fig. 1

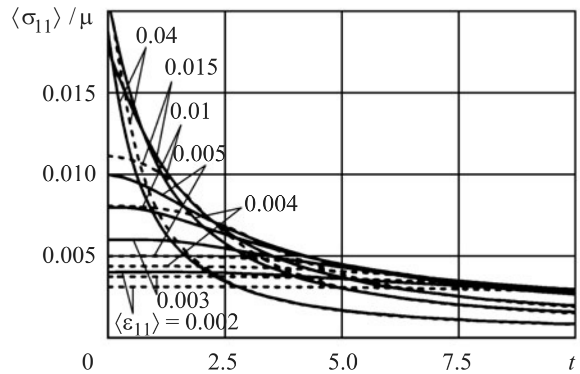

Fig. 2

$$p^{(0,n)} = \frac{a^{(0)} f^{(n)}(b^{(0)}) - b^{(0)} f^{(n)}(a^{(0)})}{f^{(n)}(b^{(0)}) - f^{(n)}(a^{(0)})}, \tag{2.5}$$

where $a^{(0)} = p_0, b^{(0)} = 1$. The subsequent approximations of the secant method are found in the iterative process

$$p^{(m,n)} = Af^{(n)}(p^{(m-1,n)})(p^{(m-1,n)}) \equiv \frac{a^{(m)} f^{(n)}(b^{(m)}) - b^{(m)} f^{(n)}(a^{(m)})}{f^{(n)}(b^{(m)}) - f^{(n)}(a^{(m)})}, \tag{2.6}$$

$$a^{(m)} = a^{(m-1)}, \quad b^{(m)} = p^{(m-1,n)} \quad \text{at} \quad f^{(n)}(a^{(m-1)}) f^{(n)}(p^{(m-1,n)}) \le 0,$$

$$a^{(m)} = p^{(m-1,n)}, \quad b^{(m)} = b^{(m-1)} \quad \text{at} \quad f^{(n)}(a^{(m-1)}) f^{(n)}(p^{(m-1,n)}) \ge 0$$

$$(m = 1, 2, \ldots),$$

which proceeds until

$$|f^{(n)}(p^{(m,n)})| < \delta, \tag{2.7}$$

where δ is the error of the root.

This theory was used to study the coupled processes of nonlinear deformation and microdamage of a homogeneous material described by the linear-hardening diagram (2.1), (2.2) with the following constants [2, 4]:

$$K = 3.33 \text{ GPa}, \quad \mu_0 = 1.11 \text{ GPa}, \quad \mu' = 0.331 \text{ GPa} \tag{2.8}$$

and the following proportional limits and minimum tensile microstrength ($\sigma_s = \sqrt{3/2}\, k_0$):

$$\sigma_0 = 0.003 \text{ GPa}, \quad \sigma_s = 0.011 \text{ GPa}. \tag{2.9}$$

If

$$\langle \varepsilon_{11} \rangle \ne 0, \quad \langle \sigma_{22} \rangle = \langle \sigma_{33} \rangle = 0, \tag{2.10}$$

then, according to (1.1), the macrostress $\langle \sigma_{11} \rangle$ is related to the macrostrain $\langle \varepsilon_{11} \rangle$ by

$$\langle \sigma_{11} \rangle = \frac{3K^* \mu^*}{K^* + \mu^*/3} \langle \varepsilon_{11} \rangle. \tag{2.11}$$

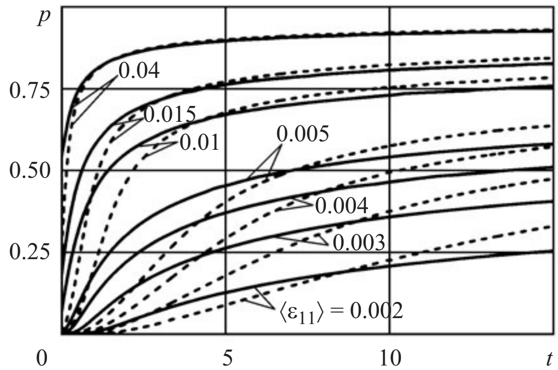

Fig. 3

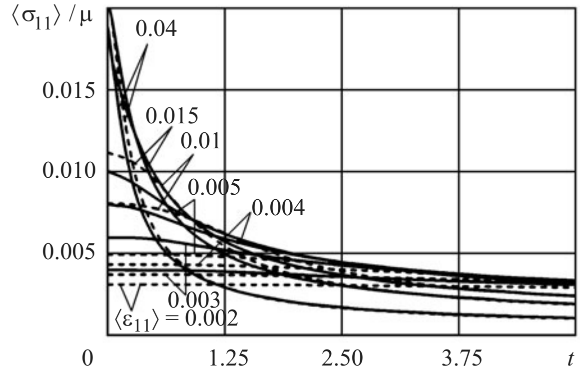

Fig. 4

It is assumed that

$$I_{\langle \varepsilon \rangle} = \sqrt{\frac{2}{3}} \frac{3K^* \langle \varepsilon_{11} \rangle}{2(K^* + \mu^* / 3)} \tag{2.12}$$

in (1.31), (1.37), which is equivalent to conditions (2.10).

Figure 1 shows (by solid lines) the porosity p of the linear-hardening material with $\psi(\bar{t})$ defined by (1.35) as a function of time $\bar{t}$ for different values of $\langle \varepsilon_{11} \rangle$. For comparison, the figure shows (by dashed lines) p versus $\bar{t}$ for a linear material. The same notation is used in Figs. 2–4. As is seen, physical nonlinearity has a significant effect on microdamage. The microdamage of the linear-hardening material occurs later (at greater values of $\bar{t}$) and more intensively than in the linear material, i.e., at great values of $\bar{t}$, the porosity of the linear-hardening material is higher than that of the linear material).

Figure 2 shows (by solid lines) the macrostress $\langle \sigma_{11} \rangle / \mu$ of the linear-hardening material with $\bar{t}$ defined by (1.35) as a function of time $\bar{t}$ for different values of $\langle \varepsilon_{11} \rangle$. As is seen, at small values of $\psi(\bar{t})$, the physical nonlinearity of the material has a significant effect on its stress state as well. At great values of $\bar{t}$, the effect of nonlinearity on the stress state is weak.

Figures 3 and 4 show the porosity p and macrostress $\langle \sigma_{11} \rangle / \mu$, respectively, of linear-hardening and linear materials with $\psi(\bar{t})$ defined by (1.40) as a function of time $\bar{t}$ for different values of $\langle \varepsilon_{11} \rangle$. As is seen, the curves are qualitatively similar to those for the function $\psi(\bar{t})$ defined by (1.35).

Conclusions. A theory of long-term damage of physically nonlinear homogeneous materials has been proposed. The damage of the material has been modeled by randomly arranged micropores. The failure criterion for a single microvolume is determined by its stress-rupture strength determined by the dependence of the time to brittle fracture on the difference between the equivalent stress and its limit, which characterizes the ultimate strength according to the Huber–Mises criterion. The equation of damage (porosity) balance of a physically nonlinear material at an arbitrary time has been formulated. Algorithms of calculating the time dependence of microdamage and macrostresses have been developed. The respective curves have been plotted. The effect of nonlinearity of the material on its macrodeformation and damage curves has been examined.

REFERENCES

1. Ya. S. Berezikovich, *Approximate Calculations* [in Russian], GITTL, Moscow–Leningrad (1949).
2. A. N. Guz, L. P. Khoroshun, G. A. Vanin, et al., *Materials Mechanics*, Vol. 1 of the three-volume series *Mechanics of Composite Materials and Structural Members* [in Russian], Naukova Dumka, Kyiv (1982).
3. L. M. Kachanov, *Fundamentals of Fracture Mechanics* [in Russian], Nauka, Moscow (1974).
4. A. F. Kregers, “Mathematical modeling of the thermal expansion of spatially reinforced composites,” *Mech. Comp. Mater.*, **24**, No. 3, 316–325 (1988).
5. V. P. Tamusz and V. S. Kuksenko, *Microfracture Mechanics of Polymeric Materials* [in Russian], Zinatne, Riga (1978).

6. L. P. Khoroshun, “Methods of theory of random functions in problems of macroscopic properties of microinhomogeneous media,” *Int. Appl. Mech.*, **14**, No. 2, 113–124 (1978).
7. L. P. Khoroshun, “Principles of the micromechanics of material damage. 1. Short-term damage,” *Int. Appl. Mech.*, **34**, No. 10, 1035–1041 (1998).
8. L. P. Khoroshun, “Principles of the micromechanics of material damage. 2. Long-term damage,” *Int. Appl. Mech.*, **43**, No. 2, 127–135 (2007).
9. L. P. Khoroshun and E. N. Shikula, “Mesomechanics of deformation and short-term damage of linear elastic homogeneous and composite materials,” *Int. Appl. Mech.*, **43**, No. 6, 591–620 (2007).
10. L. P. Khoroshun and E. N. Shikula, “Deformation and short-term damage of physically nonlinear stochastic composites,” *Int. Appl. Mech.*, **45**, No. 6, 613–634 (2009).
11. L. P. Khoroshun and E. N. Shikula, “Deformation and long-term damage of fibrous materials with the stress-rupture microstrength of the matrix described by a fractional-power function,” *Int. Appl. Mech.*, **45**, No. 11, 1196–1205 (2009).
12. L. P. Khoroshun and E. N. Shikula, “Coupled processes of deformation and long-term damage of fibrous materials with the microdurability of the matrix described by an exponential power function,” *Int. Appl. Mech.*, **46**, No. 1, 37–45 (2010).