FREE AXISYMMETRIC VIBRATIONS OF SOLID CYLINDERS: NUMERICAL PROBLEM SOLVING

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The problem of free vibrations of a solid cylinder with different boundary conditions is solved using the three-dimensional theory of elasticity and a numerical analytic approach. The spline-approximation and collocation methods are used to reduce the partial differential equations of elasticity to systems of ordinary differential equations of high order with respect to the radial coordinate. These equations are solved by stable numerical discrete orthogonalization and incremental search. Calculated results are presented for transversely isotropic and inhomogeneous materials of the cylinder and for several types of boundary conditions at its ends

Keywords: free vibrations, three-dimensional theory of elasticity, solid cylinder, finite length, spline-collocation

Introduction. Solving three-dimensional dynamic problems for elastic solids involves significant difficulties because of the complexity of the associated system of partial differential equations and the necessity to satisfy boundary conditions. There are only few publications [1–6, 12–22] that use the three-dimensional theory of elasticity to study the vibrations of finite-length cylinders.

To solve such problems in computational mathematics, mathematical physics, and mechanics, spline functions are widely used. For example, the spline approximation method was used in [7–11] to study the mechanical behavior of various plates and shells.

The present paper proposes an efficient numerical procedure to determine the natural frequencies and modes of axisymmetric vibrations of transversely isotropic solid finite-length cylinders with different boundary conditions. The procedure uses spline-approximation in one of the coordinate directions followed by solution of an eigenvalue boundary-value problem for systems of ordinary differential equations of high order with variable (in the general case) coefficients by stable numerical discrete orthogonalization in combination with incremental search. The approach makes it possible to study the free vibrations of finite-length cylinders made of inhomogeneous materials.

1. Problem Formulation. Governing Equations. Let us consider, using cylindrical coordinates r, θ, z, a solid cylinder of radius R and length L made of a transversely isotropic material with the axis of elastic symmetry aligned with the Oz-axis. The starting equations of elasticity for the problem of free axisymmetric radial and longitudinal vibrations (no torsional vibrations) of the cylinder include:

the Cauchy relations

$$e_r = \frac{\partial u_r}{\partial r}, \qquad e_\theta = \frac{u_r}{r}, \qquad e_z = \frac{\partial u_z}{\partial z}, \qquad 2e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \tag{1}$$

the generalized Hooke's law for a transversely isotropic material

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$$\sigma_r = \lambda_{11} e_r + \lambda_{12} e_\theta + \lambda_{13} e_z,$$

$$\sigma_\theta = \lambda_{12} e_r + \lambda_{11} e_\theta + \lambda_{13} e_z,$$

$$\sigma_z = \lambda_{13} e_r + \lambda_{13} e_\theta + \lambda_{33} e_z,$$

$$\sigma_{rz} = 2\lambda_{55} e_{rz}, \tag{2}$$

where the elements $\lambda_{ij} = \lambda_{ij}(r, z)$ of the stiffness matrix are continuous and differentiable functions of the coordinates r and z;

the equations of motion

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u_r}{\partial t^2},$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = \rho \frac{\partial^2 u_z}{\partial t^2}, \tag{3}$$

where t is the time coordinate, $u_r(r, z, t)$ and $u_z(r, z, t)$ are the projections of the total displacement in the directions tangential to the coordinate lines r and z, respectively; e_r, e_θ, e_z are the relative linear strains along the coordinate lines; e_{rz} are the shear strains; σ_r, σ_θ, σ_z are the normal stresses; σ_{rz} is the shear stress; the density of the cylinder material $\rho(r, z)$ is a continuous function of the coordinates r and z.

The elements λ_{ij} of the stiffness matrix are determined in terms of the elements c_{ij} of the compliance matrix:

$$\lambda_{11} = (c_{11}c_{33} - c_{13}^2)/\Delta, \quad \lambda_{12} = (c_{13}^2 - c_{12}c_{33})/\Delta,$$

$$\lambda_{13} = (c_{12}c_{13} - c_{13}c_{11})/\Delta, \quad \lambda_{33} = (c_{11}^2 - c_{12}^2)/\Delta,$$

$$\lambda_{55} = 1/c_{55}, \quad \Delta = c_{11}(c_{11}c_{33} - c_{13}^2) - c_{12}(c_{12}c_{33} - c_{13}^2) + c_{13}(c_{12}c_{13} - c_{13}c_{11}), \tag{4}$$

where

$$c_{11} = \frac{1}{E_r}, \quad c_{12} = -\frac{\nu_{r\theta}}{E_\theta}, \quad c_{13} = -\frac{\nu_{rz}}{E_z}, \quad c_{33} = \frac{1}{E_z}, \quad c_{55} = -\frac{1}{G_{rz}}, \tag{5}$$

E_r, E_θ, E_z are the elastic moduli along the r-, θ-, z–axes, respectively; G_{rz} is the shear modulus; $\nu_{r\theta}$, ν_{rz}, $\nu_{\theta z}$ are Poisson's ratios.

If the lateral surfaces of the cylinder are free of stresses, the boundary conditions are

$$\sigma_r(R, z, t) = 0, \quad \sigma_{rz}(R, z, t) = 0. \tag{6}$$

Let all points of the cylinder harmonically vibrate with frequency ω, i.e., $\{u_r(r, z, t), u_z(r, z, t)\} = \{\hat{u}_r(r, z), \hat{u}_z(r, z)\}\, e^{i\omega t}$ (hereafter the sign "^" is omitted).

An analogous problem for waves propagating along the axis of a homogeneous isotropic solid cylinder was considered by the classical works by Pochhammer and Chree. They represented the displacements as follows [14]:

$$u_r = B\frac{\alpha}{\bar{\gamma}}((2\bar{\gamma}^2 - \bar{\omega}^2)J_1(\beta)J_1(\alpha r) - 2\bar{\gamma}^2 J_1(\alpha)J_1(\beta r))\exp(i(\bar{\gamma}z - \omega t)),$$

$$u_z = B\frac{\alpha}{\bar{\gamma}}((2\bar{\gamma}^2 - \bar{\omega}^2)J_1(\beta)J_0(\alpha r) + 2\alpha\beta J_1(\alpha)J_0(\beta r))\exp(i(\bar{\gamma}z - \omega t)). \tag{7}$$

Whence follows the characteristic Pochhammer–Chree equation:

$$(\bar{\omega}^2 - 2\bar{\gamma}^2)^2 J_0(\alpha)J_1(\beta) + 4\bar{\gamma}^2\alpha\beta J_0(\beta)J_1(\alpha) - 2\bar{\omega}^2\alpha J_1(\alpha)J_1(\beta) = 0, \tag{8}$$

where $\bar{\gamma}$ is the dimensionless wave constant; $\bar{\omega}=\omega R\sqrt{\rho/G}$ is the dimensionless frequency; $\alpha^2=\bar{\omega}^2/k^2-\bar{\gamma}^2, \beta^2=\bar{\omega}^2-\bar{\gamma}^2$, $k^2=2\frac{1-\nu}{1-2\nu}$, $J_n(t)$ are the nth-order Bessel functions of the first kind. If $\bar{\gamma}=m\pi/\bar{L}$, where $\bar{L}=L/R$, then this solution and the associated characteristic equation are valid for finite-length solid isotropic cylinders with hinged ends.

To analyze the vibrations of a transversely isotropic solid cylinder with $r=0$, we prescribe the conditions

$$u_r=0,\qquad \sigma_{rz}=0, \tag{9}$$

which agree with (7) in the special case of isotropic cylinder.

Consider the following boundary conditions at the ends $z=0$ and $z=L$:

$$\text{(i)}\qquad \sigma_r=0,\quad u_r=0\quad\text{or}\quad \frac{\partial u_z}{\partial z}=0,\quad u_r=0, \tag{10}$$

$$\text{(ii)}\qquad u_z=0,\quad \sigma_{rz}=0\quad\text{or}\quad u_z=0,\quad \frac{\partial u_r}{\partial z}=0, \tag{11}$$

$$\text{(iii)}\qquad u_r=0,\quad u_z=0. \tag{12}$$

The governing system of equations for displacements is

$$\frac{\partial^2 u_r}{\partial r^2}=\left(-\frac{1}{\lambda_{11}}\frac{\partial\lambda_{12}}{\partial r}\frac{1}{r}+\frac{\lambda_{22}}{\lambda_{11}}\frac{1}{r^2}-\frac{1}{\lambda_{11}}\rho\omega^2\right)u_r-\frac{1}{\lambda_{11}}\frac{\partial\lambda_{55}}{\partial z}\frac{\partial u_r}{\partial z}-\frac{\lambda_{55}}{\lambda_{11}}\frac{\partial^2 u_r}{\partial z^2}$$

$$-\left(\frac{1}{\lambda_{11}}\frac{\partial\lambda_{11}}{\partial r}+\frac{1}{r}\right)\frac{\partial u_r}{\partial r}-\left(\frac{1}{\lambda_{11}}\frac{\partial\lambda_{13}}{\partial r}-\frac{\lambda_{23}-\lambda_{13}}{\lambda_{11}}\frac{1}{r}\right)\frac{\partial u_z}{\partial z}-\frac{1}{\lambda_{11}}\frac{\partial\lambda_{55}}{\partial z}\frac{\partial u_z}{\partial r}-\frac{\lambda_{13}+\lambda_{55}}{\lambda_{11}}\frac{\partial^2 u_z}{\partial z\partial r},$$

$$\frac{\partial^2 u_z}{\partial r^2}=-\frac{1}{\lambda_{55}}\frac{\partial\lambda_{23}}{\partial z}\frac{u_r}{r}-\left(\frac{1}{\lambda_{55}}\frac{\partial\lambda_{55}}{\partial r}+\frac{\lambda_{23}}{\lambda_{55}}\frac{1}{r}+\frac{1}{r}\right)\frac{\partial u_r}{\partial z}-\left(1+\frac{\lambda_{13}}{\lambda_{55}}\right)\frac{\partial^2 u_r}{\partial r\partial z}$$

$$-\frac{1}{\lambda_{55}}\frac{\partial\lambda_{13}}{\partial z}\frac{\partial u_r}{\partial r}-\frac{1}{\lambda_{55}}\rho\omega^2 u_z-\frac{1}{\lambda_{55}}\frac{\partial\lambda_{33}}{\partial z}\frac{\partial u_z}{\partial z}-\frac{\lambda_{33}}{\lambda_{55}}\frac{\partial^2 u_z}{\partial z^2}-\left(\frac{1}{r}+\frac{1}{\lambda_{55}}\frac{\partial\lambda_{55}}{\partial r}\right)\frac{\partial u_z}{\partial r}. \tag{13}$$

When $r=0$, it is necessary to evaluate the indeterminate form considering that

$$\frac{u_r}{r}\to\frac{du_r}{dr}\quad\text{as}\quad r\to 0. \tag{14}$$

Thus, with $r=0$, system (13) becomes

$$\frac{\partial^2 u_r}{\partial r^2}=-\frac{1}{\lambda_{11}}\rho\omega^2 u_r-\left(\frac{1}{\lambda_{11}}\frac{\partial\lambda_{11}}{\partial r}+\frac{1}{\lambda_{11}}\frac{\partial\lambda_{12}}{\partial r}\right)\frac{\partial u_r}{\partial r}-\frac{1}{\lambda_{11}}\frac{\partial\lambda_{13}}{\partial r}\frac{\partial u_z}{\partial z},$$

$$\frac{\partial^2 u_z}{\partial r^2}=-\frac{2}{\lambda_{55}}\frac{\partial\lambda_{13}}{\partial z}\frac{\partial u_r}{\partial r}-\frac{1}{\lambda_{55}}\rho\omega^2 u_z-\frac{1}{\lambda_{55}}\frac{\partial\lambda_{33}}{\partial z}\frac{\partial u_z}{\partial z}-\frac{\lambda_{33}}{\lambda_{55}}\frac{\partial^2 u_z}{\partial z^2}. \tag{15}$$

Let us transform the system of equations (13) and (15) as

$$\frac{\partial^2 u_r}{\partial r^2}=a_{11}u_r+a_{12}\frac{\partial u_r}{\partial z}+a_{13}\frac{\partial^2 u_r}{\partial z^2}+a_{14}\frac{\partial u_r}{\partial r}+a_{15}\frac{\partial u_z}{\partial z}+a_{16}\frac{\partial u_z}{\partial r}+a_{17}\frac{\partial^2 u_z}{\partial r\partial z},$$

$$\frac{\partial^2 u_z}{\partial r^2} = a_{21} u_r + a_{22} \frac{\partial u_r}{\partial z} + a_{23} \frac{\partial u_r}{\partial r} + a_{24} \frac{\partial^2 u_r}{\partial r \partial z} + a_{25} u_z + a_{26} \frac{\partial u_z}{\partial z} + a_{27} \frac{\partial^2 u_z}{\partial z^2} + a_{28} \frac{\partial u_z}{\partial r}, \tag{16}$$

where the coefficients $a_{11} = a_{11}(r, z, \omega)$, $a_{24} = a_{24}(r, z, \omega)$, and $a_{kl} = a_{kl}(r, z)$, $(k, l) \in \{(k, l) | k = 1, 2; l = 1, \ldots, 7\} \backslash \{(1,1), (2,4)\} \cup \{(2,8)\}$ are determined as follows when $r \neq 0$:

$$a_{11} = -\frac{1}{\lambda_{11}} \frac{\partial \lambda_{12}}{\partial r} \frac{1}{r} + \frac{1}{r^2} - \frac{1}{\lambda_{11}} \rho \omega^2, \quad a_{12} = -\frac{1}{\lambda_{11}} \frac{\partial \lambda_{55}}{\partial z}, \quad a_{13} = -\frac{\lambda_{55}}{\lambda_{11}},$$

$$a_{14} = -\left(\frac{1}{\lambda_{11}} \frac{\partial \lambda_{11}}{\partial r} + \frac{1}{r} \right), \quad a_{15} = -\left(\frac{1}{\lambda_{11}} \frac{\partial \lambda_{13}}{\partial r} \right), \quad a_{16} = -\frac{1}{\lambda_{11}} \frac{\partial \lambda_{55}}{\partial z}, \quad a_{17} = -\frac{\lambda_{13} + \lambda_{55}}{\lambda_{11}},$$

$$a_{21} = -\frac{1}{\lambda_{55}} \frac{\partial \lambda_{13}}{\partial z} \frac{1}{r}, \quad a_{22} = -\left(\frac{1}{\lambda_{55}} \frac{\partial \lambda_{55}}{\partial r} + \frac{\lambda_{13}}{\lambda_{55}} \frac{1}{r} + \frac{1}{r} \right), \quad a_{23} = -\frac{1}{\lambda_{55}} \frac{\partial \lambda_{13}}{\partial z}, \quad a_{24} = -\left(1 + \frac{\lambda_{13}}{\lambda_{55}} \right),$$

$$a_{25} = -\frac{1}{\lambda_{55}} \rho \omega^2, \quad a_{26} = -\frac{1}{\lambda_{55}} \frac{\partial \lambda_{33}}{\partial z}, \quad a_{27} = -\frac{\lambda_{33}}{\lambda_{55}}, \quad a_{28} = -\left(\frac{1}{r} + \frac{\lambda_{13}}{\lambda_{55}} \frac{1}{r} \right). \tag{17}$$

If $r = 0$, then (9) with (14) yields

$$\left. \frac{\partial u_r}{\partial z} \right|_{r=0} = 0, \quad \left. \frac{\partial^2 u_r}{\partial z^2} \right|_{r=0} = 0, \quad \left. \frac{\partial u_z}{\partial r} \right|_{r=0} = 0, \quad \left. \frac{\partial^2 u_z}{\partial z \partial r} \right|_{r=0} = 0, \tag{18}$$

where the coefficients $a_{kl} = a_{kl}(r, z)$ of the nonzero terms in (16) are

$$a_{11} = -\frac{1}{\lambda_{11}} \omega^2, \quad a_{14} = -\frac{1}{\lambda_{11}} \frac{\partial}{\partial r} (\lambda_{11} + \lambda_{12}), \quad a_{15} = -\frac{1}{\lambda_{11}} \frac{\partial \lambda_{13}}{\partial r},$$

$$a_{23} = -\frac{2}{\lambda_{55}} \frac{\partial \lambda_{13}}{\partial z}, \quad a_{25} = -\frac{1}{\lambda_{55}} \rho \omega^2, \quad a_{26} = -\frac{1}{\lambda_{55}} \frac{\partial \lambda_{33}}{\partial z}, \quad a_{28} = -\frac{\lambda_{33}}{\lambda_{55}}.$$

The boundary conditions (6) on the lateral surface $r = R$ become

$$\lambda_{11} \frac{\partial u_r}{\partial r} + \lambda_{12} \frac{u_r}{r} + \lambda_{13} \frac{\partial u_z}{\partial z} = 0, \quad \lambda_{55} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) = 0, \tag{19}$$

and conditions (9) with (18) take the following form when $r = 0$:

$$u_r = 0, \quad \frac{\partial u_z}{\partial r} = 0. \tag{20}$$

2. Problem-Solving Method. Problem (16) with boundary conditions can be solved by spline collocation, discrete orthogonalization, and incremental search. If the spline-collocation method is applied, the unknown functions $u_r(r, z)$ and $u_z(r, z)$ become

$$u_r = \sum_{i=0}^{N} u_{ri}(r) \varphi_i^{(1)}(z), \quad u_z = \sum_{i=0}^{N} u_{zi}(r) \varphi_i^{(2)}(z), \tag{21}$$

where $u_{ri}(r)$ and $u_{zi}(r)$ are unknown functions of the variable r, $\varphi_i^{(j)}(z)$ ($j = 1, 2$, $i = 0, 1, \ldots, N$) are linear combinations of B-splines on the uniform mesh Δ: $0 = z_0 < z_1 < \ldots < z_N = L$, the boundary conditions at $z = 0$ and $z = L$ taken into account. Since system (16) includes no higher then second-order derivatives of unknown functions with respect to the coordinate z, we can use cubic spline functions:

$$B_3^i(z)=\frac{1}{6}\begin{cases}0, & -\infty<z<z_{i-2},\\ y^3, & z_{i-2}\le z<z_{i-1},\\ -3y^3+3y^2+3y+1, & z_{i-1}\le z<z,\\ 3y^3-6y^2+4, & z_i\le z<z_{i+1},\\ (1-y)^3, & z_{i+1}\le z<z_{i+2},\\ 0, & z_{i+2}\le z<\infty,\end{cases} \tag{22}$$

where $y=(z-z_k)/h_z$ on the interval $[z_k, z_{k+1}]$, $k=\overline{i-2, i+1}$, $i=\overline{-1, N+1}$, $h_z=z_{k+1}-z_k=\text{const}$.

The functions $\varphi_i^{(j)}(z)$ are set up as follows:

(a) if the unknown function (u_r or u_z) is equal to zero at $z=0$ and $z=L$, then

$$\varphi_0^{(j)}(z)=-4B_3^{-1}(z)+B_3^0(z),\qquad \varphi_1^{(j)}(z)=B_3^{-1}(z)-\frac{1}{2}B_3^0(z)+B_3^1(z),$$

$$\varphi_i^{(j)}(z)=B_3^i(z),\qquad (i=2, 3,\ldots N-2),$$

$$\varphi_{N-1}^{(j)}(z)=B_3^{N-1}(z)-\frac{1}{2}B_3^N(z)+B_3^{N+1}(z),\qquad \varphi_N^{(j)}(z)=-4B_3^{N+1}(z)+B_3^N(z), \tag{23}$$

(b) if the derivative of the unknown function with respect to z is equal to zero at $z=0$ and $z=L$, then

$$\varphi_0^{(j)}(z)=B_3^0(z),\qquad \varphi_1^{(j)}(z)=B_3^{-1}(z)-\frac{1}{2}B_3^0(z)+B_3^1(z),$$

$$\varphi_{ji}(z)=B_3^i(z),\qquad (i=2, 3,\ldots, N-2),$$

$$\varphi_{N-1}^{(j)}(z)=B_3^{N-1}(z)-\frac{1}{2}B_3^N(z)+B_3^{N+1}(z),\qquad \varphi_N^{(j)}(z)=B_3^N(z), \tag{24}$$

(c) if the unknown function is equal to zero at $z=0$ and its derivative with respect to z is equal to zero at $z=L$, then

$$\varphi_0^{(j)}(z)=-4B_3^{-1}(z)+B_3^0(z),\qquad \varphi_1^{(j)}(z)=B_3^{-1}(z)-\frac{1}{2}B_3^0(z)+B_3^1(z),$$

$$\varphi_i^{(j)}(z)=B_3^i(z)\qquad (i=2, 3,\ldots, N-2),$$

$$\varphi_{N-1}^{(j)}(z)=B_3^{N-1}(z)-\frac{1}{2}B_3^N(z)+B_3^{N+1}(z),\qquad \varphi_N^{(j)}(z)=B_3^N(z). \tag{25}$$

Substituting (21) into Eqs. (16), we require them to be satisfied at the collocation points $\xi_k\in[0, L]$, $k=\overline{0, N}$. Let the number of mesh nodes (with z_0) be even, i.e., $N=2n+1$ ($n\ge 3$). The choice of the collocation points $\xi_{2i}\in[z_{2i}, z_{2i+1}]$, $\xi_{2i+1}\in[z_{2i}, z_{2i+1}]$ in the form $\xi_{2i}=z_{2i}+s_1h_z$ and $\xi_{2i+1}=z_{2i}+s_2h_z$ ($i=0, 1, 2, \ldots, n$), where s_1 and s_2 are the roots of second-order Legandre polynomials on the segment [0, 1] ($s_1=1/2-\sqrt{3}/6$, $s_2=1/2+\sqrt{3}/6$), is optimal and considerably improves the accuracy of approximation. The number of collocation points $\overline{N}=N+1$. As a result, we arrive at a system of $4(N+1)$ linear differential equations for the functions $u_{ri}, \tilde{u}_{ri}, u_{zi}, \tilde{u}_{zi}$ ($i=0,\ldots, N$), where $u'_{ri}=\tilde{u}_{ri}, u'_{zi}=\tilde{u}_{zi}$. Introducing the notation

$$\Phi_j=[\varphi_i^{(j)}(\xi_k)],\qquad k, i=0,\ldots, N,\qquad j=1, 2,$$

$$\bar{u}_r = \{u_{r0}, u_{r1}, \ldots, u_{rN}\}^{\mathrm{T}}, \quad \bar{\tilde{u}}_r = \{\tilde{u}_{r0}, \tilde{u}_{r1}, \ldots, \tilde{u}_{rN}\}^{\mathrm{T}}, \quad \bar{u}_z = \{u_{z0}, u_{z1}, \ldots, u_{zN}\}^{\mathrm{T}},$$

$$\bar{\tilde{u}}_z = \{\tilde{u}_{z0}, \tilde{u}_{z1}, \ldots, \tilde{u}_{zN}\}^{\mathrm{T}}, \quad \bar{a}_{kl}^{\mathrm{T}} = \{a_{kl}(r,\xi_0), a_{kl}(r,\xi_1), \ldots, a_{kl}(r,\xi_N)\},$$

$$(k,l) \in \{(k,l) | k = 1, 2, l = 1, \ldots, 7\} \setminus \{(1,1),(2,4)\} \cup \{(2,8)\}$$

$$\bar{a}_{11}^{\mathrm{T}} = \{a_{11}(r,\xi_0,\omega), a_{11}(r,\xi_1,\omega), \ldots, a_{11}(r,\xi_N,\omega)\},$$

$$\bar{a}_{24}^{\mathrm{T}} = \{a_{24}(r,\xi_0,\omega), a_{24}(r,\xi_1,\omega), \ldots, a_{24}(r,\xi_N,\omega)\}, \tag{26}$$

the matrix $A = [a_{ij}]$ $(i, j = 0, \ldots, N)$, and the vector $\bar{c} = \{c_0, c_1, \ldots, c_N\}^{\mathrm{T}}$ and denoting the matrix $[c_i a_{ij}]$ by $\bar{c} * A$, we reduce the system of ordinary differential equations for $u_{ri}, \tilde{u}_{ri}, u_{zi}, \tilde{u}_{zi}$ to the form

$$\frac{d\bar{u}_r}{dr} = \bar{\tilde{u}}_r, \quad \frac{d\bar{u}_z}{dr} = \bar{\tilde{u}}_z,$$

$$\frac{d\bar{\tilde{u}}_r}{dr} = \Phi_1^{-1}(\bar{a}_{11} * \Phi_1 + \bar{a}_{12} * \Phi_1' + \bar{a}_{13} * \Phi_1'')\bar{u}_r + \Phi_1^{-1}(\bar{a}_{14} * \Phi_1)\bar{\tilde{u}}_r + \Phi_1^{-1}(\bar{a}_{15} * \Phi_2')\bar{u}_z + \Phi_1^{-1}(\bar{a}_{16} * \Phi_2 + \bar{a}_{17} * \Phi_2')\bar{\tilde{u}}_z,$$

$$\frac{d\bar{\tilde{u}}_z}{dy} = \Phi_2^{-1}(\bar{a}_{21}\Phi_1 + \bar{a}_{22}\Phi_1')\bar{u}_r + \Phi_2^{-1}(\bar{a}_{23} * \Phi_1')\bar{\tilde{u}}_r + \Phi_2^{-1}(\bar{a}_{24} * \Phi_2 + \bar{a}_{25} * \Phi_2' + \bar{a}_{26} * \Phi_2'')\bar{u}_z + \Phi_2^{-1}(\bar{a}_{27} * \Phi_2)\bar{\tilde{u}}_z. \tag{27}$$

Let us represent system (27) as

$$\frac{d\bar{Y}}{dr} = A(r,\omega)\bar{Y} \quad (0 \le r \le R), \tag{28}$$

where $\bar{Y} = \{u_{r_0}, \ldots u_{rN}, \tilde{u}_{r0}, \ldots, \tilde{u}_{rN}, u_{z0}, \ldots, u_{zN}, \tilde{u}_{zo}, \ldots, \tilde{u}_{zN}\}^{\mathrm{T}}$ is a vector function depending on *r*, $A(r,\omega)$ is a $4(N+1) \times 4(N+1)$ matrix.

The boundary conditions at $r = R$ for this system of ordinary differential equations can be represented as

$$\bar{\lambda}_{11}\Phi_1\bar{\tilde{u}}_r + \bar{\lambda}_{12}\Phi_1\frac{1}{r}\bar{u}_r + \bar{\lambda}_{13}\Phi_2'\bar{u}_z = \bar{0},$$

$$\bar{\lambda}_{55}\Phi_1'\bar{u}_r + \bar{\lambda}_{55}\Phi_2\bar{\tilde{u}}_z = \bar{0}, \tag{29}$$

where

$$\bar{\lambda}_{1l}^{\mathrm{T}} = \{\lambda_{1l}(R,\xi_0), \lambda_{1l}(R,\xi_1), \ldots, \lambda_{1l}(R,\xi_N)\} \quad (l = 1, 2, 3),$$

$$\bar{\lambda}_{55}^{\mathrm{T}} = \{\lambda_{55}(R,\xi_0), \lambda_{55}(R,\xi_1), \ldots, \lambda_{55}(R,\xi_N)\},$$

and the boundary conditions at $r = 0$ as

$$\bar{\tilde{u}}_z = \bar{0}, \quad \bar{u}_r = \bar{0}, \tag{30}$$

or

$$B_1\bar{Y}(0) = \bar{0}, \quad B_2\bar{Y}(R) = \bar{0}, \tag{31}$$

where B_1 and B_2 are $2(N+1) \times 4(N+1)$ matrices.

The eigenvalue boundary-value problem (28), (31) can be solved by the discrete-orthogonalization method in combination with incremental search [2].

TABLE 1

$\bar{\omega}$	Hinged ends			Clamped ends, SCM
	m	Eq. (8)	SCM	
$\bar{\omega}_1$	1	1.24699	1.24699	1.2842
$\bar{\omega}_2$	3	2.99449	2.99448	3.1668
$\bar{\omega}_3$	1	3.58383	3.58383	3.7110
$\bar{\omega}_4$	5	4.02260	4.02272	4.2534
$\bar{\omega}_5$	3	4.44840	4.44840	4.4823
$\bar{\omega}_6$	1	4.47727	4.47727	4.8424

TABLE 2

L	Hollow cylinder, SCM					Solid cylinder	
	$R_1 = 0.3$	$R_1 = 0.2$	$R_1 = 0.1$	$R_1 = 0.01$	$R_1 = 0.001$	SCM	Eq. (8)
8	0.6305	0.6308	0.6309	0.6308	0.6309	0.6309	0.6309
4	1.2437	1.2455	1.2466	1.2469	1.2469	1.2470	1.2470
2	2.2279	2.2854	2.3168	2.3260	2.3260	2.3265	2.3263

3. Analysis of the Results. To estimate the accuracy of the approach, we will compare (Table 1) the dimensionless frequencies $\bar{\omega} = \omega R\sqrt{\rho / G}$ for a hinged solid isotropic cylinder with $\nu = 0.3$, $R = 1$, $L = 4$ obtained by the spline-collocation method (SCM) with $N = 40$ collocation points and from the Pochhammer–Gree equation (8) using Bessel functions for waves in an infinite-length cylinder (for $\alpha^2 < 0$ or $\beta^2 < 0$, we used the formula $J_n(\pm ix) = (\pm i)^n I_n(x)$, where $I_n(x)$ is the modified Bessel function). We considered even vibration modes (u_z and u_r are even and odd functions of the variable z, respectively) [14]. The wave constant $\bar{\gamma} = m\pi / L$ (m is the odd longitudinal wave number).

As can be seen, the frequencies m = 1 and 3 are in good agreement within the error. The maximum difference (about 0.2%) is observed for m = 5 and can be decreased by increasing the number of collocation points. Table 1 also presents frequencies for a solid cylinder with clamped ends. These frequencies are higher than those for the cylinder with hinged ends.

Table 2 compares the frequencies $\bar{\omega} = \omega R\sqrt{\rho / G}$ for solid cylinders ($\nu = 0.3$, R = 1, L = 2, 4, 8) calculated by our approach and the frequencies $\bar{\omega} = \omega H\sqrt{\rho / G}$ obtained by the SCM for hinged hollow cylinders with outer radius R = 1 and variable inner radius R_1 and length L.

As the inner radius decreases, the frequencies of the hollow cylinder approach those for the solid cylinder with the boundary conditions at $r = 0$.

We also studied how the inhomogeneity of the material influences the frequencies of a solid cylinder with $\nu = 0.3$, $R = 1$, L = 4 and constant density. Young's modulus varies as

$$E(r, z) = \left(\alpha\left(\frac{6z^2}{L^2} - \frac{6z}{L} + 1\right) + 1\right)\left(\beta\left(\frac{r}{R} - \frac{1}{2}\right) + 1\right)E_0. \tag{31}$$

TABLE 3

β	Boundary conditions	$\overline{\omega}$	α = 0	α = 0.3	α = 0.6
0	H–H	$\overline{\omega}_1$	1.2469	1.2935	1.3218
		$\overline{\omega}_2$	2.9944	2.9747	2.8972
		$\overline{\omega}_3$	3.5838	3.5615	3.4515
	C–C	$\overline{\omega}_1$	1.2842	1.3259	1.3493
		$\overline{\omega}_2$	3.1668	3.0908	2.9764
		$\overline{\omega}_3$	3.7111	3.6451	3.5061
0.4	H–H	$\overline{\omega}_1$	1.2836	1.3314	1.3604
		$\overline{\omega}_2$	2.9847	2.9651	2.8903
		$\overline{\omega}_3$	3.6054	3.5918	3.4880
	C–C	$\overline{\omega}_1$	1.3230	1.3655	1.3893
		$\overline{\omega}_2$	3.1659	3.0899	2.9767
		$\overline{\omega}_3$	3.7396	3.6800	3.5447

TABLE 4

d	$\overline{\omega}$	$L = 4$	$L = 6$	$L = 8$	$L = 10$
5	$\overline{\omega}_1$	0.8004	0.5303	0.3964	0.3164
	$\overline{\omega}_2$	2.3367	1.5723	1.1814	0.9454
	$\overline{\omega}_3$	3.6071	2.5454	1.9399	1.5614
10	$\overline{\omega}_1$	0.8078	0.5329	0.3975	0.3168
	$\overline{\omega}_2$	2.3655	1.5813	1.1850	0.9468
	$\overline{\omega}_3$	3.7572	2.5756	1.9498	1.5653
15	$\overline{\omega}_1$	0.8185	0.5390	0.4018	0.3204
	$\overline{\omega}_2$	2.4026	1.6027	1.2011	0.9601
	$\overline{\omega}_3$	3.8567	2.6267	1.9874	1.5965

The average Young's modulus remains constant for all possible values of α and β.

Table 3 summarizes the frequencies $\overline{\omega} = \omega R\sqrt{\rho / G_0}$ of a cylinder with hinged (H–H) and clamped (C–C) ends for $\beta = 0$, 0.4 and different values of α (even modes are considered). When $\alpha = 0$ and $\beta = 0$, the cylinder is isotropic.

As the parameter α is increased and β is kept constant, the first frequency increases (by 4.83%), while the second and third frequencies decrease (by 5.6%). As the parameter β is inreased and α is kept constant, the first and third frequencies increase, while the second frequency decreases.

We also studied the vibrations of transversely isotropic solid cylinders with the axis of elastic symmetry aligned with the *Oz*-axis. The cylinders are made of materials with the following mechanical parameters: $E = dE_0$, $E' = E_0$, $\nu = 0.3$, $\nu' = 0.15$, $G = dE_0 / 2.6$, $G' = E_0$, where E and E' are the elastic moduli in the isotropy plane and in the perpendicular direction, respectively; ν and ν' are the respective Poisson's ratios; and G and G' are the respective shear moduli.

Table 4 collects even [14] frequencies $\overline{\omega} = \omega R\sqrt{\rho / G'}$ of a cylinder with clamped ends for $L = 4, 6, 8, 10$ and $d = E / G' =$ 5, 10, 15 (anisotropy parameter).

As is seen, the frequency decreases with increasing length of the cylinder and increases with increasing *d*. As *d* increases, the first and second frequencies of the short cylinder ($L = 4$), weakly vary (by 1–2%), while the third frequency changes by 7%. This effect is less pronounced for longer cylinders.

Conclusions. We have solved, based on the three-dimensional theory of elasticity and a numerical analytic approach, the problem of the free vibrations of a solid cylinder with different boundary conditions at the ends. The starting partial differential equations of elasticity have been reduced, by using spline-approximation and spline-collocation, to a eigenvalue problem for systems of ordinary differential equations of high order with respect to the radial coordinate. The problem has been solved by discrete orthogonalization and incremental search. The frequencies for transversely isotropic and isotropic inhomogeneous cylinders with boundary conditions of several types have been calculated.

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