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EXACT BENDING SOLUTIONS OF ORTHOTROPIC RECTANGULAR CANTILEVER THIN PLATES SUBJECTED TO ARBITRARY LOADS

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Exact bending solutions of orthotropic rectangular cantilever thin plates subjected to arbitrary loads are derived by using a novel double finite integral transform method. Since only the basic elasticity equations for orthotropic thin plates are used, the method presented in this paper eliminates the need to predetermine the deformation function and is hence completely rational thus more accurate than conventional semi-inverse methods, which presents a breakthrough in solving plate bending problems as they have long been bottlenecks in the history of elasticity. Numerical results are presented to demonstrate the validity and accuracy of the approach as compared with those previously reported in the literature

Keywords: rectangular cantilever thin plate, exact solution, finite integral transform

1. Introduction. Orthotropic rectangular thin plates are widely used in various engineering applications such as decks of contemporary steel bridges, corrugated plates and reinforced concrete slabs stiffened by orthogonal ribs. The bending of orthotropic, especially isotropic rectangular thin plates with various combinations of boundary conditions has been investigated for many years by different authors. It is well known that explicit analytic solutions of orthotropic or isotropic rectangular thin plates are available only for the cases with two opposite sides simply supported (i.e., Navier's solution, Levy's solution, etc.) while it is, so far, difficult to get the solutions which exactly satisfy both the partial differential equation and other boundary conditions of a plate. Accordingly, various methods have been studied. One of the most commonly used methods for exact bending solutions of isotropic plates is the superposition method, which could be extended to orthotropic plates [1–5]. Meanwhile, the technique of Fourier series expansion is another procedure for solving complex structures [6] as well as accurate bending solutions of plates [7]. Besides, a number of numerical methods have been utilized by many researchers to analyze problems of plates and shells [8, 9] such as the finite-difference method [10–13], finite-element method [14, 15], finite-strip method [16], integral equation method [17], method of discrete singular convolution [18], method of differential quadrature [19], differential quadrature element method [20], meshless method [21], and spline element method [22].

A cantilever thin plate is an important structural element while its bending has been one of the most difficult problems in the theory of elastic thin plate. Some approximate methods have been utilized for the problem of isotropy. The method of finite difference was firstly used to solve a cantilever plate with concentrated edge load by Holl [10]. The problem is also solved by Barton [11], Macneal [12], Livesly and Birchall [23] separately with the same method. Besides, some other approximate analysis of the bending of a rectangular cantilever plate by uniform normal pressure was presented by Nash [13]. The generalized variational principle was applied to rectangular thin plates by Shu and Shih [24], and the principle was then used by Plass et al. [25] for deflection and vibration problems of cantilever plates. Leissa and Niedenfuhr [26] obtained the solution for uniformly loaded cantilevered square plates using the technique of point matching and the Rayleigh–Ritz method. In addition, Chang [3–5] derived exact solutions for the bending of both uniformly loaded and concentrated loaded isotropic rectangular cantilever plates by using the method of superposition, which involved a skilful superposition of several problems, yet used smart trial functions.

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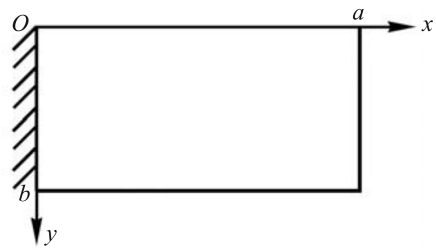

Fig. 1. An orthotropic rectangular cantilever thin plate

Integral transform is one of the best approaches to obtain exact solutions of some partial differential equations in the theory of elasticity [27]. The method has been often adopted to analyze some structural engineering problems [28]. However, to the authors' knowledge, there have been no reports on the analysis for an orthotropic or isotropic cantilever thin plate using finite integral transform.

In the present paper, a novel double finite integral transform method is adopted to acquire exact bending solutions of orthotropic rectangular cantilever thin plates under arbitrary loading. Unlike the traditional semi-inverse approaches in classical plate analysis employed by Timoshenko [1] and others such as Chang [3–5], where a trial deflection function has to be predetermined, the analysis in here is completely rational without any trial functions. The procedure of solution presented here enables one to attempt exact solutions for more problems of plates which have to hitherto be analyzed using semi-inverse method or approximate approaches. It can be not only applied to other combinations of boundary conditions but also further extended to the problems of moderately thick plates as well as buckling, vibration, etc., some of which will be reported in future. To verify the accuracy of the approach in this paper, several cases of a rectangular cantilever thin plate are examined and the results are presented for an easy comparison with those found in the previous literatures. Excellent agreement is observed thus confirming the accuracy and applicability of the present method.

2. Integral Transform and Exact Bending Solutions for an Orthotropic Rectangular Cantilever Thin Plate. The coordinate system of a thin, orthotropic rectangular cantilever plate under consideration is illustrated in Fig. 1, where $0 \le x \le a$ and $0 \le y \le b$. The governing partial differential equation for bending of the plate for which the principal directions of orthotropy coincide with the x- and y-axes [1, 29] is

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = q, \tag{1}$$

where W is the transverse deflection of the plate midplane, q is the distributed transverse load, D_x and D_y are the flexural rigidities about the y- and x-axes, $H = D_1 + 2D_{xy}$ is called the effective torsional rigidity in which $D_1 = \nu_2 D_x = \nu_1 D_y$ is defined in terms of the reduced Poisson's ratios ν_1 and ν_2, respectively.

The internal forces of the plate are

$$M_x = -\left(D_x \frac{\partial^2 W}{\partial x^2} + D_1 \frac{\partial^2 W}{\partial y^2} \right), \tag{2}$$

$$M_y = -\left(D_y \frac{\partial^2 W}{\partial y^2} + D_1 \frac{\partial^2 W}{\partial x^2} \right), \tag{3}$$

$$M_{xy} = -2D_{xy} \frac{\partial^2 W}{\partial x \partial y}, \tag{4}$$

$$Q_x = -\frac{\partial}{\partial x} \left(D_x \frac{\partial^2 W}{\partial x^2} + H \frac{\partial^2 W}{\partial y^2} \right), \tag{5}$$

$$Q_y = -\frac{\partial}{\partial y}\left(D_y \frac{\partial^2 W}{\partial y^2} + H \frac{\partial^2 W}{\partial x^2}\right), \quad (6)$$

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y}, \quad (7)$$

$$V_y = Q_y + \frac{\partial M_{xy}}{\partial x}, \quad (8)$$

where M_x, M_y, M_{xy}, Q_x, Q_y, V_x, and V_y are the bending moments, torsional moment, shear forces, and total shear forces, respectively.

The boundary conditions of the plate can be expressed as

$$W\big|_{x=0} = 0, \quad (9)$$

$$\left.\frac{\partial W}{\partial x}\right|_{x=0} = 0, \quad (10)$$

$$M_x\big|_{x=a} = 0, \quad M_y\big|_{y=0} = 0, \quad M_y\big|_{y=b} = 0, \quad (11a–c)$$

$$V_x\big|_{x=a} = 0, \quad V_y\big|_{y=0} = 0, \quad V_y\big|_{y=b} = 0, \quad (12a–c)$$

$$-2D_{xy} \left.\frac{\partial^2 W}{\partial x \partial y}\right|_{x=a,\, y=0} = 0, \quad -2D_{xy} \left.\frac{\partial^2 W}{\partial x \partial y}\right|_{x=a,\, y=b} = 0. \quad (13a, b)$$

In the particular case of isotropy, we have $\nu_1 = \nu_2 = \nu$, $D_x = D_y = H = D$, $D_1 = \nu D$, and $D_{xy} = (1-\nu)D/2$, where D is the flexural rigidity and ν is Poisson's ratio. Hence the above equations can reduce to those of an isotropic plate.

To solve the partial differential equation Eq. , a double finite integral transform approach is utilized. Since $W(x, y)$, defined within a rectangular domain $0 \le x \le a$ and $0 \le y \le b$, is a function of the independent variables x and y, we define a double finite integral transform by the equation

$$W_{mn} = \int_0^a \int_0^b W(x, y) \sin \frac{\alpha_m}{2} x \cos \beta_n y dx dy \quad (m = 1, 3, 5, \ldots, n = 0, 1, 2, \ldots). \quad (14)$$

The inversion formula can be represented as

$$W(x, y) = \frac{4}{ab} \sum_{m=1,3,5,\ldots}^{\infty} \sum_{n=0,1,2,\ldots}^{\infty} W_{mn} \delta_n \sin \frac{\alpha_m}{2} x \cos \beta_n y, \quad (15)$$

where $\alpha_m = \frac{m\pi}{a}$, $\beta_n = \frac{n\pi}{b}$, and $\delta_n = \begin{cases} 1/2, & \text{if } n = 0, \\ 1, & \text{if } n = 1, 2, 3, \ldots . \end{cases}$

The double integral transforms of higher-order partial derivatives of W appeared in Eq. are derived respectively as follows:

$$\int_0^a \int_0^b \frac{\partial^4 W}{\partial x^4} \sin \frac{\alpha_m}{2} x \cos \beta_n y dx dy$$

$$= \int_0^b \left[(-1)^{\frac{m-1}{2}} \left.\frac{\partial^3 W}{\partial x^3}\right|_{x=a} + \frac{\alpha_m}{2} \left.\frac{\partial^2 W}{\partial x^2}\right|_{x=0} - (-1)^{\frac{m-1}{2}} \frac{\alpha_m^2}{4} \left.\frac{\partial W}{\partial x}\right|_{x=a} - \frac{\alpha_m^3}{8} . W\big|_{x=0} \right] \cos \beta_n y dy + \frac{\alpha_m^4}{16} W_{mn}$$

$$=\int_0^b\left[(-1)^{\frac{m-1}{2}}\frac{\partial^3 W}{\partial x^3}\bigg|_{x=a}+\frac{\alpha_m}{2}\frac{\partial^2 W}{\partial x^2}\bigg|_{x=0}-(-1)^{\frac{m-1}{2}}\frac{\alpha_m^2}{4}\frac{\partial W}{\partial x}\bigg|_{x=a}\right]\cos\beta_n y dy+\frac{\alpha_m^4}{16}W_{mn}, \quad (16)$$

$$\int_0^a\int_0^b\frac{\partial^4 W}{\partial y^4}\sin\frac{\alpha_m}{2}x\sin\beta_n y dx dy$$

$$=\int_0^a\left[(-1)^n\frac{\partial^3 W}{\partial y^3}\bigg|_{y=b}-\frac{\partial^3 W}{\partial y^3}\bigg|_{y=0}-(-1)^n\beta_n^2\frac{\partial W}{\partial y}\bigg|_{y=b}+\beta_n^2\frac{\partial W}{\partial y}\bigg|_{y=0}\right]\sin\frac{\alpha_m}{2}x dx+\beta_n^4 W_{mn}, \quad (17)$$

$$\int_0^a\int_0^b\frac{\partial^4 W}{\partial x^2\partial y^2}\sin\frac{\alpha_m}{2}x\cos\beta_n y dx dy$$

$$=(-1)^{\frac{m-1}{2}+n}\frac{\partial^2 W}{\partial x\partial y}\bigg|_{x=a,\,y=b}-(-1)^{\frac{m-1}{2}}\frac{\partial^2 W}{\partial x\partial y}\bigg|_{x=a,\,y=0}+(-1)^n\frac{\alpha_m}{2}\frac{\partial W}{\partial y}\bigg|_{x=0,\,y=b}-\frac{\alpha_m}{2}\frac{\partial W}{\partial y}\bigg|_{x=0,\,y=0}+\frac{\alpha_m^2\beta_n^2}{4}W_{mn}$$

$$-\frac{\alpha_m^2}{4}\int_0^a\left[(-1)^n\frac{\partial W}{\partial y}\bigg|_{y=b}-\frac{\partial W}{\partial y}\bigg|_{y=0}\right]\sin\frac{\alpha_m}{2}x dx-\beta_n^2\int_0^b\left[(-1)^{\frac{m-1}{2}}\frac{\partial W}{\partial x}\bigg|_{x=a}+\frac{\alpha_m}{2}W\big|_{x=0}\right]\cos\beta_n y dy$$

$$=\frac{\alpha_m^2\beta_n^2}{4}W_{mn}-\frac{\alpha_m^2}{4}\int_0^a\left[(-1)^n\frac{\partial W}{\partial y}\bigg|_{y=b}-\frac{\partial W}{\partial y}\bigg|_{y=0}\right]\sin\frac{\alpha_m}{2}x dx-(-1)^{\frac{m-1}{2}}\beta_n^2\int_0^b\left(\frac{\partial W}{\partial x}\bigg|_{x=a}\right)\cos\beta_n y dy, \quad (18)$$

in which three boundary conditions, i.e., Eqs. and (13, a, b), have been imposed to simplify the expressions.

By performing over Eq. the double finite integral transform and the substitution of Eqs. (16)–(18), we have

$$D_x(-1)^{\frac{m-1}{2}}\int_0^b\left(\frac{\partial^3 W}{\partial x^3}\bigg|_{x=a}\right)\cos\beta_n y dy+D_y(-1)^n\int_0^a\left(\frac{\partial^3 W}{\partial y^3}\bigg|_{y=b}\right)\sin\frac{\alpha_m}{2}x dx-D_y\int_0^a\left(\frac{\partial^3 W}{\partial y^3}\bigg|_{y=b}\right)\sin\frac{\alpha_m}{2}x dx$$

$$-(-1)^n\left(D_y\beta_n^2+\frac{H\alpha_m^2}{2}\right)\int_0^a\left(\frac{\partial W}{\partial y}\bigg|_{y=b}\right)\sin\frac{\alpha_m}{2}x dx+\left(D_y\beta_n^2+\frac{H\alpha_m^2}{2}\right)\int_0^a\left(\frac{\partial W}{\partial y}\bigg|_{y=0}\right)\sin\frac{\alpha_m}{2}x dx$$

$$-(-1)^{\frac{m-1}{2}}\left(\frac{D_x\alpha_m^2}{4}+2H\beta_n^2\right)\int_0^b\left(\frac{\partial W}{\partial x}\bigg|_{x=a}\right)\cos\beta_n y dy+\frac{D_x\alpha_m}{2}\int_0^b\left(\frac{\partial^2 W}{\partial x^2}\bigg|_{x=0}\right)\cos\beta_n y dy$$

$$+\left(\frac{D_x\alpha_m^4}{16}+D_y\beta_n^4+\frac{H\alpha_m^2\beta_n^2}{2}\right)W_{mn}=q_{mn}, \quad (19)$$

where $q_{mn}=\int_0^a\int_0^b q(x,y)\sin\frac{\alpha_m}{2}x\cos\beta_n y dx dy$ represents the transform of the load function $q(x, y)$.

After single finite cosine and sine transforms over Eqs. (12*a*) and (12*b*, *c*) respectively, we obtain

$$\int_0^b\left[D_x\frac{\partial^3 W}{\partial x^3}+(H+2D_{xy})\frac{\partial^3 W}{\partial x\partial y^2}\right]_{x=a}\cos\beta_n y\, dy=0, \quad (20a)$$

$$\int_0^a \left[D_y \frac{\partial^3 W}{\partial y^3} + (H + 2D_{xy}) \frac{\partial^3 W}{\partial x^2 \partial y} \right]_{y=0} \sin \frac{\alpha_m}{2} x \, dx = 0, \tag{20b}$$

$$\int_0^a \left[D_y \frac{\partial^3 W}{\partial y^3} + (H + 2D_{xy}) \frac{\partial^3 W}{\partial x^2 \partial y} \right]_{y=b} \sin \frac{\alpha_m}{2} x \, dx = 0, \tag{20c}$$

which can be written as

$$D_x \int_0^b \left(\left. \frac{\partial^3 W}{\partial x^3} \right|_{x=a} \right) \cos \beta_n y dy = (H + 2D_{xy}) \beta_n^2 \int_0^b \left(\left. \frac{\partial W}{\partial x} \right|_{x=a} \right) \cos \beta_n y dy, \tag{21a}$$

$$D_y \int_0^a \left(\left. \frac{\partial^3 W}{\partial y^3} \right|_{y=0} \right) \sin \frac{\alpha_m}{2} x dx = \frac{(H + 2D_{xy}) \alpha_m^2}{4} \int_0^a \left(\left. \frac{\partial W}{\partial y} \right|_{y=0} \right) \sin \frac{\alpha_m}{2} x dx, \tag{21b}$$

$$D_y \int_0^a \left(\left. \frac{\partial^3 W}{\partial y^3} \right|_{y=b} \right) \sin \frac{\alpha_m}{2} x dx = \frac{(H + 2D_{xy}) \alpha_m^2}{4} \int_0^a \left(\left. \frac{\partial W}{\partial y} \right|_{y=b} \right) \sin \frac{\alpha_m}{2} x dx. \tag{21c}$$

Substituting Eqs. (21*a*–*c*) into Eq. (19) yields

$$-(-1)^n \left[D_y \beta_n^2 + \frac{(H - 2D_{xy}) \alpha_m^2}{4} \right] \int_0^a \left(\left. \frac{\partial W}{\partial y} \right|_{y=b} \right) \sin \frac{\alpha_m}{2} x dx$$

$$+ \left[D_y \beta_n^2 + \frac{(H - 2D_{xy}) \alpha_m^2}{4} \right] \int_0^a \left(\left. \frac{\partial W}{\partial y} \right|_{y=0} \right) \sin \frac{\alpha_m}{2} x dx$$

$$-(-1)^{\frac{m-1}{2}} \left[\frac{D_x \alpha_m^2}{4} + (H - 2D_{xy}) \beta_n^2 \right] \int_0^b \left(\left. \frac{\partial W}{\partial x} \right|_{x=a} \right) \cos \beta_n y dy + \frac{D_x \alpha_m}{2} \int_0^b \left(\left. \frac{\partial^2 W}{\partial x^2} \right|_{x=0} \right) \cos \beta_n y dy$$

$$+ \left(\frac{D_x \alpha_m^4}{16} + D_y \beta_n^4 + \frac{H \alpha_m^2 \beta_n^2}{2} \right) W_{mn} = q_{mn}. \tag{22}$$

Let

$$I_m = \int_0^a \left(\left. \frac{\partial W}{\partial y} \right|_{y=b} \right) \sin \frac{\alpha_m}{2} x dx, \quad J_m = \int_0^a \left(\left. \frac{\partial W}{\partial y} \right|_{y=0} \right) \sin \frac{\alpha_m}{2} x dx, \tag{23a, b}$$

$$K_n = \int_0^b \left(\left. \frac{\partial W}{\partial x} \right|_{x=a} \right) \cos \beta_n y dy, \quad L_n = \int_0^b \left(\left. \frac{\partial^2 W}{\partial x^2} \right|_{x=0} \right) \cos \beta_n y dy. \tag{23c, d}$$

Accordingly Eq. (22) is expressed in terms of the unknown constants I_m, J_m, K_n, and L_n as

$$W_{mn} = \frac{1}{\dfrac{D_x \alpha_m^4}{16} + D_y \beta_n^4 + \dfrac{H \alpha_m^2 \beta_n^2}{2}}$$

$$\times\left\{q_{mn}+(-1)^n\left[D_y\beta_n^2+\frac{(H-2D_{xy})\alpha_m^2}{4}\right]I_m-\left[D_y\beta_n^2+\frac{(H-2D_{xy})\alpha_m^2}{4}\right]J_m\right.$$

$$\left.+(-1)^{\frac{m-1}{2}}\left[\frac{D_x\alpha_m^2}{4}+(H-2D_{xy})\beta_n^2\right]K_n-\frac{D_x\alpha_m}{2}L_n\right\}. \tag{24}$$

Substituting Eq. (24) into Eq. (15), one can get the expression of $W(x, y)$ with I_m, J_m, K_n, and L_n for $m = 1, 3, 5, \ldots$ and $n = 0, 1, 2, \ldots$. Equation (24) can meet the boundary conditions described by Eqs. (9), (12*a*–*c*) and (13*a*, *b*), as indicated above.

By substituting Eq. (15) into the remaining boundary conditions represented by Eqs. and (11*a*–*c*) observing the differentiation procedure of trigonometric series [7, 30], we obtain

$$\sum_{n=1,2,\ldots}^{\infty}\sum_{m=1,3,5,\ldots}^{\infty}\frac{2\alpha_m}{ab}\delta_n W_{mn}\cos\beta_n y=0, \tag{25a}$$

$$\sum_{n=1,2,\ldots}^{\infty}\sum_{m=1,3,5,\ldots}^{\infty}\frac{4}{ab}(-1)^{\frac{m-1}{2}}\delta_n\left[D_1(-1)^n I_m-D_1J_m+D_x(-1)^{\frac{m-1}{2}}K_n-\left(D_1\beta_n^2+\frac{D_x\alpha_m^2}{4}\right)W_{mn}\right]\cos\beta_n y=0, \tag{25b}$$

$$\sum_{m=1,3,5,\ldots}^{\infty}\sum_{n=0,1,2,\ldots}^{\infty}\frac{4}{ab}\delta_n\left[D_y(-1)^n I_m-D_yJ_m+D_1(-1)^{\frac{m-1}{2}}K_n-\left(D_y\beta_n^2+\frac{D_1\alpha_m^2}{4}\right)W_{mn}\right]\sin\frac{\alpha_m}{2}x=0, \tag{25c}$$

$$\sum_{m=1,3,5,\ldots}^{\infty}\sum_{n=0,1,2,\ldots}^{\infty}\frac{4}{ab}\delta_n\left[D_yI_m-D_y(-1)^n J_m+D_1(-1)^{\frac{m-1}{2}}K_n-\left(D_y\beta_n^2+\frac{D_1\alpha_m^2}{4}\right)(-1)^n W_{mn}\right]\sin\frac{\alpha_m}{2}x=0. \tag{25d}$$

Multiplying Eqs. (25*a*, *b*) by $\cos\beta_n y dy$ followed by integration from 0 to *b* yields Eqs. (26*a*, *b*). Multiplying Eqs. (25*c*, *d*) by $\sin\frac{\alpha_m}{2}xdx$ followed by integration from 0 to *a* yields Eqs. (26*c*, *d*). They are given as follows:

$$\sum_{m=1,3,5,\ldots}^{\infty}\alpha_m W_{mn}=0 \quad (n = 0, 1, 2, \ldots), \tag{26a}$$

$$\sum_{m=1,3,5,\ldots}^{\infty}(-1)^{\frac{m-1}{2}}\left[D_1(-1)^n I_m-D_1J_m+D_x(-1)^{\frac{m-1}{2}}K_n-\left(D_1\beta_n^2+\frac{D_x\alpha_m^2}{4}\right)W_{mn}\right]=0 \quad (n = 0, 1, 2, \ldots), \tag{26b}$$

$$\sum_{n=0,1,2,\ldots}^{\infty}\delta_n\left[D_y(-1)^n I_m-D_yJ_m+D_1(-1)^{\frac{m-1}{2}}K_n-\left(D_y\beta_n^2+\frac{D_1\alpha_m^2}{4}\right)W_{mn}\right]=0 \quad (n = 1, 3, 5, \ldots), \tag{26c}$$

$$\sum_{n=0,1,2,\ldots}^{\infty}\delta_n\left[D_yI_m-D_y(-1)^n J_m+D_1(-1)^{\frac{m-1}{2}}K_n-\left(D_y\beta_n^2+\frac{D_1\alpha_m^2}{4}\right)(-1)^n W_{mn}\right]=0 \quad (n = 1, 3, 5, \ldots). \tag{26d}$$

Substituting Eq. (24) into Eqs. (26*a*–*d*), we finally arrive at

$$(-1)^n\sum_{m=1,3,5,\ldots}^{\infty}\alpha_m C_{mn}R_{mn}I_m-\sum_{m=1,3,5,\ldots}^{\infty}\alpha_m C_{mn}R_{mn}J_m+\sum_{m=1,3,5,\ldots}^{\infty}(-1)^{\frac{m-1}{2}}\alpha_m C_{mn}T_{mn}K_n$$

$$-\frac{D_x}{2}\sum_{m=1,3,5,\ldots}^{\infty}\alpha_m^2 C_{mn}L_n = -\sum_{m=1,3,5,\ldots}^{\infty}\alpha_m C_{mn}L_n \qquad (n = 0,\ 1,\ 2,\ \ldots), \tag{27}$$

$$(-1)^n \sum_{m=1,3,5,\ldots}^{\infty}\left[(-1)^{\frac{m-1}{2}}D_1 - C_{mn}E_{mn}R_{mn}\right]I_m - \sum_{m=1,3,5,\ldots}^{\infty}\left[(-1)^{\frac{m-1}{2}}D_1 - C_{mn}E_{mn}R_{mn}\right]J_m$$

$$+\sum_{m=1,3,5,\ldots}^{\infty}\left[(-1)^{m-1}D_x - (-1)^{\frac{m-1}{2}}C_{mn}E_{mn}T_{mn}\right]K_m + \frac{D_x}{2}\sum_{m=1,3,5,\ldots}^{\infty}\alpha_m C_{mn}E_{mn}L_n$$

$$= \sum_{m=1,3,5,\ldots}^{\infty}C_{mn}E_{mn}q_{mn} \qquad (n = 0,\ 1,\ 2,\ \ldots), \tag{28}$$

$$\sum_{n=0,1,2,\ldots}^{\infty}\delta_n(-1)^n(D_y - C_{mn}F_{mn}R_{mn})I_m - \sum_{n=0,1,2,\ldots}^{\infty}\delta_n(D_y - C_{mn}F_{mn}R_{mn})J_m$$

$$+(-1)^{\frac{m-1}{2}}\sum_{n=0,1,2,\ldots}^{\infty}\delta_n(D_1 - C_{mn}F_{mn}T_{mn})K_n + \frac{D_x}{2}\sum_{n=0,1,2,\ldots}^{\infty}\delta_n\alpha_m C_{mn}F_{mn}L_n = \sum_{m=0,1,2,\ldots}^{\infty}C_{mn}F_{mn}q_{mn} \qquad (n = 1,\ 3,\ 5,\ \ldots), \tag{29}$$

$$\sum_{n=0,1,2,\ldots}^{\infty}\delta_n(D_y - C_{mn}F_{mn}R_{mn})I_m - \sum_{n=0,1,2,\ldots}^{\infty}\delta_n(-1)^n(D_y - C_{mn}F_{mn}R_{mn})J_m$$

$$+(-1)^{\frac{m-1}{2}}\sum_{n=0,1,2,\ldots}^{\infty}(-1)^n\delta_n(D_1 - C_{mn}F_{mn}T_{mn})K_n + \frac{D_x\alpha_m}{2}\sum_{n=0,1,2,\ldots}^{\infty}\delta_n(-1)^n C_{mn}F_{mn}L_n$$

$$= \sum_{m=0,1,2,\ldots}^{\infty}C_{mn}F_{mn}q_{mn} \qquad (m = 1,\ 3,\ 5,\ \ldots), \tag{30}$$

where

$$C_{mn} = \frac{\alpha_m}{\dfrac{D_x\alpha_m^4}{16} + D_y\beta_n^4 + \dfrac{H\alpha_m^2\beta_n^2}{2}}, \tag{31a}$$

$$E_{mn} = \left(D_1\beta_n^2 + \frac{D_x\alpha_m^2}{4}\right)(-1)^{\frac{m-1}{2}}, \qquad F_{mn} = \left(D_y\beta_n^2 + \frac{D_1\alpha_m^2}{4}\right), \tag{31b, c}$$

$$R_{mn} = D_y\beta_n^2 + \frac{(H - 2D_{xy})\alpha_m^2}{4}, \qquad T_{mn} = \frac{D_x\alpha_m^2}{4} + (H - 2D_{xy})\beta_n^2. \tag{31d, e}$$

Equations (27)–(30) are four infinite systems of linear simultaneous equations with respect to unknown constants I_m, J_m, K_n, and L_n ($m = 1, 3, 5, \ldots, n = 0, 1, 2, \ldots$). In practice, a finite number of terms in each set of equations are considered and the resulting sets of finite number of simultaneous equations are solved to determine the constants.

The bending moments along the clamped edge $x = 0$ could be obtained conveniently using the expression

$$M_x\big|_{x=0} = -\left(D_x\frac{\partial^2 W}{\partial x^2} + D_1\frac{\partial^2 W}{\partial y^2}\right)\bigg|_{x=0} = -\left(D_x\frac{\partial^2 W}{\partial x^2}\right)\bigg|_{x=0} = -D_x\left(\frac{2}{b}\sum_{n=0,1,2,\ldots}^{\infty}L_n\delta_n\cos\beta_n y\right). \tag{32}$$

TABLE 1. Deflections and bending moments for a uniformly loaded isotropic rectangular cantilever thin plate with $\nu = 0.3$

Parameter	a/b	x, y		Ref. [3]	FEM*	Present
$W\left(\frac{qa^4}{D}\right)$	1	$x = a$	$y = 0$	0.12933	0.12708	0.12722
			$y = 0.125a$	0.12998	0.12788	0.12797
			$y = 0.25a$	0.13056	0.12851	0.12857
			$y = 0.375a$	0.13091	0.12892	0.12895
			$y = 0.5a$	0.13102	0.12905	0.12908
		$y = 0$	$x = 0$	0	0	0
			$x = 0.25a$	0.011949	0.1182	0.011771
			$x = 0.5a$	0.044327	0.043221	0.043283
			$x = 0.75a$	0.085046	0.083888	0.084085
			$x = a$	0.12933	0.12708	0.12722
	1/2	x, y		Ref. [13]	Ref. [3]	Present
		$x = a$	$y = 0$	0.135	0.12540	0.124303
			$y = 0.125b$		0.12691	0.126015
			$y = 0.25b$	0.139	0.12784	0.127106
			$y = 0.375b$		0.12825	0.127626
			$y = 0.5b$	0.141	0.12837	0.127770
$M_x(qa^2)$	1	x, y		Ref. [3]	FEM	Present
		$x = 0$	$y = 0.125b$	–0.51270	–0.50399	–0.51240
			$y = 0.25b$	–0.53353	–0.52760	–0.52959
			$y = 0.375b$	–0.53550	–0.53058	–0.53135
			$y = 0.5b$	–0.53560	–0.53092	–0.53136
	1/2	x, y		Ref. [13]	Ref. [3]	Present
		$x = 0$	$y = 0.125b$		–0.51074	–0.51542
			$y = 0.25b$	–0.5047	–0.51386	–0.51606
			$y = 0.375b$		–0.51451	–0.51424
			$y = 0.5b$	–0.5082	–0.51049	–0.51362

* The finite elements solutions in this paper are adopted from Chang [3, 5], which were offered by Mr. Wu Liang-tze of Peking University.

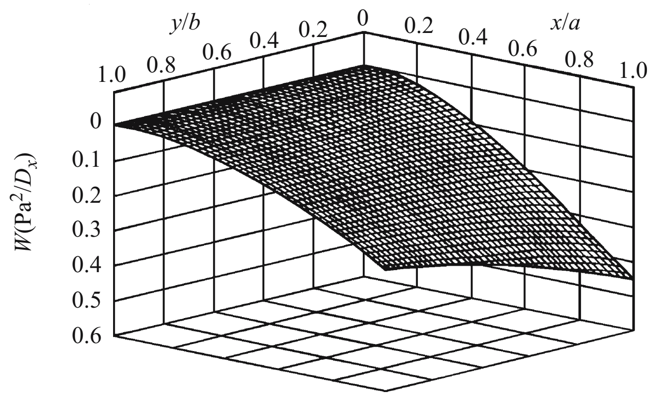

Fig. 2. The deflection surface of an orthotropic rectangular cantilever thin plate subjected to a concentrated load at the free corner $x = a,\ y = 0\,(a/b = 1)$

TABLE 2. Deflections and bending moments for an isotropic square cantilever thin plate subjected to a concentrated load at the plate center $x = a/2,\ y = b/2$, with $\nu = 0.3$

Parameter	*x*, *y*		Ref. [4]	Present
$W\left(\frac{Pa^2}{D}\right)$	$x = a$	$y = 0$	0.10353	0.105632
		$y = 0.125a$	0.10577	0.107802
		$y = 0.25a$	0.10773	0.109713
		$y = 0.375a$	0.10904	0.111051
		$y = 0.5a$	0.10957	0.111537
	$y = 0$	$x = 0$	0	0
		$x = 0.25a$	0.010044	0.010181
		$x = 0.5a$	0.037122	0.037648
		$x = 0.75a$	0.069947	0.071316
		$x = a$	0.10353	0.10563
$M_x(P)$	$x = 0$	$y = 0.125a$	–0.46448	–0.47254
		$y = 0.25a$	–0.52839	–0.53571
		$y = 0.375a$	–0.56979	–0.57703
		$y = 0.5a$	–0.58645	–0.59030

TABLE 3. Deflections and bending moments for an isotropic rectangular cantilever thin plate subjected to a concentrated load at the middle of the free edge $x = a,\ y = b/2$, with $\nu = 0.3$

Parameter	a/b	x, y		Ref. [10]	Ref. [5]	Present
$W\left(\frac{Pa^2}{D}\right)$	1/4	$x = a$	$y = 0$	0.03015	0.035158	0.034363
			$y = 0.125a$	0.04993	0.04823	0.041889
			$y = 0.25a$	0.08364	0.078971	0.073564
			$y = 0.375a$	0.13594	0.12678	0.12423
			$y = 0.5a$	0.18773	0.16991	0.16882
	1	x, y		Ref. [5]	FEM	Present
		$x = a$	$y = 0$	0.34120	0.32912	0.32908
			$y = 0.125b$	0.35006	0.33863	0.33833
			$y = 0.25b$	0.35929	0.34812	0.34774
			$y = 0.375b$	0.36769	0.35663	0.35628
			$y = 0.5b$	0.37239	0.36101	0.36104
		$y = 0$	$x = 0$	0		0
			$x = 0.25a$	0.02386		0.023158
			$x = 0.5a$	0.09940		0.096624
			$x = 0.75a$	0.2120		0.20483
			$x = a$	0.3412		0.32908
$M_x(P)$	1/4	x, y		Ref. [10]	Ref. [5]	Present
		$x = 0$	$y = 0.125b$	–0.12600	–0.13892	–0.11411
			$y = 0.25b$	–0.22672	–0.23034	–0.22175
			$y = 0.375b$	–0.37352	–0.40233	–0.39884
			$y = 0.5b$	–0.49672	–0.51798	–0.51287
	1	x, y		Ref. [5]	FEM	Present
		$x = 0$	$y = 0.125b$	–1.0042	–0.99064	–0.99090
			$y = 0.25b$	–1.1423	–1.0819	–1.0918
			$y = 0.375b$	–1.1514	–1.1186	–1.1244
			$y = 0.5b$	–1.1571	–1.1282	–1.1271

TABLE 4. Deflections and bending moments for an orthotropic rectangular cantilever thin plate subjected to a concentrated load at the free corner $x = a,\ y = 0\ (D_x = D_y, D_{xy} = 0.5D_x, \nu_1 = \nu_2 = 0.3)$

Parameter	a/b	$y = 0$	$y = 0.25b$		$y = 0.5b$		$y = 0.75b$	$y = b$
$W\left(\frac{Pa^2}{D_x}\right)$, $x = a$	1/4	0.3523	0.1231		0.03580		0.01104	0.004960
	1/2	0.3611	0.2282		0.1371		0.08764	0.06242
	1	0.4477	0.3843		0.3293		0.2857	0.2507
	2	0.7512	0.7214		0.6927		0.6657	0.6403
	4	1.447	1.432		1.417		1.403	1.388
		$x = 0$	$x = 0.25a$		$x = 0.5a$		$x = 0.75a$	$x = a$
$W\left(\frac{Pa^2}{D_x}\right)$, $y = 0$	1/4	0	0.03354		0.1170		0.2289	0.3523
	1/2	0	0.03407		0.1194		0.2343	0.3611
	1	0	0.03970		0.1439		0.2875	0.4477
	2	0	0.06316		0.2354		0.4781	0.7512
	4	0	0.1205		0.4504		0.9186	1.447
		$x = 0$	$x = 0.25a$		$x = 0.5a$		$x = 0.75a$	$x = a$
$W\left(\frac{Pa^2}{D_x}\right)$, $y = b$	1/4	0	0.0002005		0.001156		0.002854	0.004960
	1/2	0	0.002577		0.01479		0.03615	0.06242
	1	0	0.01343		0.06605		0.1507	0.2507
	2	0	0.04396		0.1862		0.3978	0.6403
	4	0	0.1084		0.4227		0.8754	1.388
		$y = 0.125b$	$y = 0.25b$	$y = 0.375b$	$y = 0.5b$	$y = 0.625b$	$y = 0.75b$	$y = 0.875b$
$M_x(P)$, $x = 0$	1/4	–0.7070	–0.3357	–0.1613	–0.08163	–0.04562	–0.02734	–0.01616
	1/2	–1.062	–0.7267	–0.5086	–0.3720	–0.2872	–0.2276	–0.1662
	1	–1.542	–1.327	–1.152	–1.026	–0.9227	–0.7983	–0.5912
	2	–2.567	–2.480	–2.342	–2.255	–2.158	–1.949	–1.503
	4	–4.357	–4.702	–4.733	–4.653	–4.485	–4.171	–3.450

After the constants I_m, J_m, K_n, and L_n obtained are substituted into Eq. (24), formula (15) gives the bending solutions of an orthotropic rectangular cantilever plate. The results are theoretically exact solutions when m and $n \to \infty$ while in practice we only take the larger ones to obtain desired accuracy.

3. Numerical Results. In order to verify the validity of the results derived in the paper, three cases of loading of an isotropic rectangular cantilever thin plate with different aspect ratios are examined:

(1) a uniform load of intensity q;

(2) a concentrated load at the center of the plate;

(3) a concentrated load at the middle of the free edge $x = a$.

Comparison with known results of Holl [10], Nash [13] and Chang [3–5] including the transverse deflections and bending moments at specific locations for cases above is presented in Tables 1–3, respectively, which displays excellent agreement.

For future comparison, exact bending solutions of an orthotropic cantilever thin plate subjected to a concentrated load at the free corner $(a, 0)$ are obtained and some numerical results are tabulated in Table 4. In addition, the deflection surface of the case is illustrated in Fig. 2 for direct viewing.

For sufficient accuracy of the solutions, we take the first 50 terms of I_m and J_m and the first 100 terms of K_n and L_n in the calculation. It should be noted that the convergence of the results is not so fast because of the double trigonometric series adopted. However, the simultaneous equations can be solved without any difficulty using mathematical packages such as MATLAB; and above all, the value of the present approach lies in its excellent ability to obtain exact bending solutions of a plate.

4. Conclusions. The present paper shows that the bending problem of an orthotropic rectangular cantilever thin plate can be solved accurately by a novel double finite integral transform method. The main advantage of the approach is it does not require the preselection of a deformation function, which, however, can scarcely be avoided in the traditional semi-inverse approaches. Also, the present approach provides an efficient procedure for more accurate results which should be of both academic and practical importance. Accordingly, it serves as a completely rational theoretical model for the bending problems of rectangular thin plates. The present method can be capable of dealing with any other combinations of boundary conditions while be further extended to the problems of moderately thick plates as well as buckling, vibration, etc., some of which will be explored in due course.

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REFERENCES

1. S. P. Timoshenko and S. W. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York (1959).
2. M. K. Huang and H. D. Conway, “Bending of a uniformly loaded rectangular plate with two adjacent edges clamped and the others either simply supported or free,” *J. Appl. Mech*., 451–460 (1952).
3. F. V. Chang, “Bending of uniformly cantilever rectangular plates,” *Appl. Math. Mech*., **1**, 371–383 (1980).
4. F. V. Chang, “Bending of a cantilever rectangular plate loaded discontinuously,” *Appl. Math. Mech*., **2**, 403–410 (1981).
5. F. V. Chang *Elastic Thin Plates* [in Chinese], 2nd Edition, Science Press, Beijing (1984).
6. Ya. M. Grigorenko, “Using discrete Fourier series to solve boundary-value stress problems for elastic bodies with complex geometry and structure,” *Int. Appl. Mech*., **45**, No. 5, 469–513 (2009).
7. M. R. Khalili, K. Malekzadeh, and R. K. Mittal, “A new approach to static and dynamic analysis of composite plates with different boundary conditions,” *Compos. Struct*., **69**, 149–155 (2005).
8. A. Ya. Grigorenko and N. P. Yaremchenko, “Stress state of nonthin orthotropic shells with varying thickness and rectangular planform,” *Int. Appl. Mech*., **44**, No. 8, 905–915 (2008).
9. E. I. Bespalova, “Solving stationary problems for shallow shells by a generalized Kantorovich–Vlasov method,” *Int. Appl. Mech*., **44**, No. 11, 1283–1293 (2008).
10. D. L. Holl, “Cantilever plate with concentrated edge load,” *J. Appl. Mech*., **4**, 8–10 (1937).
11. M. V. Barton, *Finite Difference Equations for the Analysis of Thin Rectangular Plates with Combinations of Fixed and Free Edges*, Defense Research Lab. Rep. No. 175, Univ. of Texas (1948).
12. R. H. MacNeal, “The solution of elastic plate problems by electrical analogies,” *J. Appl. Mech*., **18**, 59–67 (1951).

13. W. A. Nash, "Several approximate analysis of the bending of a rectangular cantilever plate by uniform normal pressure," *J. Appl. Mech*., 33–36 (1952).
14. O. C. Zienkiewicz and Y. K. Cheung, "The finite element method for analysis of elastic isotropic and orthotropic slabs," *Proc. Inst. Civ. Eng*., **28**, 471–488 (1964).
15. Z. C. Shih, "On the spline finite element method," *J. Comput. Math*., **1**, 50–72 (1979).
16. Y. K. Cheung, *Finite Strip Method in Structural Analysis*, Pergamon Press, Oxford (1976).
17. M. A. Jaswon and M. Maiti, "An integral equation formulation of plate bending problems," *J. Eng. Math*., **2**, 83–93 (1968).
18. Ö. Civalek, "Three-dimensional vibration, buckling and bending analyses of thick rectangular plates based on discrete singular convolution method," *Int. J. Mech. Sci*., **49**, 752–765 (2007).
19. Ö. Civalek, "Application of differential quadrature (DQ) and harmonic differential quadrature (HDQ) for buckling analysis of thin isotropic plates and elastic columns," *Eng. Struct*., **26**, 171–186 (2004).
20. J. B. Han and K. M. Liew, "Static analysis of Mindlin plates: the differential quadrature element method (DQEM)," *Comput. Meth. Appl. Mech. Eng*., **177**, 51–75 (1999).
21. B. M. Donning and W. K. Liu, "Meshless methods for shear-deformable beams and plates," *Comput. Meth. Appl. Mech. Eng*., **152**, 47–71 (1998).
22. P. Shen and P. He, "Bending analysis of rectangular moderately thick plates using spline finite element method," *Computers & Structures*, **54**, 1023–1029 (1995).
23. R. K. Livesley and P. C. Birchall, *Analysis of a Loaded Cantilever Plate by Finite Difference Methods*, Royal Aircraft Establishment Tech Note MS26, Farnsborough (1956).
24. T. K. Shu and T. T. Shih, "Variational principle of elastic thin plates and its application," *J. Beijing University of Aeronautics and Astronautics*, **1**, 27–40 (1957).
25. H. J. Plass, J. H. Gaines, and C. D. Newsom, "Application of Reissner's variational principle to cantilever plate deflection and vibration problems," *J. Appl. Mech*., **29**, 127–135 (1962).
26. A. W. Leissa and F. W. Niedenfuhr, "A study of the cantilevered square plate subjected to a uniform loading," *J. Aero/Space Sci*., **29**, 162–169 (1962).
27. I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York (1972).
28. I. N. Sneddon, *Application of Integral Transforms in the Theory of Elasticity*, McGraw-Hill, New York (1975).
29. R. Szilard, *Theory and Analysis of Plates: Classical and Numerical Methods*, Prentice-Hall, New Jersey (1974).
30. A. E. Green, "Double Fourier series and boundary value problems," *Proc. Cambridge Philos. Soc*., **40**, 222–228 (1944).