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MOTION PLANNING FOR A WHEELED ROBOTIC VEHICLE WITH NO STEERABLE WHEEL

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The motion-planning problem for a wheeled robotic vehicle with no steerable wheel is solved in both Cartesian and polar coordinate systems. The results are illustrated by examples

Keywords: wheeled robotic vehicle, optimal control, nonholonomic constraints

Introduction. The control problem for various vehicles [1, 2, 4], including systems with nonholonomic constraints [3] such as wheeled robotic vehicles [7–10] still attract the attention of researchers. The class of wheeled robotic vehicles includes simple robots without steerable wheels. For such vehicles, either path-planning problem [5] or motion-planning problem [6] can be solved. In what follows, we will formulate the motion-planning problem as a variational problem for a wheeled robotic vehicle similar to that considered in [6]. The problem will be solved in Cartesian and polar coordinate systems, followed by illustrative examples.

1. Description of the Model. Figure 1 shows a wheeled robotic vehicle without a steerable wheel. This vehicle has two drive wheels on the axle *AB* and a castoring wheel at the point *F*. The velocities of the points *A* (V_A) and *B* (V_B) are perpendicular to *AB*. The robot has mass *m*, its center of gravity is at the point *C*, and the moment of inertia about the point *C* is *J*. The distance between the wheels is *r*, and the distance between the points *C* and *D* is *d*.

The delayed action of the wheels and drives is neglected. It is assumed that the torques generated by the drives produce forces p_A and p_B applied to the points *A* and *B* and acting at a right angle to *AB*.

Let the motion of this vehicle be described by equations similar to those given in [6]:

$$\dot{\eta} = \Gamma(\eta)\nu, \qquad M\dot{\nu} + \Omega(\dot{\eta})\nu = BP,$$

$$\eta = [x \quad y \quad \theta]^{\mathrm{T}}, \qquad \nu = [V_A \quad V_B], \qquad P = [p_A \quad p_B],$$

$$\Gamma(\eta) = \frac{1}{2}\begin{bmatrix} \cos\theta & \cos\theta \\ \sin\theta & \sin\theta \\ \frac{2}{r} & -\frac{2}{r} \end{bmatrix}, \qquad M = \frac{1}{4r^2}\begin{bmatrix} m_1 & m_2 \\ m_2 & m_1 \end{bmatrix},$$

$$m_1 = mr^2 + 4(md^2 + J), \qquad m_2 = mr^2 - 4(md^2 + J),$$

$$\Omega(\eta) = \frac{md\dot{\theta}}{r}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad B = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}. \tag{1.1}$$

The superscript “T” denotes transposition.

Assuming that $V_A + V_B \neq 0$, we change variables in the second equation in (1.1):

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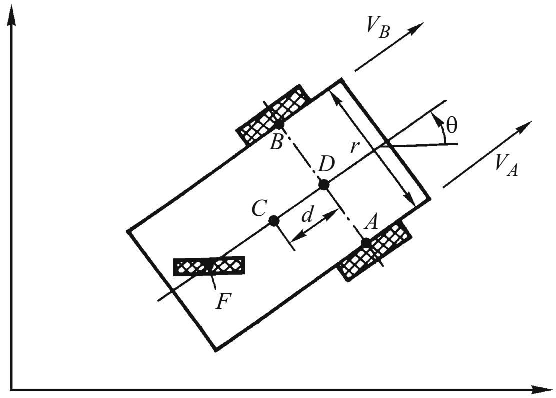

Fig. 1

$$u_1 = \frac{V_A - V_B}{V_A + V_B}, \qquad u_2 = V_A + V_B. \tag{1.2}$$

Performing appropriate transformations, we reduce the second equation (1.1) to the form

$$\dot{u} + F(\dot{\theta}, u) = B_u P, \qquad u = [u_1 u_2]^{\mathrm{T}}, \qquad F(\dot{\theta}, u) = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \dot{\theta}, \qquad B_u = \begin{bmatrix} B_{u1} \\ B_{u2} \end{bmatrix},$$

$$F_1 = -\frac{d(4u_1^2 (md^2 + J) + mr^2)}{2r(md^2 + J)}, \qquad F_2 = \frac{2du_1 u_2}{r},$$

$$B_{u1} = \frac{1}{\mu_2 (md^2 + J)} [2u_1 (md^2 + J) - mr^2 \; 2u_1 (md^2 + J) + mr^2], \qquad B_{u2} = \frac{2}{m} [11]. \tag{1.3}$$

Thus, the task is to design control algorithms (select control forces p_A and p_B) for a robotic vehicle moving as described by Eqs. (1.1) and (1.3).

Note that unlike wheeled robots with steerable wheels [10], it is impossible to decompose the original problem for vehicles with no steerable wheels into path-planning and time-scaling problems [10].

2. Problem Reduction. Let $\dot{x} > 0$. In this case, it is possible to eliminate time as an independent variable, and to use the coordinate x instead. This will show the order of the system of equations describing the motion of the robot.

Thus, dividing Eqs. (1.1) and (1.3) by the first equation of (1.1) and taking (1.2) into account, we obtain the system of equations

$$y' = \frac{dy}{dx} = \tan\theta, \qquad \theta' = \frac{d\theta}{dx} = \frac{2u_1}{r\cos\theta},$$

$$u_1' = \frac{du_1}{dx} = \frac{2}{u_2 \cos\theta} B_{u1} P - F_1 \frac{2u_1}{r\cos\theta},$$

$$u_2' = \frac{du_2}{dx} = \frac{2}{u_2 \cos\theta} B_{u2} P - F_2 \frac{2u_1}{r\cos\theta}. \tag{2.1}$$

Here and later on (except in Sec. 5), the prime denotes differentiation with respect to x.

3. Two-Point Problem. Given values of the vector $\xi = \begin{bmatrix} y & \theta & u_1 & u_2 \end{bmatrix}^{\mathrm{T}}$ at the points $x = 0$ and $x = x_{\mathrm{f}}$, find control forces p_A and p_B that would steer system (2.1) from a position $\xi(0)$ to a position $\xi(x_{\mathrm{f}})$. Considering that

$$y'' = \frac{\theta'}{\cos^2\theta} = \frac{2u_1}{r\cos^3\theta}, \qquad y''' = \frac{2u_1'}{r\cos^3\theta} + \frac{12u_1^2\sin\theta}{r^2\cos^5\theta}, \tag{3.1}$$

we rearrange system (2.1) into

$$y''' = \gamma, \qquad \gamma = \frac{4}{ru_2\cos^4\theta}B_{u1}P - \frac{4u_1}{r^2\cos^4\theta}F_1 + \frac{12u_1^2\sin\theta}{r^2\cos^5\theta},$$

$$u_2' = \mu, \qquad \mu = \frac{2}{u_2\cos\theta}B_{u2}P - \frac{u_1}{r\cos\theta}F_2. \tag{3.2}$$

According to (2.1) and (3.1), we get

$$\theta = \arctan(y'), \qquad u_1 = \frac{y'' r\cos^3\theta}{2}. \tag{3.3}$$

Note that the problem can be solved in two stages, which follows from (3.2). At the first stage, we find the function $\gamma(x)$ that would make the first three components of the vector ξ equal to each other at the point x_f. Next, we use the second equation in (3.2) to find the function $\mu(x)$ that would make $u_2(x_f)$ equal to a given value. At the second stage, we use the functions $\gamma(x)$ and $\mu(x)$ found at the first stage and the corresponding values $y(x), \theta(x), u_1(x), u_2(x)$ to find the components of the vector P, i.e., the functions $p_A(x), p_B(x)$, from a system of linear equations. This is how the two-point boundary-value problem can be solved. In fact, we have decomposed it into two similar problems of lower dimension.

4. Optimization Procedure. Let us consider the first stage of solution for the first equation of system (3.2). As in [10], it is expedient to optimize the system with respect to some quadratic performance criterion.

Considering (3.3), we formulate boundary conditions for the phase vector $w^{\mathrm{T}} = [y \;\; y' \;\; y'']$ at the initial and final points, rather than for y, θ, u_1. Rearranging the first equation in (3.2) into

$$w' = Aw + B\gamma, \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.1}$$

we formulate the motion-planning problem as follows. Let the vector w satisfy (4.1). Given $w(0)$ and $w(x_f)$, find the function $\gamma(x)$ that would minimize the following quadratic functional:

$$I = \int_0^{x_f} (w^{\mathrm{T}}Qw + \gamma^2)\,dx \qquad (Q \ge 0). \tag{4.2}$$

The solution of problem (4.1), (4.2) is known (see, for example, [4]) and found in several stages.

The vector of conjugate variables $\lambda = [\lambda_1\lambda_2\lambda_3]^{\mathrm{T}}$ is introduced, and the Hamiltonian matrix H of the variational problem (4.1), (4.2) is formed:

$$H = \begin{bmatrix} A & -T \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \qquad (T = BB^{\mathrm{T}}). \tag{4.3}$$

Then the expression for $\gamma(x)$ becomes

$$\gamma(x) = -B^{\mathrm{T}}\lambda(x), \tag{4.4}$$

where w and λ are such that

$$z(x) = \Phi(x) z(0), \qquad z = \begin{bmatrix} w \\ \lambda \end{bmatrix}, \qquad \Phi(x) = e^{Hx}. \tag{4.5}$$

The value $\lambda(0)$ depends on $w(0)$ and $w(x_f)$ as follows:

$$\lambda(0) = (\Phi_{12}(x_f))^{-1} (w(x_f) - \Phi_{11}(x_f) w(0)),$$

$$\Phi(x_f) = \begin{bmatrix} \Phi_{11}(x_f) & \Phi_{12}(x_f) \\ \Phi_{21}(x_f) & \Phi_{22}(x_f) \end{bmatrix}. \tag{4.6}$$

Thus, formulas (2.8)–(2.10) give the solutions of the problem posed because the functions $w(x)$ and $\gamma(x)$ determine the parameters $y(x)$, $\theta(x)$, $u_1(x)$.

Note that a similar approach can be followed to find the function $\mu(x)$ that would allow the boundary condition for u_2 to be satisfied at the point $x = x_f$. Since the second equation in (3.2) is simple, we can use a simpler approach to solving this problem. Let $u_2(0)$ and $u_2(x_f)$ be given, and let the function $u_2(x)$ have a simple (linear) form:

$$u_2(x) = \alpha x + u_2(0), \qquad \alpha = \frac{u_2(x_f) - u_2(0)}{x_f}. \tag{4.7}$$

According to (3.2), we have

$$u_2'(x) = \alpha. \tag{4.8}$$

The system of linear equations below follows from (3.2):

$$\frac{4}{r u_2 \cos^4 \theta} B_{u1} P = \gamma + \frac{4 u_1}{r^2 \cos^4 \theta} F_1 + \frac{12 u_1 \sin \theta}{r^2 \cos^5 \theta},$$

$$\frac{2}{u_2 \cos \theta} B_{u2} P = \mu + \frac{u_1}{r \cos \theta} F_2, \tag{4.9}$$

It determines the components of the vector P.

Thus, formulas (4.3)–(4.9) give the solution of the two-point boundary-value problem.

Recall that the original system was reduced in Sec. 2 by replacing time t as an independent variable by the coordinate x. To establish how the variables found by solving the problem depend on time, we can use the time dependence of the coordinate x. This dependence is defined by

$$t = \int_0^x \frac{2dx}{u_2 \cos \theta}. \tag{4.10}$$

Thus, having the functions $u_2(x)$ and $\theta(x)$, we can establish the dependence $t(x)$ according to (4.10).

5. Polar Coordinates. In the previous sections, we described the motion of the robot in Cartesian coordinates. Sometimes, however, it is expedient to use polar coordinates. For example, it was assumed in Sec. 2 that $\dot{x} = \frac{u_1}{u_2} \cos \theta > 0$ when the robot moves. However, this may not be so when, for example, $\theta = \pm \pi / 2$. In such problems, it makes sense to use polar coordinates ($x = R \cos \varphi$, $y = R \sin \varphi$). The first three equations in (1.1) have the following polar analogs [10]:

$$\dot{R} = \frac{u_2}{2} \sin \varepsilon, \qquad \dot{\varphi} = \frac{u_2}{2R} \cos \varepsilon, \qquad \dot{\varepsilon} = \dot{\varphi} - \frac{2 u_1 R}{r \cos \varepsilon}, \qquad \varepsilon = \frac{\pi}{2} + \varphi - \theta. \tag{5.1}$$

To reduce the problem, assuming that $\dot{\varphi} > 0$, we choose the angle φ as an independent variable. In view of (5.1), we have the following analog of Eqs. (2.1):

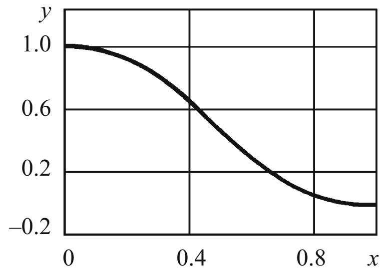

Fig. 2

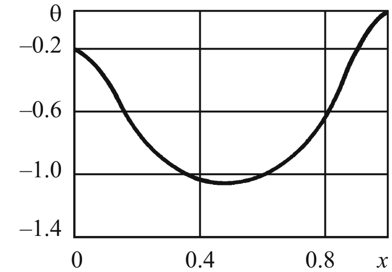

Fig. 3

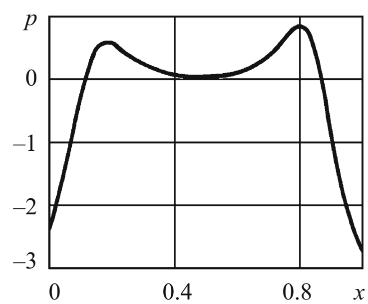

Fig. 4

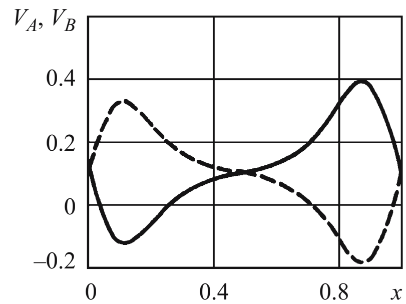

Fig. 5

$$R' = \frac{dR}{d\varphi} = R\tan\varepsilon, \qquad \varepsilon' = \frac{d\varepsilon}{d\varphi} = 1 - \frac{2u_1 R}{r\cos\varepsilon},$$

$$u_1' = \frac{du_1}{d\varphi} = \frac{2R}{u_2\cos\varepsilon} B_{u1} P - \frac{2u_1 R}{r\cos\varepsilon} F_1,$$

$$u_2' = \frac{2R}{u_2\cos\varepsilon} B_{u2} P - \frac{2u_1 R}{r\cos\varepsilon} F_2. \tag{5.2}$$

Here, the prime denotes differentiation with respect to φ. We have

$$R'' = R\tan^2\varepsilon + \frac{R}{\cos^2\varepsilon} - \frac{2R^2 u_1}{r\cos^2\varepsilon}, \tag{5.3}$$

$$R''' = a + bu_1', \tag{5.4}$$

$$a = -\frac{R}{r^2\cos^5\varepsilon}(\sin\varepsilon\cos^2\varepsilon r^2(\cos^2\varepsilon - 6) + 6u_1 R\sin\varepsilon(3r\cos\varepsilon - 2u_1 R)), \qquad b = -\frac{2R^2}{r\cos^3\varepsilon}.$$

In polar coordinates, the two-point problem is formulated in much the same way as in Sec. 3. Given values of the vector $\xi = [R\varepsilon u_1 u_2]^{\mathrm{T}}$ at the points $\varphi = 0$ and $\varphi = \varphi_{\mathrm{f}}$, find control forces p_A and p_B that would steer system (5.2) from a position $\xi(0)$ to a position $\xi(\varphi_{\mathrm{f}})$. Considering (5.3) and (5.4), we rearrange system (5.2) into

$$R''' = \gamma, \qquad \gamma = \frac{2Rb}{u_2\cos\varepsilon} B_{u1} P + \frac{2Ra}{u_2\cos\varepsilon} - \frac{2u_1 R}{r\cos\varepsilon} F_1,$$

$$u_2' = \mu, \qquad \mu = \frac{2R}{u_2\cos\varepsilon} B_{u2} P - \frac{2u_1 R}{r\cos\varepsilon} F_2. \tag{5.5}$$

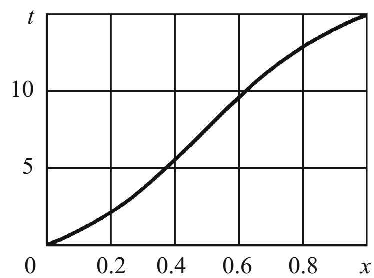

Fig. 6

According to (5.2) and (5.3), the "physical" parameters ε and u_1 appearing in (5.5) are expressed in terms of R', R'' as follows:

$$\varepsilon = \arctan\left(\frac{R'}{R}\right),$$

$$u_1 = -\frac{r\cos^3\varepsilon}{2R^2}\left(R\tan^2\varepsilon + \frac{R}{\cos^2\varepsilon} - R''\right). \tag{5.6}$$

Note that Eqs. (5.5) are similar in structure to Eqs. (3.2); therefore, the algorithm outlined in Sec. 4 can be used to solve the two-point problem formulated above.

Note that formula (4.10) has the following polar analog:

$$t = \int_0^{\varphi} \frac{d\varphi}{\omega}, \qquad \omega = \frac{u_2 \cos\varepsilon}{2R}.$$

6. Examples. Let us consider examples of a robot, similar to that in [6] moving in Cartesian (example 1) and polar (example 2) coordinate systems. The vehicle has the following dynamic parameters: m = 30, J = 15, r = 1.5. Let $d = 0 (F_1 = 0, F_2 = 0)$ for the sake of simplicity.

Example 1. Let $y = 1$, $\theta = -0.2$, $V_A = V_B$ at time zero ($x = 0$), i.e., $u_1 = 0$. The following boundary conditions must be satisfied at the final point ($x = 1$): $y = 0, \theta = 0, u_1 = 0$. Let Q in functional (4.2) be a zero matrix. Moreover, let $u_2(0) = u_2(1) = 0.2$, i.e., $\alpha = 0, \mu = 0$ according to (4.7) and (4.8). Then $p_A = -p_B = p$ follows from the second equation in (4.9). The solution of this problem found by the algorithm of Sec. 4 is shown in Figs. 2–5.

Figure 5 shows the velocities V_A (solid line) and V_B (dashed line) determined by (1.2). Figure 6 depicts the curve of the function $t(x)$ plotted by (4.10). This function makes it possible, if needed, to plot the time-dependence of the dynamic parameters the robot has during the maneuver.

Figure 3 demonstrates that $\pi/2 > \theta(x) > -\pi/2$ over the entire range of x; i.e., the assumption that $\dot{x} = \dfrac{u_2\cos\theta}{2} > 0$ (Sec. 2) remains valid during the maneuver.

Example 2. Let $R = 1, \theta = \pi/2, V_A = V_B$ at time zero ($\varphi = 0$), i.e., $\varepsilon = 0, u_1 = 0$. The following boundary conditions must be satisfied at the final point ($\varphi = \pi$): $R = 1, u_1 = 0, \varepsilon = 0$, i.e., $\theta = 3\pi/2$.

Let $u_2(0) = u_2(\pi) = 0.4$, i.e., $\alpha = 0, \mu = 0$ according to (4.7) and (4.8), as in Example 1. Then $p_A = -p_B = p$ follows from the second equation in (4.9). Let, unlike Example 1, Q in (4.2) be a nonzero matrix of the form $Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Figures 7–10 show the solution of this problem.

Figure 10 shows the velocities V_A (solid line) and V_B (dashed line). Note that $\cos\theta = 0$ at the initial and final points, i.e., $\dot{x} = 0$.

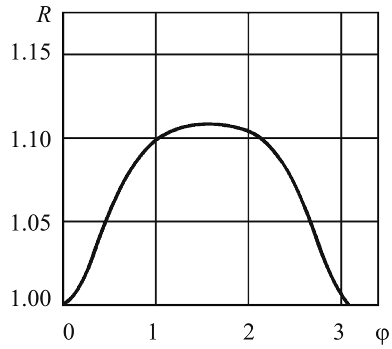

Fig. 7

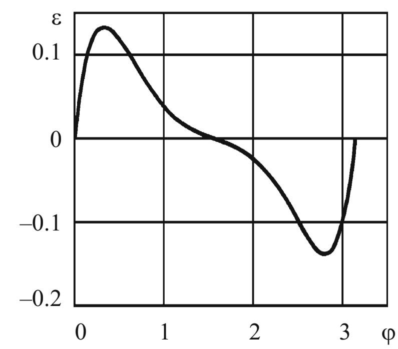

Fig. 8

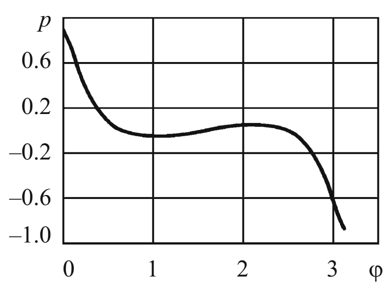

Fig. 9

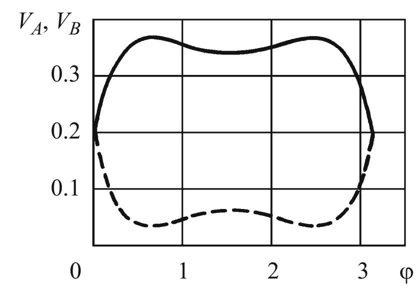

Fig. 10

Conclusions. We have solved the motion-planning problem for a wheeled robotic vehicle without a steerable wheel in Cartesian and polar coordinates. The results have been illustrated by examples.

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