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AN ASYMPTOTIC LINEAR THIN-WALLED ROD MODEL COUPLING TWIST AND BENDING

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A linear one-dimensional model for thin-walled rods with open strongly curved cross-section, obtained by asymptotic methods is presented. A dimensional analysis of the linear three-dimensional equilibrium equations yields dimensionless numbers that reflect the geometry of the structure and the level of applied forces. For a given force level, the order of magnitude of the displacements and the corresponding one-dimensional model are deduced by asymptotic expansions. In the case of low force levels, we obtain a one-dimensional model whose kinematics, traction, and twist equations correspond to the Vlassov ones. However, this model couples twist and bending effects in the bending equations, unlike the Vlassov model where the twist angle and the bending displacement are uncoupled

Keywords: thin-walled rod model, linear elasticity, asymptotic methods

1. Introduction. Thin and thin-walled structures (plates, shells, rods and thin-walled rods) are widely used in industry because they provide maximum stiffness with minimum weight. However, there exists many different models in the literature. Therefore, engineers must know a priori their respective domain of validity and what model to use in function of the given data of the problem (geometry of the structure, applied loads, boundary conditions).

Classical models (the Kirchhoff–Love, Koiter, Bernouilli, Vlassov, etc.) are generally obtained from three-dimensional equilibrium equations by making a priori (kinematic and static) assumptions on the unknowns of the problem. Therefore, the domain of validity of these classical models with respect to the given data of the problem is difficult to specify rigorously.

Asymptotic methods enable to deduce rigorously plate, shell, and rod models from the three-dimensional equations without making any a priori assumption. In linear plate and shell theory, since the pioneering work of Goldenveizer [11], there exists a large literature on the subject [2, 7, 38–41].

In the linear theory of rods, the first works on the subject are due to Rigolot [33]. More recently, other justifications of linear and nonlinear rod models by asymptotic expansion were developed in [3, 20–22, 42]. Let us also cite the synthesis [46] of previous works [44, 45], which recall the different possible approaches in the linear theory of elastic rods (displacement formulation and mixed formulation in stress-displacements).

These results then have been extended to thin-walled rods. The approach used is based on the asymptotic behavior of the Poisson equation in a thin domain when the thickness tends to zero [34, 35, 46]. This way, Rodriguez and Viaño [36] have justified a linear elastic model of Vlassov for a thin-walled rod by asymptotic method similar to the Vlassov one. However, their approach uses "a priori" scaling assumptions on the displacement field, which is an unknown of the problem. Moreover, it is based on an expansion at the second order of the equations with respect to the diameter ε and then the relative thickness η is assumed to tend to zero. These two operations do not a priori commute and the result depends on the choice made (see Fig. 1).

This is a classical result well known for multi-scales asymptotic approaches. It is encountered in shell theory (with the relative thickness and the shallowness as small parameters), in homogenization of composite or periodic structures [1, 9, 19].

We propose in this paper to use the constructive approach based on asymptotic expansions, already developed by the authors for plates [24–31], shells [8, 16, 17] and thin-walled rods [12, 13, 18], to deduce a linear model for thin-walled rod from

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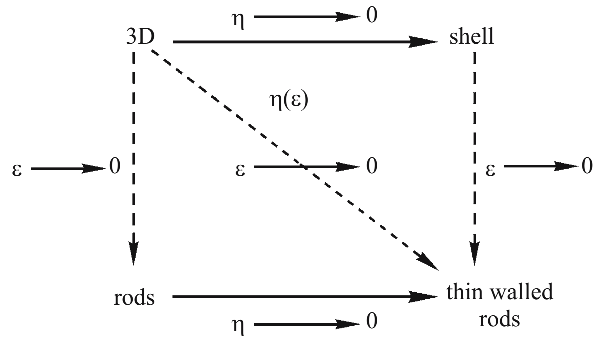

Fig. 1. Existing asymptotic approaches

three-dimensional equations. The approach used is based on a decomposition of the three-dimensional equations on Frenet basis of the initial configuration. Then a dimensional analysis of equilibrium equations lets appear pertinent dimensionless numbers characterizing the geometry and the applied loads. These numbers are measurable and enable to define the domain of validity of the obtained model. Thus the order of magnitude of the displacements and the corresponding asymptotic model are directly deduced from the level of applied forces. This constitutes the constructive character of our approach.

In this paper we limit our analysis to thin-walled rods with strongly curved profile subjected to low force levels. In Lemma 1, we begin with deducing the order of magnitude of the displacements from the level of applied forces. Then the asymptotic expansion of equations leads to the kinematics and to the one-dimensional equilibrium equations of results 1 to 4. The kinematics and the one-dimensional traction and twist equations correspond exactly to Vlassov ones [47]. However, whereas Vlassov theory relies on a priori physical assumptions, in the approach developed here the unknowns of the problem are directly deduced from the three-dimensional equations.

On the other hand, the one-dimensional bending equations obtained in result 4 differ from Vlassov ones. They involve a supplementary term coupling bending and torsion effects, whereas they are uncoupled in Vlassov model. (Such a limitation of Vlassov theory has already been noticed by other authors [5, 6, 23, 43]).

We recall that in linear elastic theory, the thin-walled rods possess the following particular property: an external bending loading whose resultant induces a torque, will generally induce not only a bending displacement but also a twist. In contrary, a torque will induce only a twist, but no bending, at the difference from the model obtained in this paper. That is why we call it model "with coupling between twist and bending." However, let us notice that such a coupling between twist and bending effects exists in the models used for flexural-torsional buckling or in dynamics models for flexural and torsional vibration analysis (see for example [14, 15, 32, 37, 48]), but not for classical linear elastic analysis.

2. The Three-Dimensional Problem. We assume once and for all that an origin O and an orthonormal basis (e_1, e_2, e_3) have been chosen in $\mathbb{R}^3$. We index by a star (*) all dimensional variables and the variables without a star will denote dimensionless variables. Let ω^* be an open cylindrical surface of $\mathbb{R}^3$, (Oe_3) its axis, whose length is L and diameter d. We note γ_g^* and γ_d^* its lateral boundary, $\gamma_1^* = \omega^* \times \{0\}$ and $\gamma_2^* = \omega^* \times \{L\}$ its extremities.

Let us consider now a thin-walled rod with open cross-section and $2h$ thickness, whose middle surface is ω^*. The thin-walled rod occupies the set $\overline{\Omega}^* = \overline{\omega}^* \times [-h, h]$ of $\mathbb{R}^3$ in its reference configuration. We call $\Gamma_1^* = \gamma_1^* \times]-h, h[$ and $\Gamma_2^* = \gamma_2^* \times]-h, h[$ the extreme faces, $\Gamma_g^* = \gamma_g^* \times]-h, h[$ and $\Gamma_d^* = \gamma_d^* \times]-h, h[$ the lateral faces, $\Gamma_\pm^* = \omega^* \times \{\pm h\}$ the upper and lower faces. Let M^* be a generic point of the beam. We decompose the vector $\overrightarrow{OM^*}$ as follows:

$$\overrightarrow{OM^*} = x_3^* e_3 + \overrightarrow{G^* C^*} + \overrightarrow{C^* m^*} + r^* n, \tag{1}$$

where x_3^* is the coordinate of the current cross-section containing M^* on the axis (Ox_3^*), G^* the point of intersection between the axis (Ox_3^*) and the current cross-section, C^* an arbitrary chosen point in the plane of the cross-section (see Fig. 2) located by its

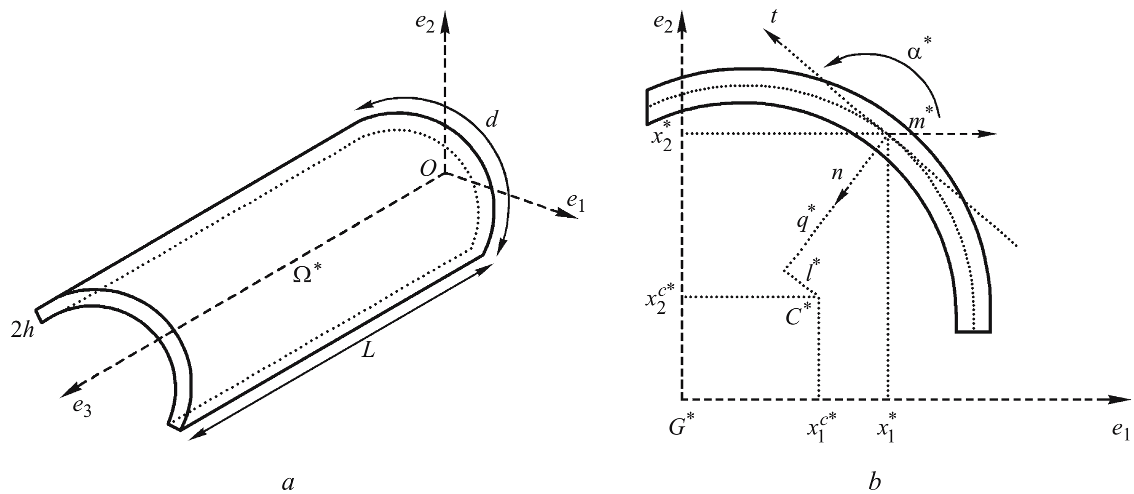

Fig. 2. Scheme of the rod and of the profile in the plane of a section

cartesian coordinates (x_1^{c*}, x_2^{c*}), and r^* the thickness variable. We call C^* the intersection curve between ω^* and the cross-section. The orthogonal projection m^* of M^* on the middle surface is located by its cartesian coordinates $x^* = (x_1^*, x_2^*)$ or by its curvilinear abscissa s^* along C^*. The origin s_0^* of the curvilinear abscissa is an arbitrary chosen point of C^*. We note n the unit normal and t the unit tangent vector of C^*. Moreover, we call l^* and q^* the coordinates of the vector $\overrightarrow{C^* m^*}$ in the basis (t, n). Finally, we call α^* the angle (e_1, t) and c^* the curvature of the curve C^* (see Fig. 2).

In what follows, we consider only thin-walled rods such as $\frac{d}{L} << 1$, $\frac{h}{d} << 1$ and $h||c^*||_\infty << 1$. We assume that the rod is subjected to the applied body forces $f^* = f_t^* t + f_n^* n + f_3^* e_3 : \overline{\Omega}^* \to \mathbb{R}^3$ and to the applied surface forces $g^{*\pm} = g_t^{*\pm} t + g_n^{*\pm} n + g_3^{*\pm} e_3 : \overline{\Gamma}_{0\pm}^* \to \mathbb{R}^3$. Moreover, the rod is assumed to be clamped on its extremities Γ_1^* and Γ_2^*, and free on its lateral faces Γ_g^* and Γ_d^*. The unknown of the problem is then the displacement $U^* : \overline{\Omega}^* \to \mathbb{R}^3$. Within the framework of linear elasticity, the displacement U^* and the Cauchy stress tensor σ^* satisfy the linear equilibrium equations:

$$\begin{cases} \text{Div}^* \sigma^* &= -f^* \quad \text{in} \quad \Omega^*, \\ U^* &= 0 \quad \text{on} \quad \Gamma_{0,1}^*, \\ \sigma^*.N &= g^{\pm *} \quad \text{on} \quad \Gamma_\pm^*, \\ \sigma^*.T &= 0 \quad \text{on} \quad \Gamma_{g,d}^*, \end{cases} \tag{2}$$

where N and T denote the unit outward normal vector to the upper and lower faces and to the lateral extremities respectively. Within the framework of linear elasticity, the constitutive law of the Hookean material considered writes $\sigma^* = \lambda Tr(e^*)I + 2\mu e^*$, where $e^* = \frac{1}{2}\left(\frac{\partial U^*}{\partial M^*} + \overline{\frac{\partial U^*}{\partial M^*}}\right)$ denotes the linear strain tensor, λ and μ denote the Lamé constants of the material, and the overbar the transposition operator. Finally the boundary conditions on $\Gamma_g^* \cup \Gamma_d^*$ are considered on average over the thickness, in order the twist to be of the same order as the bending in the asymptotic model obtained.

3. Dimensional Analysis of Equilibrium Equations and Reduction to a One-Scale Problem. First, we decompose the equations such as to separate the axial components from the components in the plane of the cross-section. To do this, let us decompose U^* on Frenet basis (t, n, e_3) of the initial configuration as follows:

$$U^* = u_t^* t + u_n^* n + u_3^* e_3 . \tag{3}$$

Then the gradient of the vector U^* can be decomposed in the basis (t, n, e_3) on the following form:

$$\frac{\partial U^*}{\partial M^*} = \left[k^* \left(\frac{\partial U^*}{\partial s^*} + c^* \Lambda U^* \right) \quad \frac{\partial U^*}{\partial r^*} \quad \frac{\partial U^*}{\partial x_3^*} \right] = \begin{bmatrix} k^* \left(\frac{\partial u_t^*}{\partial s^*} - c^* u_n^* \right) & \frac{\partial u_t^*}{\partial r^*} & \frac{\partial u_t^*}{\partial x_3^*} \\ k^* \left(\frac{\partial u_n^*}{\partial s^*} + c^* u_t^* \right) & \frac{\partial u_n^*}{\partial r^*} & \frac{\partial u_n^*}{\partial x_3^*} \\ k^* \frac{\partial u_3^*}{\partial s^*} & \frac{\partial u_3^*}{\partial r^*} & \frac{\partial u_3^*}{\partial x_3^*} \end{bmatrix},$$

where $k^* = \dfrac{1}{1 - r^* c^*}$ and where $\Lambda = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ denotes the two-dimensional matrix of the wedge product. In the same way, the three-dimensional equilibrium equations can be decomposed in the basis (t, n, e_3) and writes in Ω^*:

$$\frac{\partial \sigma_{tn}^*}{\partial r^*} + k^* \left(\frac{\partial \sigma_{tt}^*}{\partial s^*} - 2c^* \sigma_{tn}^* \right) + \frac{\partial \sigma_{t3}^*}{\partial x_3^*} = -f_t^*,$$

$$\frac{\partial \sigma_{nn}^*}{\partial r^*} + k^* \left(\frac{\partial \sigma_{tn}^*}{\partial s^*} + c^* \sigma_{tt}^* - c^* \sigma_{nn}^* \right) + \frac{\partial \sigma_{n3}^*}{\partial x_3^*} = -f_n^*,$$

$$\frac{\partial \sigma_{n3}^*}{\partial r^*} + k^* \left(\frac{\partial \sigma_{t3}^*}{\partial s^*} - c^* \sigma_{n3}^* \right) + \frac{\partial \sigma_{33}^*}{\partial x_3^*} = -f_3^*.$$

The detailed expression of the components of σ^* will be given directly in their dimensionless form (4). The associated boundary conditions on the upper and lower faces $\Gamma_\pm^*$ are given by:

$$\sigma_{tn}^* = g_t^{\pm *}, \quad \sigma_{nn}^* = g_n^{\pm *}, \quad \sigma_{n3}^* = g_3^{\pm *}$$

and the boundary conditions on the lateral extremities $\Gamma_{g,d}^*$ reduce to:

$$\sigma_{tt}^* = 0, \quad \sigma_{tn}^* = 0, \quad \sigma_{t3}^* = 0.$$

It is important to notice that a boundary layer with respect to the shear stress σ_{t3}^* appears on the free lateral extremities. This is a classical phenomenon in plate and shell theory. In order to avoid this boundary layer which is not the subject of this paper, we relax the boundary conditions on $\Gamma_{g,d}^*$ as follows:

$$\int_{-h}^{h} \sigma_{t3}^* \, dr = 0.$$

3.1. Dimensional Analysis of Equations. Let us define the following dimensionless physical data and dimensionless unknowns of the problem:

$$u_t = \frac{u_t^*}{u_{rt}}, \quad u_n = \frac{u_n^*}{u_{rn}}, \quad u_3 = \frac{u_3^*}{u_{r3}}, \quad x_3 = \frac{x_3^*}{L}, \quad s = \frac{s^*}{d}, \quad r = \frac{r^*}{h}, \quad c = \frac{c^*}{c_r},$$

$$f_t = \frac{f_t^*}{f_{rt}}, \quad f_n = \frac{f_n^*}{f_{rn}}, \quad f_3 = \frac{f_3^*}{f_{r3}}, \quad g_t = \frac{g_t^*}{g_{rt}}, \quad g_n = \frac{g_n^*}{g_{rn}}, \quad g_3 = \frac{g_3^*}{g_{r3}},$$

where the variables indexed by $(_r)$ are the reference ones. The new variables which appear (without a star) are dimensionless. To avoid any assumption on the order of magnitude of the displacement components, the reference scales u_{tr}, u_{nr} and u_{3r} are firstly assumed to be equal to *h*. Thus we a priori allow small displacements in the framework of the theory of linear elasticity.

In a natural way we introduce $c_r = \|c^*\|_\infty$ which denotes the maximum of curvature (the smaller radius of curvature) of the middle surface ω^*. As in shell theory, the order of magnitude of the curvature is a fundamental data in the asymptotic expansion of equations. Therefore we will have to distinguish the rods with shallow cross profile from the rods with strongly curved profile.

First the dimensional analysis of the coefficient k^* leads to $k = \dfrac{1}{1 - hc_r rc}$. Setting $\nu = hc_r$, the assumption of thin walled-rod ensures that $\nu < 1$. We then have the following expansion $k = 1 + \nu rc + (\nu rc)^2 + \ldots$.

On the other hand, the dimensional analysis of the stress tensor leads to:

$$\sigma = \begin{bmatrix} \sigma_{tt} & \sigma_{tn} & \sigma_{t3} \\ \sigma_{tn} & \sigma_{nn} & \sigma_{n3} \\ \sigma_{t3} & \sigma_{n3} & \sigma_{33} \end{bmatrix}$$

with

$$\sigma_{tt} = \beta \frac{\partial u_n}{\partial r} + (\beta + 2)[1 + \nu rc + (\nu rc)^2 + \ldots]\left(\eta \frac{\partial u_t}{\partial s} - \nu c u_n\right) + \beta\eta\varepsilon \frac{\partial u_3}{\partial x_3},$$

$$\sigma_{nn} = (\beta + 2)\frac{\partial u_n}{\partial r} + \beta[1 + \nu rc + (\nu rc)^2 + \ldots]\left(\eta \frac{\partial u_t}{\partial s} - \nu c u_n\right) + \beta\eta\varepsilon \frac{\partial u_3}{\partial x_3},$$

$$\sigma_{33} = \beta \frac{\partial u_n}{\partial r} + \beta[1 + \nu rc + (\nu rc)^2 + \ldots]\left(\eta \frac{\partial u_t}{\partial s} - \nu c u_n\right) + (\beta + 2)\eta\varepsilon \frac{\partial u_3}{\partial x_3},$$

$$\sigma_{tn} = \frac{\partial u_t}{\partial r} + [1 + \nu rc + (\nu rc)^2 + \ldots]\left(\eta \frac{\partial u_n}{\partial s} + \nu c u_t\right),$$

$$\sigma_{t3} = [1 + \nu rc + (\nu rc)^2 + \ldots]\eta \frac{\partial u_3}{\partial s} + \eta\varepsilon \frac{\partial u_t}{\partial x_3}, \quad \sigma_{n3} = \frac{\partial u_3}{\partial r} + \eta\nu \frac{\partial u_n}{\partial x_3}, \tag{4}$$

where we set $\sigma^* = \mu\sigma$, $\varepsilon = d / L$, $\eta = h / d$, and $\beta = \lambda / \mu$. Now let us denote ω the dimensionless middle surface obtained from ω^*, whose current point will be noted *m*. Its associated curvature *C* is obtained by dimensional analysis of C^*. Then the dimensional analysis of the three-dimensional linear equilibrium equations leads in $\Omega = \omega \times]-1, 1[$ to:

$$\frac{\partial \sigma_{tn}}{\partial r} + (1 + \nu rc + (\nu rc)^2 + \ldots)\left(\eta \frac{\partial \sigma_{tt}}{\partial s} - 2\nu c \sigma_{tn}\right) + \eta\varepsilon \frac{\partial \sigma_{t3}}{\partial x_3} = -F_t f_t,$$

$$\frac{\partial \sigma_{nn}}{\partial r} + (1 + \nu rc + (\nu rc)^2 + \ldots)\left(\eta \frac{\partial \sigma_{tn}}{\partial s} + \nu c \sigma_{tt} - \nu c \sigma_{nn}\right) + \eta\varepsilon \frac{\partial \sigma_{n3}}{\partial x_3} = -F_n f_n,$$

$$\frac{\partial \sigma_{n3}}{\partial r} + (1 + \nu rc + (\nu rc)^2 + \ldots)\left(\eta \frac{\partial \sigma_{t3}}{\partial s} - \nu c \sigma_{n3}\right) + \eta\nu \frac{\partial \sigma_{33}}{\partial x_3} = -F_t f_3. \tag{5}$$

The associated boundary conditions on the upper and lower faces $\Gamma_\pm$ become:

$$\sigma_{tn} = G_t g_t^\pm, \quad \sigma_{nn} = G_n g_n^\pm, \quad \sigma_{n3} = G_3 g_3^\pm. \tag{6}$$

Therefore, this dimensional analysis naturally reveals the following dimensional numbers characterizing the thin-walled rod problems in linear elasticity (they are measurable data of the problem and must be considered as given data):

$$\varepsilon = \frac{d}{L}, \quad \eta = \frac{h}{d}, \quad \nu = hc_r, \quad F_t = \frac{hf_{tr}}{\mu}, \quad F_n = \frac{hf_{nr}}{\mu}, \quad F_3 = \frac{hf_{3r}}{\mu}, \quad G_t = \frac{g_{tr}}{\mu}, \quad G_n = \frac{g_{nr}}{\mu}, \quad G_3 = \frac{g_{3r}}{\mu},$$

(i) The shape ratio ε characterizes the inverse of the shooting-pain of the rod. This is a known parameter of the problem which satisfies $\varepsilon < 1$.

(ii) The dimensional number η denotes the ratio between the thickness h of the rod to the length of its profile. This number is also a data of the problem which satisfies $\eta < 1$.

(iii) The shape ratio $\nu = hc_r$ is the ratio between the thickness to the smaller radius of curvature of the middle surface ω of the rod. Its is a given geometrical data of the problem.

(iv) The force ratios F_i, G_i $(i \in \{t, n, 3\})$ represent respectively the ratio of the resultant on the thickness of the body forces (respectively of the surface forces) to μ considered as a reference stress. These numbers only depend on known physical quantities and must be considered as known data of the problem.

3.2. One-Scale Assumption. To reduce the problem to a one-scale problem, ε is chosen as the small reference parameter of the problem. (If not we have multi-scale problems which are much more complicated. It is not the subject of this paper).

The other dimensional numbers are then linked to ε, or more precisely to the powers of ε. In a natural way, as in shell theory, we have to distinguish thin-walled rods:

– with strongly curved profile, where $\nu = \varepsilon$;

– with shallow profile, where $\nu = \varepsilon^2$.

This distinction is fundamental because these two families of thin-walled rods do not have the same asymptotic behavior.

On the other hand, three cases can be distinguished and studied:

– the thick rods, where $\eta = 1$. This is not the subject of this paper;

– the thin-walled rods, where $\eta = \varepsilon$. It is the case studied here;

– the very thin-walled rods, where $\eta = \varepsilon^p$, $p > 1$. This case is not studied in this paper.

Finally, the applied loads are an essential given data of the problem. In the framework of a one-scale asymptotic expansion, the force ratios must be linked also to ε. This is equivalent to fix the order of magnitude of the applied forces which are given data. In the case of thin-walled rods with strongly curved profile, we will consider applied forces such as: $F_t = G_t = \varepsilon^6$, $F_n = G_n = \varepsilon^6$, $F_3 = G_3 = \varepsilon^5$.

These force ratios, which characterize the level of applied forces, are chosen in order all kinds of loading to be involved at the same order in the asymptotic one-dimensional equilibrium equations.

In the sequel, we shall consider a thin-walled rod with a strongly curved profile corresponding to $\eta = \nu = \varepsilon$, submitted to force levels such as $F_t = F_n = \varepsilon^6$, $G_t = G_n = \varepsilon^6$, and $F_3 = G_3 = \varepsilon^5$. The problem then reduces to a dimensionless one-scale problem, which can be easily written from (5) and (6), using the expressions (4) of the stresses.

4. Asymptotic Expansion of Equations. The standard asymptotic technique then proceeds as follows. First we postulate that the solution $U = (u_t, u_n, u_3)$ of the problem admits a formal expansion with respect to the powers of ε:

$$(u_t, u_n, u_3) = (u_t^0, u_n^0, u_3^0) + \varepsilon(u_t^1, u_n^1, u_3^1) + \varepsilon^2(u_t^2, u_n^2, u_3^2) + \dots . \tag{7}$$

The expansion of U with respect to ε implies an expansion of the components of the stresses σ with respect to ε as well. Then we replace u_t, u_n, u_3 by their expansions in equilibrium equations and we equate to zero the factor of the successive powers of ε. This way we obtain a succession of coupled problems $P_0, P_1, P_2, \dots$. Its resolution leads to the search asymptotic one-dimensional model corresponding to the force level considered.

It is important to notice that with the approach developed here, the order of magnitude of the displacements (which are unknowns of the problem) are directly deduced from the level of applied forces. In particular, for the force levels considered here, the axial displacement is one order smaller then the other ones. This is the result of the following lemma:

Lemma 1. For force levels such as $F_t = F_n = \varepsilon^6$, $G_t = G_n = \varepsilon^6$, and $F_3 = G_3 = \varepsilon^5$, we have $u_3^0 = 0$.

The proof of this lemma is rather long and technical. The demarche is similar to the proof of result 1 and is not detailed here. Hence, for the level forces considered, the reference scales of the axial displacement $u_{3r} = h$ is not properly chosen. In order for u_3 to be of the order of one unit, the reference scales of the displacement must satisfy $u_{3r} = \varepsilon h$. Therefore the new reference scale for the axial displacement u_3^* that we have to consider is $u_{3r} = \varepsilon h$. The other reference scales for the tangential and normal displacements $u_{tr} = u_{nr} = h$ stay unchanged.

Remark 1. It is important to notice that this lemma only leads to the right scalings for the displacements corresponding to the level of applied forces considered. However, it would have been possible to start directly from these right scalings or reference scales for the displacements, as it is often made in the literature.

5. The One-Dimensional Model. In the last section, we have determined the right reference scales (or equivalently the order of magnitude) of the displacements corresponding to the force levels considered. In this section, we perform the asymptotic expansion of equations which leads to the search one-dimensional model.

According to the force levels considered $F_t = F_n = \varepsilon^6$, $G_t = G_n = \varepsilon^6$, and $F_3 = G_3 = \varepsilon^5$, the dimensionless equilibrium equations must be written again with $u_{3r} = \varepsilon h$ and $u_{tr} = u_{nr} = h$ as reference scales. The dimensionless components of the displacement will still be noted with u_t, u_n, and u_3. Thus for the level forces considered here, the new dimensionless equilibrium equations are the same as the previous ones (4)–(6). Only u_3 must be changed into εu_3 in the new expressions of the components of the stresses. Then we assume again that there exists a formal expansion with respect to ε, similar to (7), of the new dimensionless solution (u_t, u_n, u_3).

5.1. A Vlassov Kinematics. **Result 1:** For applied force levels such as $F_t = F_n = \varepsilon^6, G_t = G_n = \varepsilon^6$, and $F_3 = G_3 = \varepsilon^5$, the leading term (u_t^0, u_n^0, u_3^0) is a displacement of Vlassov type which satisfies:

$$\tilde{u}_t^0 = \bar{u}_1^c \cos(\alpha) + \bar{u}_2^c \sin(\alpha) - q(s)\overline{\Theta}^0,$$

$$\tilde{u}_n^0 = -\bar{u}_1^c \sin(\alpha) + \bar{u}_2^c \cos(\alpha) + l(s)\overline{\Theta}^0,$$

$$\tilde{u}_3^0 = \bar{u}_3 - x_1 \frac{d\bar{u}_1^c}{dx_3} - x_2 \frac{d\bar{u}_2^c}{dx_3} - \omega \frac{d\overline{\Theta}^0}{dx_3},$$

where $\bar{u}_3$ denotes the axial or traction displacement; $\bar{u}_1^c$ and $\bar{u}_2^c$ denote the tangential displacements of the point C; $\overline{\Theta}^0$ denotes the angle of rotation around the axis (C, e_3); ω is called the sectorial area defined as follows; $d\omega / ds = -q$.

Proof. The asymptotic expansion of the new dimensionless equations leads again to problems $P_0, P_1, P_2, \ldots$.

Problem P_0. The cancellation of the factor of ε^0 leads to P_0 which can be written:

$$\text{in } \Omega \quad \begin{cases} \dfrac{\partial \sigma_{tn}^0}{\partial r} = 0 \\ \dfrac{\partial \sigma_{nn}^0}{\partial r} = 0 \end{cases} \quad \text{for } r = \pm 1 \quad \begin{cases} \sigma_{tn}^0 = 0, \\ \sigma_{nn}^0 = 0. \end{cases}$$

Therefore, we get $\sigma_{tn}^0 = \sigma_{nn}^0$ in Ω which implies that all the components of σ^0 are equal to zero. Then writing the components of the stresses in terms of displacements, we obtain $\frac{\partial u_t^0}{\partial r} = 0$ and $\frac{\partial u_n^0}{\partial r} = 0$ in Ω, or in an equivalent way (in the next, for the simplicity of the notations, we will adopt the following ones: a function u which depends only of (s, x_3) will be noted $\tilde{u}$; a function u which depends only on (x_3) will be noted $\bar{u}$):

$$u_t^0 = \tilde{u}_t^0(s, x_3), \quad u_n^0 = \tilde{u}_n^0(s, x_3). \tag{8}$$

Let us now prove that $u_3^0 = \bar{u}_3^0(x_3)$.

Problem P_1. The cancellation of the factor of ε leads to problem P_1 which easily implies that $\sigma_{tn}^1 = \sigma_{nn}^1 = \sigma_{n3}^1 = 0.$ Writing the stresses in terms of displacements, we obtain in Ω:

$$u_t^1 = -\tilde{\psi}_t^0 r + \tilde{u}_t^1, \tag{9}$$

$$u_n^1 = -\frac{\beta}{\beta+2}\tilde{\psi}_n^0 r + \tilde{u}_n^1, \tag{10}$$

$$u_3^0 = \tilde{u}_3^0 \tag{11}$$

with

$$\tilde{\psi}_t^0 = \frac{\partial \tilde{u}_n^0}{\partial s} + c\tilde{u}_t^0, \qquad \tilde{\psi}_n^0 = \frac{\partial \tilde{u}_t^0}{\partial s} - c\tilde{u}_n^0. \tag{12}$$

From the last expressions, the components of the stresses at order 1 reduce to:

$$\sigma_{tt}^1 = \beta\frac{\partial u_n^1}{\partial r} + (2+\beta)\tilde{\psi}_n^0 = 4\frac{\beta+1}{\beta+2}\tilde{\psi}_n^0, \tag{13}$$

$$\sigma_{33}^1 = \beta\frac{\partial u_n^1}{\partial r} + \beta\tilde{\psi}_n^0 = 2\frac{\beta}{\beta+2}\tilde{\psi}_n^0, \tag{14}$$

$$\sigma_{t3}^1 = \frac{\partial \tilde{u}_3^0}{\partial s}. \tag{15}$$

The boundary conditions on the lateral surfaces at order one for $s = s_+$ and $s = s_g$ write $\sigma_{tt}^1 = 0$ and

$$\int_{-1}^{1} \sigma_{t3}^1 \, dr = 0. \tag{16}$$

Problem P_2. The cancellation of the factor of ε^2 leads to problem P_2 which reduces in Ω to

$$\frac{\partial \sigma_{tn}^2}{\partial r} + \frac{\partial \sigma_{tt}^1}{\partial s} = 0, \qquad \frac{\partial \sigma_{nn}^2}{\partial r} + c\sigma_{tt}^1 = 0, \qquad \frac{\partial \sigma_{n3}^2}{\partial r} = 0 \tag{17}$$

with the associated boundary conditions for $r = \pm 1$

$$\sigma_{tn}^2 = 0, \qquad \sigma_{nn}^2 = 0, \qquad \sigma_{n3}^2 = 0. \tag{18}$$

Let us integrate Eq. (17) over the thickness. With the boundary condition (18), we obtain $\int_{-1}^{1} \sigma_{tt}^1 dr = 0$. Replacing σ_{tt}^1 with its expression (13) in terms of displacement, we get $\tilde{\psi}_n^0 = 0$. Then, from (9)–(14) we deduce that $\sigma_{tt}^1 = \sigma_{33}^1 = 0$ and $u_n^1 = \tilde{u}_n^1$.

Then problem P_2 leads, according to the boundary conditions, to $\sigma_{tn}^2 = \sigma_{nn}^2 = \sigma_{n3}^2 = 0$. The last equations are equivalent in terms of displacements to:

$$u_t^2 = -\tilde{\psi}_t^1 r + \tilde{u}_t^2, \quad u_n^2 = \frac{\beta}{\beta+2}\frac{\partial \tilde{\psi}_t^0}{\partial s}\frac{r^2}{2} - \frac{\beta}{\beta+2}\tilde{\psi}_n^1 r + \tilde{u}_n^2, \quad u_3^1 = -\frac{\partial \tilde{u}_n^0}{\partial x_3} r + \tilde{u}_3^1 \tag{19}$$

with $\tilde{\psi}_t^1 = \frac{\partial \tilde{u}_n^1}{\partial s} + c\tilde{u}_t^1$ and $\tilde{\psi}_n^1 = \frac{\partial \tilde{u}_t^1}{\partial s} - c\tilde{u}_n^1$.

From the last expressions of the displacements, we obtain the following expressions of the components of the stresses σ^2 at order two:

$$\sigma_{tt}^{2}=-4\frac{\beta+1}{(\beta+2)}\frac{\partial\tilde{\psi}_{t}^{0}}{\partial s}r+4\frac{\beta+1}{\beta+2}\tilde{\psi}_{n}^{1}, \tag{20}$$

$$\sigma_{33}^{2}=-2\frac{\beta}{(\beta+2)}\frac{\partial\tilde{\psi}_{t}^{0}}{\partial s}r+2\frac{\beta}{\beta+2}\tilde{\psi}_{n}^{1}, \tag{21}$$

$$\sigma_{t3}^{2}=\frac{\partial\tilde{u}_{3}^{0}}{\partial s}+\frac{\partial\tilde{u}_{t}^{0}}{\partial x_{3}}. \tag{22}$$

The associated boundary conditions on the lateral surface $s=s_{-}$ and $s=s_{+}$ write $\sigma_{tt}^{2}=0$ and

$$\int_{-1}^{1}\sigma_{t3}^{2}\,dr=0. \tag{23}$$

That leads, in terms of displacements, to $\tilde{\psi}_{n}^{1}(s_{-},0)=\tilde{\psi}_{n}^{1}(s_{+},0)=0$ and

$$\frac{\partial\tilde{\psi}_{t}^{0}}{\partial s}(s_{-})=\frac{\partial\tilde{\psi}_{t}^{0}}{\partial s}(s_{+})=0,\qquad\left[\frac{\partial\tilde{u}_{3}^{0}}{\partial s}+\frac{\partial\tilde{u}_{t}^{0}}{\partial x_{3}}\right](s_{-})=\left[\frac{\partial\tilde{u}_{3}^{0}}{\partial s}+\frac{\partial\tilde{u}_{t}^{0}}{\partial x_{3}}\right](s_{+})=0. \tag{24}$$

Problem P_3. The cancellation of the factor of ε^3 leads to problem P_3 which reduces in Ω to:

$$\frac{\partial\sigma_{tn}^{3}}{\partial r}+\frac{\partial\sigma_{tt}^{2}}{\partial s}=0, \tag{25}$$

$$\frac{\partial\sigma_{nn}^{3}}{\partial r}+c\sigma_{tt}^{2}=0, \tag{26}$$

$$\frac{\partial\sigma_{n3}^{3}}{\partial r}+\frac{\partial\sigma_{t3}^{2}}{\partial s}=0 \tag{27}$$

with the associated boundary conditions for $r=\pm1$

$$\sigma_{tn}^{3}=0,\qquad\sigma_{nn}^{3}=0, \tag{28}$$

$$\sigma_{n3}^{3}=0. \tag{29}$$

As previously, let us integrate Eq. (26) over the thickness. With the boundary condition (28), we obtain $\int_{-1}^{1}\sigma_{tt}^{2}dr=0$. According the the expression (20) of σ_{tt}^{2}, we get $\tilde{\psi}_{n}^{1}=0$. Thus expressions (20) and (21) of σ_{tt}^{2} and σ_{33}^{2} reduce to:

$$\sigma_{tt}^{2}=-4\frac{\beta+1}{(\beta+2)}\frac{\partial\tilde{\psi}_{t}^{0}}{\partial s}r,\qquad\sigma_{33}^{2}=-2\frac{\beta}{(\beta+2)}\frac{\partial\tilde{\psi}_{t}^{0}}{\partial s}r. \tag{30}$$

In the same way, we shall now integrate (27) over the thickness. Using (29), and then (23) and (22), we obtain:

$$\frac{\partial\tilde{u}_{3}^{0}}{\partial s}+\frac{\partial\tilde{u}_{t}^{0}}{\partial x_{3}}=0, \tag{31}$$

which is nothing else than the non-distorsion Vlassov assumption obtained for the leading term of the expansion of the displacement. Using the previous results obtained, the expressions of the stresses at order three reduce to

$$\sigma_{tn}^3 = 4\frac{\beta+1}{\beta+2}\frac{\partial^2\tilde{\psi}_t^0}{\partial s^2}\frac{r^2-1}{2}, \quad \sigma_{nn}^3 = 4c\frac{\beta+1}{\beta+2}\frac{\partial\tilde{\psi}_t^0}{\partial s}\frac{r^2-1}{2}, \quad \sigma_{n3}^3 = 0. \tag{32}$$

On the other hand, according to (32), the boundary conditions at order three $\sigma_{tn}^3(s_-, x_3) = \sigma_{tn}^3(s_+, x_3) = 0$, leads in terms of displacements to:

$$\frac{\partial^2\tilde{\psi}_t^0}{\partial s^2}(s_-, x_3) = \frac{\partial^2\tilde{\psi}_t^0}{\partial s^2}(s_+, x_3) = 0. \tag{33}$$

Problem P_4. The cancellation of the factor of ε^4 leads to problem P_4 which reduces in Ω to:

$$\frac{\partial\sigma_{tn}^4}{\partial r} + \frac{\partial\sigma_{tt}^3}{\partial s} - 2c\sigma_{tn}^3 + rc\frac{\partial\sigma_{tt}^2}{\partial s} = 0, \tag{34}$$

$$\frac{\partial\sigma_{nn}^4}{\partial r} + \frac{\partial\sigma_{tn}^3}{\partial s} + c\sigma_{tt}^3 - c\sigma_{nn}^3 + rc^2\sigma_{tt}^2 = 0, \quad \frac{\partial\sigma_{n3}^4}{\partial r} + \frac{\partial\sigma_{t3}^3}{\partial s} + \frac{\partial\sigma_{33}^2}{\partial x_3} = 0, \tag{35}$$

with the boundary conditions for $r = \pm 1$

$$\sigma_{tn}^4 = 0, \tag{36}$$

$$\sigma_{nn}^4 = 0, \tag{37}$$

$$\sigma_{n3}^4 = 0. \tag{38}$$

Using the boundary conditions (36) and (37), an integration of Eqs. (34) and (35) over the thickness lead to:

$$\int_{-1}^{1}\left(\frac{\partial\sigma_{tt}^3}{\partial s} - 2c\sigma_{tn}^3 + rc\frac{\partial\sigma_{tt}^2}{\partial s}\right)dr = 0, \tag{39}$$

$$\int_{-1}^{1}\left(\frac{\partial\sigma_{tn}^3}{\partial s} + c\sigma_{tt}^3 - c\sigma_{nn}^3 + rc^2\sigma_{tt}^2 dr\right)dr = 0. \tag{40}$$

In the same way, after multiplying Eqs. (25) and (26) with *rc*, the integration over the thickness leads to

$$\int_{-1}^{1} c\sigma_{tn}^3 dr = \int_{-1}^{1} rc\frac{\partial\sigma_{tt}^2}{\partial s}dr, \tag{41}$$

$$\int_{-1}^{1} c\sigma_{nn}^3 dr = \int_{-1}^{1} rc^2\sigma_{tt}^2 dr. \tag{42}$$

We then use (41) [respectively (42)] to simplify (39)[respectively (40)] which reduce to

$$\int_{-1}^{1}\left(\frac{\partial\sigma_{tt}^3}{\partial s} - c\sigma_{tn}^3\right)dr = 0 \tag{43}$$

$$\int_{-1}^{1}\left(\frac{\partial\sigma_{tn}^3}{\partial s} + c\sigma_{tt}^3\right)dr = 0. \tag{44}$$

On the other hand, let us derive (44) with respect to *s*. We have:

$$\int_{-1}^{1}\left(\frac{\partial^2\sigma_{tn}^3}{\partial s^2}+\frac{dc}{ds}\sigma_{tt}^3+c\frac{\partial\sigma_{tt}^3}{\partial s}\right)dr=0. \tag{45}$$

Now using (43) and (44) to eliminate σ_{tt}^3 in (45), we obtain according to (32): $\frac{\partial^4\tilde{\psi}_t^0}{\partial s^4}-\frac{1}{c}\frac{dc}{ds}\frac{\partial^3\tilde{\psi}_t^0}{\partial s^3}+c^2\frac{\partial^2\tilde{\psi}_t^0}{\partial s^2}=0,$ whose general solution is given by $\frac{\partial^2\tilde{\psi}_t^0}{\partial s^2}=A\cos(\alpha)+B\sin(\alpha),$ with $c(s)=\frac{d\alpha}{ds}$. Using the boundary conditions (33) and (24), we obtain $\frac{\partial\tilde{\psi}_t^0}{\partial s}=0$ or equivalently $\tilde{\psi}_t^0=\overline{\Theta}(x_3)$. Therefore the tangential displacements are solution of the following differential system:

$$\begin{array}{lll}\tilde{\psi}_n^0 & =0 & =0\Rightarrow \\ \tilde{\psi}_t^0= & \overline{\Theta}^0 & \end{array}\quad \begin{array}{l}\frac{\partial\tilde{u}_t^0}{\partial s}-c\tilde{u}_n^0=0,\\ \frac{\partial\tilde{u}_n^0}{\partial s}+c\tilde{u}_t^0=\overline{\Theta}^0.\end{array}$$

In a Cartesian basis, we get after a few calculations $\tilde{u}_1^0=\overline{u}_1^c-(x_2-x_2^c)\overline{\Theta}^0$ and $\tilde{u}_2^0=\overline{u}_2^c+(x_1-x_1^c)\overline{\Theta}^0$, where $\overline{u}_1^c$ and $\overline{u}_2^c$ represents at the leading order the displacements of the arbitrary point C in the directions e_1 and e_2. The angle $\overline{\Theta}^0$ characterizes the rotation of the section around the axis (C,e_3). The point C is generally identified to the shear center of the sections. In the basis (*t*, *n*), we then have:

$$\begin{cases}\tilde{u}_t^0=\overline{u}_1^c\cos(\alpha)+\overline{u}_2^c\sin(\alpha)-q(s)\overline{\Theta}^0,\\ \tilde{u}_n^0=-\overline{u}_1^c\sin(\alpha)+\overline{u}_2^c\cos(\alpha)+l(s)\overline{\Theta}^0,\end{cases}$$

with

$$\left|\begin{array}{l}l(s)=(x_1-x_1^c)\cos\alpha+(x_2-x_2^c)\sin\alpha,\\ q(s)=-(x_1-x_1^c)\sin\alpha+(x_2-x_2^c)\cos\alpha.\end{array}\right.$$

This last expression characterizes a rigid displacement in the plane of the sections and is similar to Vlassov kinematics. (Excepted for the sign of q in the expression of $\tilde{u}_t^0$. This is due to an orientation of the normal n opposite to Vlassov one). Moreover, the axial displacement u_3^0 can be determined from (31). We obtain the expresion of $\tilde{u}_3^0$ of result 1.

5.2. Traction Equation. **Result 2:** For applied level forces such as $F_t=F_n=\varepsilon^6, G_t=G_n=\varepsilon^6$, and $F_3=G_3=\varepsilon^5$, the leading terms of the displacements $\overline{u}_3$, $\overline{\Theta}^0$, $\overline{u}_1^c$, and $\overline{u}_2^c$ satisfy the following traction equation:

$$ES\frac{d^2\overline{u}_3}{dx_3^2}-ES_1\frac{d^3\overline{u}_1^c}{dx_3^3}-ES_2\frac{d^3\overline{u}_2^c}{dx_3^3}-ES_\omega\frac{d^3\overline{\Theta}^0}{dx_3^3}=-\mu P_3,$$

where E and μ are respectively Young modulus and Lamé coefficient of the material, and where:

$$S=\int_{s_-}^{s_+}\int_{-1}^{1}drds,\quad S_\omega=\int_{s_-}^{s_+}\int_{-1}^{1}\omega drds,\quad S_1=\int_{s_-}^{s_+}\int_{-1}^{1}x_1drds,$$

$$S_2=\int_{s_-}^{s_+}\int_{-1}^{1}x_2drds,\quad P_3=\int_{s_-}^{s_+}\int_{-1}^{1}f_3drds+\int_{s_-}^{s_+}[g_3^+-g_3^-]ds.$$

Proof. We just proved that $\widetilde{\psi}_t^0 = \overline{\Theta}^0$. So we have $\sigma_{tt}^2 = \sigma_{33}^2 = 0$ and $\sigma_{tn}^3 = \sigma_{nn}^3 = 0$, that leads to the following expressions of the stresses at order three:

$$\sigma_{tt}^3 = -4\frac{\beta+1}{(\beta+2)}\frac{\partial\widetilde{\psi}_t^1}{\partial s}r + 4\frac{\beta+1}{\beta+2}\widetilde{\psi}_n^2 + 2\frac{\beta}{\beta+2}\frac{\partial\widetilde{u}_3^0}{\partial x_3}, \tag{46}$$

$$\sigma_{33}^3 = -2\frac{\beta}{(\beta+2)}\frac{\partial\widetilde{\psi}_t^1}{\partial s}r + 2\frac{\beta}{\beta+2}\widetilde{\psi}_n^2 + 4\frac{\beta+1}{\beta+2}\frac{\partial\widetilde{u}_3^0}{\partial x_3}, \qquad \sigma_{t3}^3 = \frac{\partial\widetilde{u}_3^1}{\partial s} + \frac{\partial\widetilde{u}_t^1}{\partial x_3} - 2\frac{d\overline{\Theta}^0}{dx_3}r. \tag{47}$$

Problem P_4 then reduces in Ω to:

$$\frac{\partial\sigma_{tn}^4}{\partial r} + \frac{\partial\sigma_{tt}^3}{\partial s} = 0, \qquad \frac{\partial\sigma_{nn}^4}{\partial r} + c\sigma_{tt}^3 = 0, \tag{48}$$

$$\frac{\partial\sigma_{n3}^4}{\partial r} + \frac{\partial\sigma_{t3}^3}{\partial s} = 0. \tag{49}$$

Using (37), the integration of Eq. (48) over the thickness leads to $\int_{-1}^{1}\sigma_{tt}^3 dr = 0$. Then replacing σ_{tt}^3 with its expression (46), we get $4\frac{\beta+1}{\beta+2}\widetilde{\psi}_n^2 + 2\frac{\beta}{\beta+2}\frac{\partial\widetilde{u}_3^0}{\partial x_3} = 0$. On the other hand, using (38) the integration of Eq. (49) over the thickness leads to $\int_{-1}^{1}\sigma_{t3}^3 dr = 0$. According to (47), we have equivalently in terms of displacements $(\partial\widetilde{u}_3^1)/\partial s + (\partial\widetilde{u}_t^1)/\partial x_3 = 0$, and the expressions of the stresses reduce to:

$$\sigma_{tt}^3 = -4\frac{\beta+1}{(\beta+2)}\frac{\partial\widetilde{\psi}_t^1}{\partial s}r, \qquad \sigma_{33}^3 = -2\frac{\beta}{(\beta+2)}\frac{\partial\widetilde{\psi}_t^1}{\partial s}r + \frac{3\beta+2}{\beta+1}\frac{\partial\widetilde{u}_3^0}{\partial x_3}, \qquad \sigma_{t3}^3 = -2\frac{d\overline{\Theta}^0}{dx_3}r. \tag{50}$$

This last equation leads to $\sigma_{n3}^4 = 0$ according to (38) and (49).

Problem P_5. The cancellation of the factor of ε^5 leads to problem P_5 which reduces in Ω to

$$\frac{\partial\sigma_{tn}^5}{\partial r} + \frac{\partial\sigma_{tt}^4}{\partial s} - 2c\sigma_{tn}^4 + rc\frac{\partial\sigma_{tt}^3}{\partial s} + \frac{\partial\sigma_{t3}^3}{\partial x_3} = 0, \tag{51}$$

$$\frac{\partial\sigma_{nn}^5}{\partial r} + \frac{\partial\sigma_{tn}^4}{\partial s} + c\sigma_{tt}^4 - c\sigma_{nn}^4 + rc^2\sigma_{tt}^3 = 0, \tag{52}$$

$$\frac{\partial\sigma_{n3}^5}{\partial r} + \frac{\partial\sigma_{t3}^4}{\partial s} + \frac{\partial\sigma_{33}^3}{\partial x_3} = -f_3, \tag{53}$$

with the boundary conditions for $r = \pm 1$

$$\sigma_{tn}^5 = 0, \qquad \sigma_{nn}^5 = 0, \qquad \sigma_{n3}^5 = g_3^{\pm}. \tag{54}$$

Using the boundary condition (54), the integration of (53) over the thickness leads to:

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(\frac{\partial\sigma_{t3}^4}{\partial s} + \frac{\partial\sigma_{33}^3}{\partial x_3}\right)drds = -\int_{s_-}^{s_+}\int_{-1}^{1} f_3\,drds - \int_{s_-}^{s_+}[g_3^+ - g_3^-]\,ds.$$

Using the boundary condition $\int_{-1}^{1}\sigma_{t3}^4 dr = 0$ on the free lateral surface for $s = s_-$ et $s = s_+$, we obtain:

$$\int_{s_-}^{s_+}\int_{-1}^{1}\frac{\partial\sigma_{33}^3}{\partial x_3}\,drds=-\int_{s_-}^{s_+}\int_{-1}^{1}f_3\,drds-\int_{s_-}^{s_+}[g_3^+-g_3^-]\,ds.$$

Finally replacing σ_{33}^2 with its expressions (50), we obtain the traction equation of result 2.

5.3. Twist Equation. **Result 3:** For applied forces such as $F_t=F_n=\varepsilon^6$, $G_t=G_n=\varepsilon^6$, and $F_3=G_3=\varepsilon^5$, the leading terms of the displacement $\bar{u}_3$, $\overline{\Theta}^0$, $\bar{u}_1^c$, and $\bar{u}_2^c$ satisfy the following twist equation:

$$\frac{E}{\mu}S_\omega\frac{d^3\bar{u}_3}{dx_3^3}-\frac{E}{\mu}J_{1\omega}\frac{d^4\bar{u}_1^c}{dx_3^4}-\frac{E}{\mu}J_{2\omega}\frac{d^4\bar{u}_2^c}{dx_3^4}-\frac{E}{\mu}J_{\omega\omega}\frac{d^4\overline{\Theta}^0}{dx_3^4}+J_{\omega d}\frac{d^2\overline{\Theta}^0}{dx_3^2}=-M_t-\frac{dM_3}{dx_3},$$

where

$$S_\omega=\int_{s_-}^{s_+}\int_{-1}^{1}\omega\,drds,\quad J_{\omega\omega}=\int_{s_-}^{s_+}\int_{-1}^{1}\omega^2\,drds,\quad J_{1\omega}=\int_{s_-}^{s_+}\int_{-1}^{1}x_1\omega\,drds,\quad J_{2\omega}=\int_{s_-}^{s_+}\int_{-1}^{1}x_2\omega\,drds,$$

$$J_{\omega d}=\int_{s_-}^{s_+}\int_{-1}^{1}2r^2(1-cq)\,drds,\quad M_3=\int_{s_-}^{s_+}\int_{-1}^{1}\omega f_3\,drds+\int_{s_-}^{s_+}\omega[g_3^+-g_3^-]\,ds,$$

$$M_t=\int_{s_-}^{s_+}\int_{-1}^{1}lf_n\,drds+\int_{s_-}^{s_+}l[g_n^+-g_n^-]ds-\int_{s_-}^{s_+}\int_{-1}^{1}qf_t\,drds-\int_{s_-}^{s_+}q[g_t^+-g_t^-]ds.$$

Proof. Let us follow step by step for Eq. (51) and (52) the same demarche as for problem P_4. We can prove in the same way that $\tilde{\psi}_t^1$ does not depend on x_3 and we set $\tilde{\psi}_t^1=\overline{\Theta}^1(x_3)$. Thus the displacement at order 1 has the same form as the displacement at the leading order. On the other hand, according to the previous result, problem P_5 reduces in Ω to:

$$\frac{\partial\sigma_{tn}^5}{\partial r}+\frac{\partial\sigma_{tt}^4}{\partial s}+\frac{\partial\sigma_{t3}^3}{\partial x_3}=0, \tag{55}$$

$$\frac{\partial\sigma_{nn}^5}{\partial r}+c\sigma_{tt}^4=0, \tag{56}$$

$$\frac{\partial\sigma_{n3}^5}{\partial r}+\frac{\partial\sigma_{t3}^4}{\partial s}+\frac{\partial\sigma_{33}^3}{\partial x_3}=-f_3, \tag{57}$$

with the following expressions of the stresses at order three: $\sigma_{tt}^3=0$ and

$$\sigma_{33}^3=\frac{3\beta+2}{\beta+1}\frac{\partial\tilde{u}_3^0}{\partial x_3}, \tag{58}$$

$$\sigma_{t3}^3=-2\frac{d\overline{\Theta}^0}{dx_3}r. \tag{59}$$

Problem P_6. The cancellation of the factor of ε^6 leads to the following tangential and normal equations of problem P_6 which write in Ω:

$$\frac{\partial\sigma_{tn}^{6}}{\partial r}+\frac{\partial\sigma_{tt}^{5}}{\partial s}-2c\sigma_{tn}^{5}+rc\frac{\partial\sigma_{tt}^{4}}{\partial s}+\frac{\partial\sigma_{t3}^{4}}{\partial x_3}=-f_t, \tag{60}$$

$$\frac{\partial\sigma_{nn}^{6}}{\partial r}+\frac{\partial\sigma_{tn}^{5}}{\partial s}+c\sigma_{tt}^{5}-c\sigma_{nn}^{5}+rc^2\sigma_{tt}^{4}=-f_n, \tag{61}$$

with the boundary conditions for $r=\pm 1$

$$\sigma_{tn}^{6}=g_t^{\pm}, \tag{62}$$

$$\sigma_{nn}^{6}=g_n^{\pm}. \tag{63}$$

Let us integrate Eqs. (60) and (61) over the thickness. Using the boundary conditions (62) and (63), we obtain the system:

$$\int_{-1}^{1}\left(\frac{\partial\sigma_{tt}^{5}}{\partial s}-2c\sigma_{tn}^{5}+rc\frac{\partial\sigma_{tt}^{4}}{\partial s}+\frac{\partial\sigma_{t3}^{4}}{\partial x_3}\right)dr=-\int_{-1}^{1}f_t dr-[g_t^{+}-g_t^{-}], \tag{64}$$

$$\int_{-1}^{1}\left(\frac{\partial\sigma_{tn}^{5}}{\partial s}+c\sigma_{tt}^{5}-c\sigma_{nn}^{5}+rc^2\sigma_{tt}^{4}\right)dr=-\int_{-1}^{1}f_n dr-[g_n^{+}-g_n^{-}]. \tag{65}$$

Let us now use equations of problem P_5. First multiplying Eqs. (55) and (56) with rc, we obtain:

$$\int_{-1}^{1}\left(rc\frac{\partial\sigma_{tn}^{5}}{\partial r}+rc\frac{\partial\sigma_{tt}^{4}}{\partial s}+rc\frac{\partial\sigma_{t3}^{3}}{\partial x_3}\right)dr=0,\qquad \int_{-1}^{1}\left(rc\frac{\partial\sigma_{nn}^{5}}{\partial r}+rc^2\sigma_{tt}^{4}\right)dr=0.$$

An integration by parts of the previous equations leads to:

$$\int_{-1}^{1}\left(-c\sigma_{tn}^{5}+rc\frac{\partial\sigma_{tt}^{4}}{\partial s}+rc\frac{\partial\sigma_{t3}^{3}}{\partial x_3}\right)dr=0, \tag{66}$$

$$\int_{-1}^{1}(-c\sigma_{nn}^{5}+rc^2\sigma_{tt}^{4})\,dr=0. \tag{67}$$

Then replacing (66) and (67) in (64) and (65) respectively, we get:

$$\int_{-1}^{1}\left(\frac{\partial\sigma_{tt}^{5}}{\partial s}-c\sigma_{tn}^{5}-rc\frac{\partial\sigma_{t3}^{3}}{\partial x_3}+\frac{\partial\sigma_{t3}^{4}}{\partial x_3}\right)dr=-\int_{-1}^{1}f_t dr-[g_t^{+}-g_t^{-}], \tag{68}$$

$$\int_{-1}^{1}\left(\frac{\partial\sigma_{tn}^{5}}{\partial s}+c\sigma_{tt}^{5}\right)dr=-\int_{-1}^{1}f_n dr-[g_n^{+}-g_n^{-}]. \tag{69}$$

Now we shall multiply (68) with $q(s)$ and (69) with $l(s)$. We obtain:

$$\int_{-1}^{1}\left(q\frac{\partial\sigma_{tt}^{5}}{\partial s}-qc\sigma_{tn}^{5}-qrc\frac{\partial\sigma_{t3}^{3}}{\partial x_3}+q\frac{\partial\sigma_{t3}^{4}}{\partial x_3}\right)dr=-\int_{-1}^{1}qf_t dr-q[g_t^{+}-g_t^{-}], \tag{70}$$

$$\int_{-1}^{1}\left(l\frac{\partial\sigma_{tn}^5}{\partial s}+lc\sigma_{tt}^5\right)dr=-\int_{-1}^{1} lf_n\,dr-l[g_n^+-g_n^-]. \tag{71}$$

Using the following equalities:

$$q\frac{\partial\sigma_{tt}^5}{\partial s}=\frac{\partial(q\sigma_{tt}^5)}{\partial s}-\frac{\partial q}{\partial s}\sigma_{tt}^5,\qquad l\frac{\partial\sigma_{tn}^5}{\partial s}=\frac{\partial(l\sigma_{tn}^5)}{\partial s}-\frac{\partial l}{\partial s}\sigma_{tn}^5$$

we reduce Eqs. (70) and (71) to

$$\int_{-1}^{1}\left(\frac{\partial(q\sigma_{tt}^5)}{\partial s}-\frac{\partial q}{\partial s}\sigma_{tt}^5-qc\sigma_{tn}^5-qrc\frac{\partial\sigma_{t3}^3}{\partial x_3}+q\frac{\partial\sigma_{t3}^4}{\partial x_3}\right)dr=-\int_{-1}^{1} qf_t\,dr-q[g_t^+-g_t^-],$$

$$\int_{-1}^{1}\left(\frac{\partial(l\sigma_{tn}^5)}{\partial s}-\frac{\partial l}{\partial s}\sigma_{tn}^5+lc\sigma_{tt}^5\right)dr=-\int_{-1}^{1} lf_n\,dr-h[g_n^+-g_n^-].$$

Now let us integrate the previous equations with respect to s after subtraction. We obtain

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(\frac{\partial(q\sigma_{tt}^5)}{\partial s}-\frac{\partial l\sigma_{tn}^5}{\partial s}-\left[\frac{\partial q}{\partial s}+cl\right]\sigma_{tt}^5+\left[\frac{\partial l}{\partial s}-cq\right]\sigma_{tn}^5-rcq\frac{\partial\sigma_{t3}^3}{\partial x_3}+q\frac{\partial\sigma_{t3}^4}{\partial x_3}\right)drds=M_t, \tag{72}$$

where M_t, whose expression is given in result 3, denotes the twist torque calculated at point C. To simplify the previous equations, we use on one hand the geometrical properties $\frac{\partial q}{\partial s}+cl=0$ and $\frac{\partial l}{\partial s}-cq=1$, and on the other hand the boundary conditions $\sigma_{tt}^5=0$ and $\sigma_{tn}^5=0$ on $s=s_-$ and $s=s_+$. Then Eq. (72) reduces to

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(\sigma_{tn}^5-rcq\frac{\partial\sigma_{t3}^3}{\partial x_3}+q\frac{\partial\sigma_{t3}^4}{\partial x_3}\right)drds=M_t.$$

Now we multiply Eq. (55) by r and integrate it over a section. We get:

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(r\frac{\partial\sigma_{tn}^5}{\partial r}+r\frac{\partial\sigma_{tt}^4}{\partial s}+r\frac{\partial\sigma_{t3}^3}{\partial x_3}\right)drds=0.$$

Using the boundary condition $\sigma_{tt}^4=0$ on $s=s_-$ and $s=s_+$, an integration by part of the first term leads to:

$$\int_{s_-}^{s_+}\int_{-1}^{1}\sigma_{tn}^5\,drds=\int_{s_-}^{s_+}\int_{-1}^{1} r\frac{\partial\sigma_{t3}^3}{\partial x_3}\,drds. \tag{73}$$

On the other hand, we shall multiply Eq. (57) with the sectorial area ω and integrate the result over a section. We get:

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(\omega\frac{\partial\sigma_{n3}^5}{\partial r}+\omega\frac{\partial\sigma_{t3}^4}{\partial s}+\omega\frac{\partial\sigma_{33}^3}{\partial x_3}\right)drds=-\int_{s_-}^{s_+}\int_{-1}^{1}\omega f_3\,drds. \tag{74}$$

Using the property

$$\omega\frac{\partial\sigma_{t3}^4}{\partial s}=\frac{\partial(\omega\sigma_{t3}^4)}{\partial s}-\frac{d\omega}{ds}\sigma_{t3}^4=\frac{\partial(\omega\sigma_{t3}^4)}{\partial s}+q\sigma_{t3}^4$$

and the boundary condition (54), we obtain

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(\frac{\partial(\omega\sigma_{t3}^4)}{\partial s}+q\sigma_{t3}^4+\omega\frac{\partial\sigma_{33}^3}{\partial x_3}\right)drds=-\int_{s_-}^{s_+}\int_{-1}^{1}\omega f_3\ drds-\int_{s_-}^{s_+}\omega[g_3^+-g_3^-]\,ds.$$

With the boundary condition $\int_{-1}^{1}\sigma_{t3}^4 dr=0$ on $s=s_-$ and $s=s_+$, the last equation reduces to:

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(q\sigma_{t3}^4+\omega\frac{\partial\sigma_{33}^3}{\partial x_3}\right)drds=-\int_{s_-}^{s_+}\int_{-1}^{1}\omega f_3\,drds-\int_{s_-}^{s_+}\omega[g_3^+-g_3^-]\,ds.$$

Now let us derive the last equation with respect to x_3. We obtain the relation:

$$\int_{s_-}^{s_+}\int_{-1}^{1}q\frac{\partial\sigma_{t3}^4}{\partial x_3}\,drds=-\int_{s_-}^{s_+}\int_{-1}^{1}\omega\frac{\partial^2\sigma_{33}^3}{\partial x_3^2}\,drds-\frac{dM_3}{dx_3}, \tag{75}$$

where the expression of M_3 is given in result 3. To finish let us replace σ_{tn}^5 and σ_{t3}^4 with their expressions (73) and (75) in Eq. (72). We get:

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left((1-cq)r\frac{\partial\sigma_{t3}^3}{\partial x_3}-\omega\frac{\partial^2\sigma_{33}^3}{\partial x_3^2}\right)drds=M_t+\frac{dM_3}{dx_3}.$$

Finally, replacing σ_{t3}^3 and σ_{33}^3 with their expressions (58)–(59), we obtain the twist equilibrium equation of result 3.

5.4. Bending Equations. **Result 4:** For force levels such as $F_t=F_n=\varepsilon^6, G_t=G_n=\varepsilon^6$, and $F_3=G_3=\varepsilon^5$, the leading terms of the displacements $\bar{u}_3$, $\overline{\Theta}^0$, $\bar{u}_1^c$, and $\bar{u}_2^c$ are solutions of the following bending equations:

$$\frac{E}{\mu}S_1\frac{d^3\bar{u}_3}{dx_3^3}-\frac{E}{\mu}J_{11}\frac{d^4\bar{u}_1^c}{dx_3^4}-\frac{E}{\mu}J_{12}\frac{d^4\bar{u}_2^c}{dx_3^4}-\frac{E}{\mu}J_{1\omega}\frac{d^4\overline{\Theta}^0}{dx_3^4}+J_{1d}\frac{d^2\overline{\Theta}^0}{dx_3^2}=-P_1-\frac{dM_{31}}{dx_3},$$

$$\frac{E}{\mu}S_2\frac{d^3\bar{u}_3}{dx_3^3}-\frac{E}{\mu}J_{12}\frac{d^4\bar{u}_1^c}{dx_3^4}-\frac{E}{\mu}J_{22}\frac{d^4\bar{u}_2^c}{dx_3^4}-\frac{E}{\mu}J_{2\omega}\frac{d^4\overline{\Theta}^0}{dx_3^4}+J_{2d}\frac{d^2\overline{\Theta}^0}{dx_3^2}=-P_2-\frac{dM_{32}}{dx_3},$$

where

$$S_1=\int_{s_-}^{s_+}\int_{-1}^{1}x_1drds,\quad S_2=\int_{s_-}^{s_+}\int_{-1}^{1}x_2drds,\quad J_{11}=\int_{s_-}^{s_+}\int_{-1}^{1}x_1^2drds,$$

$$J_{22}=\int_{s_-}^{s_+}\int_{-1}^{1}x_2^2drds,\quad J_{12}=\int_{s_-}^{s_+}\int_{-1}^{1}x_1x_2drds,\quad J_{1\omega}=\int_{s_-}^{s_+}\int_{-1}^{1}x_1\omega drds,$$

$$J_{2\omega}=\int_{s_-}^{s_+}\int_{-1}^{1}x_2\omega drds,\quad J_{1d}=\int_{s_-}^{s_+}\int_{-1}^{1}2r^2c\cos\alpha drds,\quad J_{2d}=\int_{s_-}^{s_+}\int_{-1}^{1}2r^2c\sin\alpha drds$$

and

$$P_1=\int_{s_-}^{s_+}\int_{-1}^{1}\cos\alpha f_t\,drds+\int_{s_-}^{s_+}\cos\alpha[g_t^+-g_t^-]ds-\int_{s_-}^{s_+}\int_{-1}^{1}\sin\alpha f_n\,drds-\int_{s_-}^{s_+}\sin\alpha[g_n^+-g_n^-]ds,$$

$$P_2=\int_{s_-}^{s_+}\int_{-1}^{1}\sin\alpha f_t\,drds+\int_{s_-}^{s_+}\sin\alpha[g_t^+-g_t^-]ds+\int_{s_-}^{s_+}\int_{-1}^{1}\cos\alpha f_n\,drds+\int_{s_-}^{s_+}\cos\alpha[g_n^+-g_n^-]ds,$$

$$M_{31}=\int_{s_-}^{s_+}\int_{-1}^{1}x_1 f_3\,drds+\int_{s_-}^{s_+}x_1[g_3^+-g_3^-]ds,\qquad M_{32}=\int_{s_-}^{s_+}\int_{-1}^{1}x_2 f_3\,drds+\int_{s_-}^{s_+}x_2[g_3^+-g_3^-]ds.$$

Proof. Let us start again from Eqs. (68)–(69). We have:

$$\int_{-1}^{1}\left(\frac{\partial\sigma_{tt}^5}{\partial s}-c\sigma_{tn}^5-rc\frac{\partial\sigma_{t3}^3}{\partial x_3}+\frac{\partial\sigma_{t3}^4}{\partial x_3}\right)dr=-\int_{-1}^{1}f_t\,dr-[g_t^+-g_t^-], \tag{76}$$

$$\int_{-1}^{1}\left(\frac{\partial\sigma_{tn}^5}{\partial s}+c\sigma_{tt}^5\right)dr=-\int_{-1}^{1}f_n\,dr-[g_n^+-g_n^-]. \tag{77}$$

We shall multiply them respectively with $\cos\alpha$ and $\sin\alpha$. Then an integration over a section leads to:

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(\cos\alpha\frac{\partial\sigma_{tt}^5}{\partial s}-c\cos\alpha\sigma_{tn}^5-rc\cos\alpha\frac{\partial\sigma_{t3}^3}{\partial x_3}+\cos\alpha\frac{\partial\sigma_{t3}^4}{\partial x_3}\right)drds=-\int_{s_-}^{s_+}p_t\cos\alpha\,ds,$$

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(\sin\alpha\frac{\partial\sigma_{tn}^5}{\partial s}+c\sin\alpha\sigma_{tt}^5\right)drds=-\int_{s_-}^{s_+}p_n\sin\alpha\,ds,$$

with $p_t=\int_{-1}^{1}f_t\,dr+[g_t^+-g_t^-]$ and $p_n=\int_{-1}^{1}f_n\,dr+[g_n^+-g_n^-]$. Using the following properties:

$$\cos\alpha\frac{\partial\sigma_{tt}^5}{\partial s}=\frac{\partial(\cos\alpha\sigma_{tt}^5)}{\partial s}+c\sin\alpha\sigma_{tt}^5,$$

$$\sin\alpha\frac{\partial\sigma_{tn}^5}{\partial s}=\frac{\partial(\sin\alpha\sigma_{tn}^5)}{\partial s}-c\cos\alpha\sigma_{tn}^5,$$

we get

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(\frac{\partial(\cos\alpha\sigma_{tt}^5)}{\partial s}+c\sin\alpha\sigma_{tt}^5-c\cos\alpha\sigma_{tn}^5-rc\cos\alpha\frac{\partial\sigma_{t3}^3}{\partial x_3}+\cos\alpha\frac{\partial\sigma_{t3}^4}{\partial x_3}\right)drds$$

$$=-\int_{s_-}^{s_+}p_t\cos\alpha\,ds\int_{s_-}^{s_+}\int_{-1}^{1}\left(\frac{\partial(\sin\alpha\sigma_{tn}^5)}{\partial s}-c\cos\alpha\sigma_{tn}^5+c\sin\alpha\sigma_{tt}^5\right)drds=-\int_{s_-}^{s_+}p_n\sin\alpha\,ds.$$

By substraction, we obtain finally:

$$\int_{s_-}^{s_+}\int_{-1}^{1}-rc\cos\alpha\frac{\partial\sigma_{t3}^3}{\partial x_3}+\cos\alpha\frac{\partial\sigma_{t3}^4}{\partial x_3}\,drds=-P_1. \tag{78}$$

On the other hand, let us multiply Eq. (57) with x_1 and integrate the result over a section. We get

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(x_1\frac{\partial\sigma_{n3}^5}{\partial r}+x_1\frac{\partial\sigma_{t3}^4}{\partial s}+x_1\frac{\partial\sigma_{33}^3}{\partial x_3}\right)drds=-\int_{s_-}^{s_+}\int_{-1}^{1}x_1 f_3\,drds.$$

Then using the equality $x_1\frac{\partial\sigma_{t3}^4}{\partial s}=\frac{\partial(x_1\sigma_{t3}^4)}{\partial s}-\cos\alpha\sigma_{t3}^4$, the boundary condition (54) and $\sigma_{t3}^4=0$ for $s=s_\pm$, we obtain:

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(-\cos\alpha\sigma_{t3}^4+x_1\frac{\partial\sigma_{33}^3}{\partial x_3}\right)drds=-\int_{s_-}^{s_+}\int_{-1}^{1}x_1 f_3\,drds-\int_{s_-}^{s_+}x_1[g_3^+-g_3^-]\,ds. \tag{79}$$

Now let us derive Eq. (79) with respect to x_3. We get

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(-\cos\alpha\frac{\partial\sigma_{t3}^4}{\partial x_3}+x_1\frac{\partial^2\sigma_{33}^3}{\partial x_3^2}\right)drds=-\frac{dM_{31}}{dx_3}. \tag{80}$$

Adding Eqs. (78) and (80), we have

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(-rc\cos\alpha\frac{\partial\sigma_{t3}^3}{\partial x_3}+x_1\frac{\partial^2\sigma_{33}^3}{\partial x_3^2}\right)drds=-P_1-\frac{dM_{31}}{dx_3}.$$

Finally, replacing σ_{t3}^3 and σ_{33}^3 by their respective expressions, we obtain the bending equation in the direction e_1 of result 4. The bending equation in the direction e_2 is obtained in the same way, by permutation of the indices.

6. Comparison with Vlassov Model. To compare the one-dimensional thin-walled beam model obtained at results 1 to 4 to Vlassov model, we shall first go back to the initial dimensional domain Ω^* and to the dimensional variables u_t^*, u_n^*, u_3^*, f^*, and g^*. To do this, let us define

$$u_t^{*0}=u_{tr}u_t^0=hu_t^0,\quad u_n^{*0}=u_{nr}u_n^0=hu_n^0,\quad u_3^{*0}=u_{3r}u_t^0=\varepsilon hu_3^0. \tag{81}$$

We then have the following result:

Result 5: For force levels such as $F_t=F_n=\varepsilon^6$, $G_t=G_n=\varepsilon^6$, and $F_3=G_3=\varepsilon^5$, the displacement $(u_t^{*0}, u_n^{*0}, u_3^{*0})$ is of Vlassov type:

$$\tilde{u}_t^{*0}=\bar{u}_1^{*c}\cos(\alpha)+\bar{u}_2^{c*}\sin(\alpha)-q^*(s)\bar{\Theta}^{*0},$$

$$\tilde{u}_n^{*0}=-\bar{u}_1^{*c}\sin(\alpha)+\bar{u}_2^{c*}\cos(\alpha)+l^*(s)\bar{\Theta}^{*0},$$

$$\tilde{u}_3^{*0}=\bar{u}_3^*-x_1^*\frac{d\bar{u}_1^{*c}}{dx_3^*}-x_2^*\frac{d\bar{u}_2^{*c}}{dx_3^*}-\omega^*\frac{d\bar{\Theta}^{*0}}{dx_3^*}. \tag{82}$$

Starting from result 1 and from the dimensional analysis on the geometric parameters performed, the proof of this result does not constitute any difficulty and is left to the reader. We just need to set $\bar{u}_1^{*c}=h\bar{u}_1^c$, $\bar{u}_2^{*c}=h\bar{u}_2^c$, $\bar{u}_3^*=\varepsilon h\bar{u}_3$, $\omega^*=d^2\omega$, $\bar{\Theta}^{*0}=\varepsilon\bar{\Theta}^0$ and to use the relations $h/L=\varepsilon^2$ and $hL=d^2$ between the small parameters.

In the same way, we shall go back to dimensional variables in traction, twist and bending equations of results 2 to 4. However, we will not give here the complete dimensional equations, but only the reduced ones which are sufficient for a comparison with Vlassov model. We recall that the one-dimensional equations reduce to a much more simple form if they are written in a particular base, called "reduced basis". In this reduced basis, the directions e_1 and e_2 correspond to the principal inertial axis of the profile. Moreover the origin of the frame coincides with the center of gravity of the profile and the origin of the sectorial area with the shear center. We then have the following result whose proof does not constitute any difficulty and is left to the reader:

Result 6: For force levels such as $F_t = F_n = \varepsilon^6$, $G_t = G_n = \varepsilon^6$, and $F_3 = G_3 = \varepsilon^5$, the leading terms of the displacements $\overline{u}_3^*$, $\overline{\Theta}^{*0}$, $\overline{u}_1^{*c}$, and $\overline{u}_2^{*c}$ are solution of the following reduced one-dimensional equilibrium equations:

$$ES^* \frac{d^2 \overline{u}_3^{*0}}{dx_3^{*2}} = -P_3^*, \tag{83}$$

$$EJ^*_{\omega^*\omega^*} \frac{d^4 \overline{\Theta}^{*0}}{dx_3^{*4}} - \mu J^*_{\omega^* d} \frac{d^2 \overline{\Theta}^{*0}}{dx_3^{*2}} = M_t^* + \frac{dM_3^*}{dx_3^*}, \tag{84}$$

$$EJ^*_{11} \frac{d^4 \overline{u}_1^{*c}}{dx_3^{*4}} - \mu J^*_{1d} \frac{d^2 \overline{\Theta}^{*0}}{dx_3^{*2}} = P_1^* + \frac{dM_{31}^*}{dx_3^*}, \tag{85}$$

$$EJ^*_{22} \frac{d^4 \overline{u}_2^{*c}}{dx_3^{*4}} - \mu J^*_{2d} \frac{d^2 \overline{\Theta}^{*0}}{dx_3^{*2}} = P_2^* + \frac{dM_{32}^*}{dx_3^*}. \tag{86}$$

The dimensional expressions of the forces and of the geometric constants involved in result 6 may be obtained easily fom results 2 to 4. We shall quote that the kinematics, the one-dimensional reduced traction and twist equilibrium equations of results 5 and 6 correspond exactly to Vlassov ones [47]. However the one-dimensional bending equations (85), (86) differ from Vlassov ones which write (in the reduced basis):

$$EJ^*_{11} \frac{d^4 \overline{u}_1^{*c}}{dx_3^{*4}} = P_1^* + \frac{dM_{31}^*}{dx_3^*}, \tag{87}$$

$$EJ^*_{22} \frac{d^4 \overline{u}_2^{*c}}{dx_3^{*4}} = P_2^* + \frac{dM_{32}^*}{dx_3^*}. \tag{88}$$

Therefore, at the difference from Vlassov model, the bending equations (85), (86) contain a supplementary term coupling twist and bending effects. This coupling term is linked to the new geometrical constants J^*_{1d} and J^*_{2d} and does not seem to have any equivalent in the literature. It corresponds most probably to a correction at the second order of Vlassov model. Thus the model obtained by asymptotic expansion in this paper should improve Vlassov one where the twist angle and the bending displacements are uncoupled. (We recall that from Vlassov model, an external bending loading whose resultant induces a torque, will induce not only a bending displacement but also a twist. In contrary, a torque will induce only a twist, but no bending, unlike the model obtained in this paper where these two effects are coupled).

Let us quote that such a limitation of Vlassov theory (lack of coupling) already have been noticed by other authors [5, 6, 23, 43]. To improve Vlassov model, the authors proposed to add directly supplementary terms characterizing coupling effects in equilibrium equations.

7. Conclusion. In this paper we deduced by asymptotic expansion a one-dimensional linear model for thin-walled rods obtained for a strongly curved profile subjected to low force levels. The obtained kinematics, the one-dimensional traction and twist equilibrium equations of results 1 to 3 correspond exactly to Vlassov ones [47]. However, whereas Vlassov approach relies on a priori physical assumptions, with our approach the kinematics and equilibrium equations are directly deduced from the three-dimensional equilibrium equations for the level of applied forces considered. Thus the domain of validity of the obtained model can be specified precisely thanks to the dimensionless numbers introduced.

Another major result is that that this asymptotic approach leads to an explicit analytical expression of the geometrical constants involved in the one-dimensional equilibrium equations. In particular, we obtain a general analytical expression of the twist rigidity $J^*_{\omega d}$, whereas in the literature only an approximate expression depending on an empiric coefficient is given [47].

Finally, it is important to notice that the one-dimensional bending equations of result 4 differ from Vlassov ones. At the difference from Vlassov model, we obtain a supplementary term coupling twist and bending effects. This coupling is due to the new geometrical constants J_{1d}^{*} and J_{2d}^{*} and does not seem to have any equivalent in the literature.

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