International Applied Mechanics, Vol. 46, No. 3, 2010

GENERAL LOVE SOLUTION IN THE LINEAR ISOTROPIC INHOMOGENEOUS THEORY OF RADIUS-DEPENDENT ELASTICITY

M. Yu. Kashtalyan[1] and J. J. Rushchitsky[2]

A general Love solution for the inhomogeneous linear isotropic theory of elasticity with the elastic constants dependent on the coordinate *r* is proposed. The axisymmetric case is analyzed and cylindrical coordinates are used. This is the fourth publication in the series on general solutions in the inhomogeneous theory of elasticity. The new results are promising for the modern theory of functionally graded materials. The key steps of deriving the Love solutions are described for further use of the derivation procedure. The procedure of generalizing the Love solutions to the inhomogeneous theory of elasticity is detailed. The results obtained are discussed

Keywords: linear inhomogeneous isotropic elasticity, radially variable elastic parameters, general Love solution, functionally graded material

Introduction. Cylindrical objects are surprisingly often found in nature, engineering, and even in the home. A classical natural example is the trunk of a tree (for example, bamboo). A bolt is another example from engineering; people always use something cylindrical in their everyday life, from an ordinary stick, a pencil, a water pipe to a rolling pin. When is service, all these objects are subject to various mechanical loads—they are stretched, compressed, bent, twisted, cut, etc.

This is why the mechanics of materials and structures has always put emphasis on cylindrical bodies. Thousands of scientific publications study their mechanical behavior. As a rule, solid or hollow cylinders are assumed homogeneous in mechanical properties. According to real observations, however, cylindrical objects are highly inhomogeneous. Such inhomogeneity is often manifested as radial variation in density and other mechanical properties. It is appropriate to mention bamboo because it is denser and stronger on the outside surface, and its density and mechanical characteristics such as tensile and shear moduli decrease with distance from this surface. Not only natural materials are inhomogeneous, but technologically it seems to make sense to introduce artificially inhomogeneity into materials. The recently formulated and actively developing theory of functionally gradient materials (FGMs) focuses on artificial inhomogeneous materials and is the main consumer of achievements in the analysis of inhomogeneous materials.

Remark 1. The following significant publications confirm that the FGM theory is successful and relevant: the pioneering studies [24–26, 32], the recent review [6] in the world's best survey journal, the two comprehensive monographs [30, 35], the typical papers [10, 27, 28, 34] published in 2008.

While the mechanics of homogeneous bodies may be considered a well-developed science, the mechanics of inhomogeneous bodies abounds in poorly studied fragments. This is especially true of the analytic mechanics of inhomogeneous bodies that develops rigorous mathematical models leading to differential or integral equations (which are solved by analytic methods).

It is one of the fragments mentioned above that is examined here. The present paper addresses a general Love-type solution in the axisymmetric inhomogeneous isotropic theory of radius-dependent elasticity.

[1]Centre for Micro- and Nanomechanics, University of Aberdeen, Scotland, AB24 3UE Aberdeen, Great Britain, e-mail: m.kashtalyan@abdn.ac.uk. [2]S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterov St., Kyiv, Ukraine 03057, e-mail: rushch@inmech.kiev.ua. Translated from Prikladnaya Mekhanika, Vol. 46, No. 3, pp. 3–13, March 2010. Original article submitted February 15, 2009.

1063-7095/10/4603-0245

It should be noted that inhomogeneity due to vertically varying mechanical properties is addressed in relatively many publications concerned with general solutions. The radius dependence of the elastic parameters is yet to be studied adequately. The general Love-type solution proposed here has not been studied at all.

Remark 2. The present paper is the fourth one in the series of publications [1, 2, 23] on general solutions in the inhomogeneous theory of elasticity. We believe that this series is closely related to the two previous series of publications in which we participated (one on the FGM theory [19–22] and the other on the theory of nanocomposites [7, 8, 10–18]) and hope that these results will be even more closely related in the future.

1. Love Solution in the Classical Linear Isotropic Theory of Elasticity. Let us outline the standard procedure of introducing the Love function. To this end, we write the necessary relations in circular cylindrical coordinates for the axisymmetric problem of the linear isotropic theory of elasticity.

Remark 3. The Love solution is one of the oldest and has successfully been applied to axisymmetric problems in the classical theory of elasticity. It was for the first time used in Love's classical book [29, Sec. 188]. More recent books on the theory of elasticity define the Love solution differently: some introduce it directly, assuming a relationship between the components of the displacement vector and the Love function (see, e.g., [3, Sec. 12]); some start with the Helmholtz solution and only then introduce the Love function (see, e.g., [31, Sec. 5.4]); some derive the Love solution from the Boussinesq–Galerkin solution, assuming that the first two components of the Boussinesq–Galerkin function are zero (see, e.g., [18, Sec. 5.1.4] and [5, Sec. 1.10]). Sometimes, the Love solution is expressed in terms of two harmonic functions and then reduced to one biharmonic Love function (see [3, Sec. 12]).

The axisymmetric state will be described in circular cylindrical coordinates (r, ϑ, z) where the z-axis is the axis of symmetry and the coordinate ϑ is absent. Then the displacement vector has only two nonzero components $u = (u_r, u_\vartheta = 0, u_z)$;

the strain tensor has four nonzero components,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_{\vartheta\vartheta} = \frac{u_r}{r}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \qquad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \qquad \varepsilon_{r\vartheta} = 0, \qquad \varepsilon_{\vartheta z} = 0,$$

the expression for the dilatation also becomes simpler:

$$e = \varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{zz} = \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z},$$

the linear stress tensor has four nonzero components,

$$\sigma = (\sigma_{rr}, \sigma_{\vartheta\vartheta}, \sigma_{zz}, \sigma_{rz}, \sigma_{r\vartheta} = 0, \sigma_{\vartheta z} = 0),$$

the constitutive equations are

$$\sigma_{rr} = \lambda e + 2\mu\varepsilon_{rr}, \qquad \sigma_{\vartheta\vartheta} = \lambda e + 2\mu\varepsilon_{\vartheta\vartheta}, \qquad \sigma_{zz} = \lambda e + 2\mu\varepsilon_{zz}, \qquad \sigma_{rz} = 2\mu\varepsilon_{rz}, \tag{1}$$

the system of equilibrium equations (without body forces) consists of two equations:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\vartheta\vartheta}}{r} = 0, \qquad \frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0. \tag{2}$$

The system of Lame equations also consists of two equations:

$$\mu\left(\Delta - \frac{1}{r^2}\right)u_r + (\lambda+\mu)\frac{\partial e}{\partial r} = 0, \qquad \mu\Delta u_z + (\lambda+\mu)\frac{\partial e}{\partial z} = 0, \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{3}$$

The same equations can be written without dilatation:

$$(\lambda+2\mu)\left(\Delta - \frac{1}{r^2}\right)u_r - (\lambda+\mu)\frac{\partial^2 u_r}{\partial z^2} + (\lambda+\mu)\frac{\partial^2 u_z}{\partial r\partial z} = 0, \tag{4}$$

$$\mu\Delta u_z + (\lambda+\mu)\frac{\partial^2 u_z}{\partial z^2} + (\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) = 0. \tag{5}$$

The way from the system of equations (4), (5) for two unknown functions u_r, u_z to the Love function takes many steps. First, the functions u_r and u_z should be replaced by new functions $R(r, z)$ and $Z(r, z)$,

$$u_r = \frac{\partial R}{\partial r}, \quad u_z = Z. \tag{6}$$

Remark 4. This procedure is just interesting for the homogeneous theory of elasticity, but it should be followed carefully in the inhomogeneous theory of elasticity, which is why we present here the fragments of the analysis that are usually omitted in the homogeneous theory of elasticity.

Next, (6) should be substituted into Eq. (4):

$$\mu\left(\Delta - \frac{1}{r^2}\right)\frac{\partial R}{\partial r} + (\lambda+\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{\partial Z}{\partial z}\right) = 0.$$

The next step is to integrate this equation over *r*, considering the two equalities

$$\left(\Delta - \frac{1}{r^2}\right)\frac{\partial R}{\partial r} = \left(\frac{\partial^2}{\partial r^2}\frac{\partial R}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial R}{\partial r} + \frac{\partial^2}{\partial z^2}\frac{\partial R}{\partial r} - \frac{1}{r^2}\frac{\partial R}{\partial r}\right) = \frac{\partial}{\partial r}\Delta R,$$

$$e = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{\partial Z}{\partial z} \tag{7}$$

and equate the constant of integration to zero without loss of generality. As result, the equation

$$\frac{\partial}{\partial r}\left\{\mu\Delta R + (\lambda+\mu)\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{\partial Z}{\partial z}\right)\right\} = 0$$

yields the equation

$$\left[(\lambda+2\mu)\Delta - (\lambda+\mu)\frac{\partial^2}{\partial z^2}\right]R + (\lambda+\mu)\frac{\partial Z}{\partial z} = 0. \tag{8}$$

Thus, Eq. (8) is derived by transforming and integrating Eq. (4).

Remark 5. It should be noted that general solutions proposed by many authors are not in fact general, but rather they are partial to an extent. To be general, the new equations should be made completely adequate to the Lame equations. For example, when the additional functions due to integration are ignored, as done in deriving Eq. (8), the resulting solution loses a piece of generality.

The second equation (Eq. (5)) is transformed similarly:

$$\mu\Delta u_z + (\lambda+\mu)\frac{\partial e}{\partial z} = 0 \to \mu\Delta Z + (\lambda+\mu)\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{\partial Z}{\partial z}\right] = 0 \to$$

$$\to (\lambda+\mu)\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)\right] + \left[\mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial z^2}\right]Z = 0 \to$$

$$\to (\lambda+\mu)\frac{\partial}{\partial z}\left[\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \frac{\partial^2 R}{\partial z^2} - \frac{\partial^2 R}{\partial z^2}\right] + \left[\mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial z^2}\right]Z = 0 \to$$

$$\to (\lambda+\mu)\frac{\partial}{\partial z}\left[\Delta-\frac{\partial^2}{\partial z^2}\right]R+\left[\mu\Delta+(\lambda+\mu)\frac{\partial^2}{\partial z^2}\right]Z=0. \tag{9}$$

This step yields a new system of equations (8), (9), which is equivalent to system (4), (5), for the functions $R(r, z)$ and $Z(r, z)$.

Next, it is necessary to introduce the Love function $\chi(r, z)$ by the formulas

$$R=-\frac{\partial}{\partial z}\chi, \quad Z=\left(\frac{\lambda+2\mu}{\lambda+\mu}\Delta-\frac{\partial^2}{\partial z^2}\right)\chi. \tag{10}$$

Conceptually, the Love function satisfies Eq. (8) identically as follows:
– the operator before R is used to represent Z in terms of the new function χ;
– the operator before Z is used to represent R in terms of the new function χ;

$$\left[(\lambda+2\mu)\Delta-(\lambda+\mu)\frac{\partial^2}{\partial z^2}\right]R+(\lambda+\mu)\frac{\partial Z}{\partial z}=0\to\left(\frac{\lambda+2\mu}{\lambda+\mu}\Delta-\frac{\partial^2}{\partial z^2}\right)R+\frac{\partial Z}{\partial z}=0\to$$

$$\to\left(\frac{\lambda+2\mu}{\lambda+\mu}\Delta-\frac{\partial^2}{\partial z^2}\right)\left[R=-\frac{\partial}{\partial z}\chi\right]+\frac{\partial}{\partial z}\left[Z=\left(\frac{\lambda+2\mu}{\lambda+\mu}\Delta-\frac{\partial^2}{\partial z^2}\right)\chi\right]\equiv 0.$$

To derive the equation for the Love function $\chi(r, z)$, it is necessary to substitute (10) into Eq. (9):

$$(\lambda+\mu)\frac{\partial}{\partial z}\left[\Delta-\frac{\partial^2}{\partial z^2}\right]R+\left[\mu\Delta+(\lambda+\mu)\frac{\partial^2}{\partial z^2}\right]Z=0\to$$

$$\to-(\lambda+\mu)\frac{\partial}{\partial z}\left[\Delta-\frac{\partial^2}{\partial z^2}\right]\frac{\partial}{\partial z}\chi+\left[\mu\Delta+(\lambda+\mu)\frac{\partial^2}{\partial z^2}\right]\left(\frac{\lambda+2\mu}{\lambda+\mu}\Delta-\frac{\partial^2}{\partial z^2}\right)\chi=0\to$$

$$\to-\frac{\partial^2}{\partial z^2}\left[\cancel{\Delta}-\cancel{\frac{\partial^2}{\partial z^2}}\right]\chi+\left(\frac{\mu}{\lambda+\mu}\frac{\lambda+2\mu}{\lambda+\mu}\Delta\Delta+\cancel{\frac{\lambda+2\mu}{\lambda+\mu}\Delta\frac{\partial^2}{\partial z^2}-\frac{\mu}{\lambda+\mu}\Delta\frac{\partial^2}{\partial z^2}}-\cancel{\frac{\partial^4}{\partial z^4}}\right)\chi=0\to$$

$$\to\frac{\mu}{\lambda+\mu}\frac{\lambda+2\mu}{\lambda+\mu}\Delta\Delta\chi=0\to\Delta\Delta\chi=0. \tag{11}$$

The crossed terms are canceled in pairs.

We can obtain the general Love solution in terms of one biharmonic function $\chi(r, z)$ for the displacement components:

$$u_r(r,z)=-\frac{\partial^2}{\partial r\partial z}\chi(r,z), \quad u_z(r,z)=\left(\frac{\lambda+2\mu}{\lambda+\mu}\Delta-\frac{\partial^2}{\partial z^2}\right)\chi(r,z). \tag{12}$$

As is seen from (11), the Love function does not depend on the elastic constants, the radial displacement does not depend on them too by the first equation in (12), and the axial displacement depends on Poisson's ratio by the second equation in (12):

$$u_z(r,z)=\left(2(1-\nu)\Delta-\frac{\partial^2}{\partial z^2}\right)\chi(r,z).$$

The strains are also dependent on Poisson's ratio alone:

$$e=\varepsilon_{rr}+\varepsilon_{\vartheta\vartheta}+\varepsilon_{zz}=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z}=\frac{\partial}{\partial z}\left\{-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right\}\chi+\frac{\partial}{\partial z}\left(\frac{\lambda+2\mu}{\lambda+\mu}\Delta-\frac{\partial^2}{\partial z^2}\right)\chi$$

$$= \frac{\partial}{\partial z}\left\{ -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) + \left(\frac{\lambda + 2\mu}{\lambda + \mu}\Delta - \frac{\partial^2}{\partial z^2} \right) \right\}\chi$$

$$= \frac{\partial}{\partial z}\left\{ -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)\chi + \frac{\lambda + 2\mu}{\lambda + \mu}\Delta\chi \right\} = \frac{\mu}{\lambda + \mu}\Delta\frac{\partial \chi}{\partial z} \to e = \frac{\mu}{\lambda + \mu}\Delta\frac{\partial \chi}{\partial z}. \tag{13}$$

Thus, only the stresses depend on all the elastic constants:

$$\sigma_{rr} = \lambda e + 2\mu\varepsilon_{rr} = \frac{\lambda\mu}{\lambda + \mu}\Delta\frac{\partial \chi}{\partial z} - 2\mu\frac{\partial}{\partial r}\frac{\partial^2}{\partial r \partial z}\chi = 2\mu\frac{\partial}{\partial z}\left(\nu\Delta\chi - \frac{\partial^2 \chi}{\partial r^2} \right),$$

$$\sigma_{\vartheta\vartheta} = \lambda e + 2\mu\varepsilon_{\vartheta\vartheta} = \frac{\lambda\mu}{\lambda + \mu}\Delta\frac{\partial \chi}{\partial z} - 2\mu\frac{1}{r}\frac{\partial^2}{\partial r \partial z}\chi = 2\mu\frac{\partial}{\partial z}\left(\nu\Delta\chi - \frac{1}{r}\frac{\partial \chi}{\partial r} \right),$$

$$\sigma_{zz} = \lambda e + 2\mu\varepsilon_{zz} = \frac{\lambda\mu}{\lambda + \mu}\Delta\frac{\partial \chi}{\partial z} + 2\mu\frac{\partial}{\partial z}\left(\frac{\lambda + 2\mu}{\lambda + \mu}\Delta - \frac{\partial^2}{\partial z^2} \right)\chi = 2\mu\frac{\partial}{\partial z}\left[(2 - \nu)\Delta - \frac{\partial^2}{\partial z^2} \right]\chi,$$

$$\sigma_{rz} = 2\mu\varepsilon_{rz} = \mu\left[-\frac{\partial}{\partial z}\frac{\partial^2}{\partial r \partial z}\chi + \frac{\partial}{\partial r}\left(\frac{\lambda + 2\mu}{\lambda + \mu}\Delta - \frac{\partial^2}{\partial z^2} \right)\chi \right] = 2\mu\frac{\partial}{\partial r}\left((1 - \nu)\Delta\chi - \frac{\partial^2 \chi}{\partial z^2} \right). \tag{14}$$

Let us also outline a procedure of deriving a Love pseudosolution. It differs from the classical procedure in that Eq. (4) is not integrated because this integration is impossible in the inhomogeneous theory of elasticity.

Let us consider Eq. (4) that has not been integrated:

$$\left[(\lambda + 2\mu)\left(\Delta - \frac{1}{r^2} \right) - (\lambda + \mu)\frac{\partial^2}{\partial z^2} \right]\frac{\partial R}{\partial r} + (\lambda + \mu)\frac{\partial^2}{\partial r \partial z}Z = 0. \tag{15}$$

To follow the classical procedure, we will introduce a new (Love) function as follows:

$$R = -(\lambda + \mu)\frac{\partial^2 \chi}{\partial r \partial z}, \qquad Z = \left[(\lambda + 2\mu)\left(\Delta - \frac{1}{r^2} \right) - (\lambda + \mu)\frac{\partial^2}{\partial z^2} \right]\frac{\partial \chi}{\partial r}. \tag{16}$$

Remark 6. Formulas (16) differ from the classical formulas (10) even if we denote $\partial\chi / \partial r \equiv \hat{\chi}$.

Next, we substitute (16) into Eq. (9) to obtain an equation for the Love pseudofunction $\hat{\chi} \equiv \partial\chi / \partial r$:

$$\Delta\Delta\hat{\chi} - \frac{1}{r^2}\left[\Delta\hat{\chi} + \frac{\lambda + \mu}{\lambda + 2\mu}\frac{\partial^2 \hat{\chi}}{\partial z^2} \right] = 0. \tag{17}$$

This pseudofunction is not biharmonic.

2. Love Solution in the Inhomogeneous Linear Isotropic Theory of Radius-Dependent Elasticity. Let us solve the problem of the previous section in the case where the Lame constants λ and μ are at least two times differentiable functions of the coordinate r:

$$\lambda(r) = \lambda_h l(r), \quad \mu(r) = \mu_h m(r) \quad \left(\lambda_h, \mu_h = \text{const}, \nu_h = \frac{\lambda_h}{2(\lambda_h + \mu_h)} \right).$$

The first fundamental difference from the homogeneous theory of elasticity is in the constitutive equations:

$$\sigma_{rr}(r, z) = \lambda(r)e(r, z) + 2\mu(r)\varepsilon_{rr}(r, z), \qquad \sigma_{\vartheta\vartheta}(r, z) = \lambda(r)e(r, z) + 2\mu(r)\varepsilon_{\vartheta\vartheta}(r, z),$$

$$\sigma_{zz}(r, z) = \lambda(r)e(r, z) + 2\mu(r)\varepsilon_{zz}(r, z), \qquad \sigma_{rz}(r, z) = 2\mu(r)\varepsilon_{rz}(r, z). \tag{18}$$

It is dependences (18) that complicate the procedure of going over from the equilibrium equations (2) to the system of Lame-type equations of the inhomogeneous theory of elasticity. This system also consists of two equations:

$$(\lambda(r)+2\mu(r))\left(\Delta u_r-\frac{u_r}{r^2}\right)-(\lambda(r)+\mu(r))\frac{\partial^2 u_r}{\partial z^2}+\underline{\lambda'(r)\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)}+\underline{2\mu'(r)\frac{\partial u_r}{\partial r}}$$

$$+(\lambda(r)+\mu(r))\frac{\partial^2 u_z}{\partial r\partial z}+\underline{\lambda'(r)\frac{\partial u_z}{\partial z}}=0, \tag{19}$$

$$[\mu(r)+\lambda(r)]\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)+\underline{\mu'(r)\frac{\partial u_r}{\partial z}}+\mu(r)\Delta u_z+[\mu(r)+\lambda(r)]\frac{\partial^2 u_z}{\partial z^2}+\underline{\mu'(r)\frac{\partial u_z}{\partial r}}=0. \tag{20}$$

The underlined terms distinguish system (19), (20) from the respective system (4), (5) in the homogeneous theory.

We assume that the Love solutions (6) are also valid for the inhomogeneous theory. Then substituting (6) into (19) and (20) and repeating the procedure of Sec. 1, we obtain an equation for the Love function $\chi(r, z)$. At the first step of the procedure, Eqs. (19) and (20) are transformed into the following two equations:

$$\left\{[\lambda(r)+2\mu(r)]\frac{\partial}{\partial r}\Delta-[\lambda(r)+\mu(r)]\frac{\partial^3}{\partial r\partial z^2}+\underline{\lambda'(r)\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial}{\partial r}}+\underline{2\mu'(r)\frac{\partial^2}{\partial r^2}}\right\}R$$

$$+\left\{[\lambda(r)+\mu(r)]\frac{\partial^2}{\partial r\partial z}+\underline{\lambda'(r)\frac{\partial}{\partial z}}\right\}Z=0, \tag{21}$$

$$\left\{[\mu(r)+\lambda(r)]\frac{\partial}{\partial z}\left(\Delta-\frac{\partial^2}{\partial z^2}\right)+\underline{\mu'(r)\frac{\partial}{\partial z}\frac{\partial}{\partial r}}\right\}R+\left\{\mu(r)\Delta+[\mu(r)+\lambda(r)]\frac{\partial^2}{\partial z^2}+\underline{\mu'(r)\frac{\partial}{\partial r}}\right\}Z=0. \tag{22}$$

Remark 7. System (21), (22) is no longer analogous to system (8), (9) of the homogeneous theory; Eq. (21) cannot be transformed into a new differential equation by integrating over the radial coordinate; therefore, we have to follow a procedure of obtaining a Love pseudosolution.

Let us introduce the Love-type function $\chi(r, z)$ in a standard manner where Eq. (21) is satisfied identically:

$$R=-\left\{[\lambda(r)+\mu(r)]\frac{\partial^2}{\partial r\partial z}+\underline{\lambda'(r)\frac{\partial}{\partial z}}\right\}\chi,$$

$$Z=\left\{[\lambda(r)+2\mu(r)]\frac{\partial}{\partial r}\Delta-[\lambda(r)+\mu(r)]\frac{\partial^3}{\partial r\partial z^2}+\underline{\lambda'(r)\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial}{\partial r}}+\underline{2\mu'(r)\frac{\partial^2}{\partial r^2}}\right\}\chi, \tag{23}$$

and Eq. (22) is transformed into an equation for the Love function:

$$\left\{[\mu(r)+\lambda(r)]\frac{\partial}{\partial z}\Delta-[\mu(r)+\lambda(r)]\frac{\partial^3}{\partial z^3}+\underline{\mu'(r)\frac{\partial^2}{\partial r\partial z}}\right\}$$

$$\times\left\{[\lambda(r)+\mu(r)]\frac{\partial^2}{\partial r\partial z}+\underline{\lambda'(r)\frac{\partial}{\partial z}}\right\}\chi+\left\{\mu(r)\Delta+[\mu(r)+\lambda(r)]\frac{\partial^2}{\partial z^2}+\underline{\mu'(r)\frac{\partial}{\partial r}}\right\}$$

$$\times\left\{[\lambda(r)+2\mu(r)]\left(\Delta-\frac{1}{r^2}\right)\frac{\partial}{\partial r}-[\lambda(r)+\mu(r)]\frac{\partial^3}{\partial r\partial z^2}+\underline{\lambda'(r)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)}+\underline{2\mu'(r)\frac{\partial^2}{\partial r^2}}\right\}\chi=0. \tag{24}$$

After labor-intensive transformations, Eq. (24) becomes

$$\Delta\Delta\frac{\partial\chi}{\partial r}+k_{4rz}(r)\frac{\partial^4\chi}{\partial r^2\partial z^2}+k_{\Delta\Delta}(r)\Delta\Delta\chi+k_{\Delta r}(r)\Delta\frac{\partial^2\chi}{\partial r^2}$$

$$+k_{3rz}(r)\frac{\partial^3\chi}{\partial r\partial z^2}+k_{\Delta}(r)\Delta\chi+k_{2z}(r)\frac{\partial^2\chi}{\partial z^2}+k_{2r}(r)\frac{\partial^2\chi}{\partial r^2}+k_{1r}(r)\frac{\partial\chi}{\partial r}=0, \tag{25}$$

where

$$k_{4rz}(r)=-2\frac{\lambda'(r)+\mu'(r)}{\lambda(r)+2\mu(r)},\quad k_{\Delta\Delta}(r)=\frac{\lambda'(r)}{\lambda(r)+2\mu(r)},$$

$$k_{\Delta r}(r)=\frac{\lambda(r)\mu'(r)+2\lambda(r)\lambda'(r)+3\mu(r)\lambda'(r)+8\mu(r)\mu'(r)}{\mu(r)[\lambda(r)+2\mu(r)]},$$

$$k_{3rz}(r)=-\frac{3\mu(\lambda''-\mu'')+4\lambda'\mu'+\mu(\lambda'+\mu')\frac{1}{r}+(\mu+\lambda)(\lambda+2\mu)\frac{1}{r^2}}{\mu(r)[\lambda(r)+2\mu(r)]},$$

$$k_{\Delta}(r)=\frac{\mu(r)\lambda'''(r)+\mu'(r)\lambda''(r)+\frac{1}{r}\mu(r)\lambda''(r)}{\mu(r)[\lambda(r)+2\mu(r)]},$$

$$k_{2z}(r)=-\frac{\mu(r)\lambda'''(r)+\frac{1}{r}\mu(r)\lambda''(r)-2\mu'(r)\lambda''(r)}{\mu(r)[\lambda(r)+2\mu(r)]},$$

$$k_{2r}(r)=2\frac{\mu'\mu''+(\lambda\mu'+2\mu\lambda'+4\mu\mu')+\mu\mu'''+\frac{1}{r}\mu\mu''}{\mu(r)[\lambda(r)+2\mu(r)]},$$

$$k_{1r}(r)=-\frac{\mu(3\lambda''+2\mu'')\frac{1}{r^2}+\mu'(\lambda'+2\mu')\frac{1}{r^2}+3\mu(\lambda'+2\mu')\frac{1}{r^3}}{\mu(r)[\lambda(r)+2\mu(r)]}. \tag{26}$$

Equation (25) includes a Laplasian. To separate variables, it is sometimes convenient to represent the Laplasian as $\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}=\Delta_r+\frac{\partial^2}{\partial z^2}$. Then the operators that include the Laplasian can be rearranged into

$$\Delta\Delta=\Delta_r\Delta_r+2\Delta_r\frac{\partial^2}{\partial z^2}+\frac{\partial^4}{\partial z^4},\quad \Delta\frac{\partial^2}{\partial r^2}=\Delta_r\frac{\partial^2}{\partial r^2}+\frac{\partial^4}{\partial r^2\partial z^2},\quad \Delta\frac{\partial}{\partial r}=\Delta_r\frac{\partial}{\partial r}+\frac{\partial^3}{\partial r\partial z^2}$$

and Eq. (25) becomes

$$\left(\Delta_r\Delta_r+2\Delta_r\frac{\partial^2}{\partial z^2}+\frac{\partial^4}{\partial z^4}\right)\frac{\partial\chi}{\partial r}+[k_{4rz}(r)+k_{\Delta 2r}(r)]\frac{\partial^4\chi}{\partial r^2\partial z^2}$$

$$+k_{\Delta\Delta}(r)\left(\Delta_r\Delta_r\chi+2\Delta_r\frac{\partial^2\chi}{\partial z^2}+\frac{\partial^4\chi}{\partial z^4}\right)+k_{\Delta 2r}(r)\Delta_r\frac{\partial^2\chi}{\partial r^2}+k_{\Delta r}\Delta_r\frac{\partial\chi}{\partial r}+[k_{3rz}(r)+k_{\Delta r}]\frac{\partial^3\chi}{\partial r\partial z^2}$$

$$+k_{\Delta}(r)\Delta_r\chi+[k_{2z}(r)+k_{\Delta}(r)]\frac{\partial^2\chi}{\partial z^2}+k_{2r}(r)\frac{\partial^2\chi}{\partial r^2}+k_{1r}(r)\frac{\partial\chi}{\partial r}=0. \tag{27}$$

Equation (27) can be represented as three groups of terms: the first group includes only operators with respect to r (first and second rows), the second group only operators with respect to z (third row), and the third group mixed operators (fourth row):

$$\Delta_r \Delta_r \frac{\partial \chi}{\partial r} + k_{\Delta\Delta}(r) \Delta_r \Delta_r \chi + k_{\Delta 2r}(r) \Delta_r \frac{\partial^2 \chi}{\partial r^2} + k_{\Delta r} \Delta_r \frac{\partial \chi}{\partial r}$$

$$+ k_{\Delta}(r) \Delta_r \chi + k_{2r}(r) \frac{\partial^2 \chi}{\partial r^2} + k_{1r}(r) \frac{\partial \chi}{\partial r}$$

$$+ k_{\Delta\Delta}(r) \left(2\Delta_r \frac{\partial^2 \chi}{\partial z^2} + \frac{\partial^4 \chi}{\partial z^4} \right) + [k_{2z}(r) + k_{\Delta}(r)] \frac{\partial^2 \chi}{\partial z^2}$$

$$+ 2\Delta_r \frac{\partial^3 \chi}{\partial r \partial z^2} + \frac{\partial^5 \chi}{\partial r \partial z^4} + [k_{4rz}(r) + k_{\Delta 2r}(r)] \frac{\partial^4 \chi}{\partial r^2 \partial z^2} + [k_{3rz}(r) + k_{\Delta r}] \frac{\partial^3 \chi}{\partial r \partial z^2} = 0. \quad (28)$$

Remark 8. What complicates Eqs. (25), (27), and (28) compared with the respective equation of the homogeneous theory is typical of the inhomogeneous theory [1, 2, 4, 23, 33].

3. Special Cases. The inhomogeneous isotropic theory of elasticity traditionally examines the special case where Poisson's ratio is constant. In particular, when inhomogeneity is incorporated into the Lame parameters $\lambda(r) = \lambda_h l(r)$ and $\mu(r) = \mu_h m(r)$, both parameters vary in a similar manner: $l(r) = m(r)$. Then Poisson's ratio is indeed constant due to the well-known formulas $\nu = \frac{\lambda}{2(\lambda + \mu)} = \frac{\lambda_h m(r)}{2[\lambda_h m(r) + \mu_h m(r)]} = \frac{\lambda_h}{2(\lambda_h + \mu_h)}$. This considerably simplifies the relations of the inhomogeneous isotropic theory of elasticity derived using various general solutions.

In the case being considered, the coefficients of the equation for the Love-type function (25) are much simpler—it can be expressed in terms of one elastic constant ν_h and four functions $m(r)$, $m'(r)$, $m''(r)$, $m'''(r)$ and its coefficients include functions of the radial coordinate r. For example, the first two coefficients are expressed as

$$k_{4rz}(r) = -2 \frac{\lambda'(r) + \mu'(r)}{\lambda(r) + 2\mu(r)} = -2 \frac{\lambda_h + \mu_h}{\lambda_h + 2\mu_h} \frac{m'(r)}{m(r)} = -2 \frac{1}{2 - \frac{\lambda_h}{\lambda_h + \mu_h}} = -\frac{1}{1 - \nu_h} [\ln m(r)]',$$

$$k_{\Delta\Delta}(r) = \frac{\lambda'(r)}{\lambda(r) + 2\mu(r)} = \frac{\lambda_h}{\lambda_h + 2\mu_h} \frac{\lambda_h + \mu_h}{\lambda_h + \mu_h} \frac{m'(r)}{m(r)} = \frac{\nu_h}{1 - \nu_h} [\ln m(r)]'.$$

For greater clarity, we will examine the classical (for the inhomogeneous theory) special case of exponential dependence $m(z) = e^{\alpha z}$. This case is convenient in that, given various general solutions, it leads to an equation with constant coefficients.

Let us check how much the coefficients of Eq. (26) will change. To this end, they should be rearranged considering that $m(z) = e^{\alpha z}$, $m'(z) = \alpha e^{\alpha z}$, $m''(z) = \alpha^2 e^{\alpha z}$, $m'''(z) = \alpha^3 e^{\alpha z}$. Then

$$k_{4rz}(r) = -\frac{\alpha}{1 - \nu_h}, \quad k_{\Delta\Delta}(r) = \frac{\alpha \nu_h}{1 - \nu_h}, \quad k_{\Delta 2r}(r) = 4\alpha \frac{1 - \nu_h}{1 - 2\nu_h}, \quad k_{\Delta r} = \alpha^2 \frac{2 + \nu_h}{1 - \nu_h} + \frac{\alpha}{r} - \frac{1}{r^2},$$

$$k_{3rz}(r) = -\alpha^2 \frac{-13 + 6\nu_h}{2(1 - \nu_h)(1 - 2\nu_h)} - \frac{\alpha}{2r(1 - \nu_h)} - \frac{1}{r^2 (1 - 2\nu_h)}, \quad k_{\Delta} = \left(2\alpha^3 + \frac{1}{r} \alpha^2 \right) \frac{\nu_h}{1 - \nu_h},$$

$$k_{2z}(r) = \left(\alpha^3 - \frac{\alpha^2}{r} \right) \frac{\nu_h}{1 - \nu_h}, \quad k_{2r}(r) = 2 \left[\left(2\alpha^3 + \frac{\alpha^2}{r} \right) \frac{1 - 2\nu_h}{2(1 - \nu_h)} + \frac{\alpha}{r^2} \frac{2 - \nu_h}{2(1 - \nu_h)} \right],$$

$$k_{1r}(r) = -\frac{2\alpha^2}{r^2}\frac{1}{1-\nu} - \frac{3\alpha}{r^3}. \quad (29)$$

These formulas confirm that the coefficients are constant. Hence, Eq. (25) transforms into an equation with constant coefficients, and the inhomogeneity affects the Love-type function only through the initial value of Poisson's ratio ν_h and the coefficients $\alpha, \alpha^2, \alpha^3$.

Remark 9. When the parameter α is quite small, one more possible simplification is to neglect its square and cube. Then coefficients (29) become even simpler, some equaling zero.

Discussion. The general solution proposed above has a property typical of all the general solutions obtained earlier in the inhomogeneous theory of elasticity: unlike the classical homogeneous theory, the new (Love-type) function in the inhomogeneous theory is not biharmonic (the biharmonic equation has many additional terms). These terms include derivatives of the Love-type function with respect to both coordinates (vertical and radial) with coefficients that include derivatives of the elastic parameters, which are functions of the radial coordinate.

The experience of applying such complicated equations to functionally gradient materials [19–22] suggests that in these cases, it is possible to separate variables and, thus, to overcome the arising mathematical difficulties. Therefore, the results obtained are planned to be used in the theory of functionally gradient materials in which one of the authors is somewhat experienced. Also, the general solutions proposed can be used in the theory of micro- and nanocomposites, in which the other author is somewhat experienced [7, 8, 11–18].

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