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RESONANT VIBRATIONS OF A HINGED THERMOVISCOELASTIC RECTANGULAR PLATE WITH SENSORS AND ACTUATORS

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The paper discusses the active damping of the resonant flexural vibrations of a hinged thermoviscoelastic rectangular plate with distributed piezoelectric sensors and actuators. The thermoviscoelastic behavior of the passive and active materials is described using the concept of complex characteristics. The interaction of mechanical and thermal fields is taken into account. The Bubnov–Galerkin method is used. The effect of dissipative heating, the dimensions of the piezoelectric inclusions, and the feedback factor on the effectiveness of active damping of resonance vibrations of the plate is studied

Keywords: resonant flexural vibrations, thermoviscoelastic rectangular plate, hinged edges, active damping, sensor, actuator, dissipative heating

Introduction. Thin viscoelastic composite rectangular plates made of passive and piezoactive materials are widely used as structural elements in various fields of modern engineering. When they are subject to harmonic mechanical loads with a frequency close to a resonant frequency, there is a risk of failure because of the high amplitude of vibrations. Therefore, it is necessary to use passive and active damping of the forced resonance vibrations of such plates. One of the main methods for the active damping of the vibration of thin-walled elements employs both sensors and actuators. The principal advantage of this method is the possibility to change the damping factor during the operation of the structure. Relevant studies are reviewed in [15–18]. The effectiveness of this type of active damping depends on many factors, including self-heating temperature, arrangement and dimensions of sensors and actuators, etc. In turn, the arrangement and dimensions of piezoelectric inclusions are strongly dependent on the mechanical boundary conditions. The basic equations of the theory of viscoelastic plates with distributed sensors and actuators are presented and relevant studies are briefly reviewed in [5–11].

The present paper studies the influence of dissipative heating on the active damping of the forced resonant vibrations of hinged thin viscoelastic rectangular plates with sensors and actuators. The associated problem will be solved by the Bubnov–Galerkin method. By analyzing numerical results, we will study how the effectiveness of the active damping of a hinged rectangular plate depends on the feedback factor, self-heating temperature, and the dimensions of the piezoinclusions.

1. Problem Formulation. The general theory of active damping of forced flexural vibrations of thin plates with piezoelectric sensors and actuators is outlined in [7]. Let us present those equations of [7] that will be used below. We will use the same notation and hypotheses as in [7].

We will restrict ourselves to the case of flexural vibrations of a sandwich plate with a passive core of thickness h_0 and piezoactive face layers having thickness h_1 each and polarized in opposite directions. The properties of these piezolayers only differ by the sign of the piezoelectric constant. In the case of flexural vibrations, the constitutive equations given in [7] become simpler:

$$M_1 = D_{11}\kappa_1 + D_{12}\kappa_2 + M_0,$$

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$$M_2 = D_{12}\kappa_1 + D_{22}\kappa_2 + M_0,$$

$$H = D_{66}\kappa_{12}. \tag{1.1}$$

Expressions for the stiffness characteristics and M_0 in (1.1) are given in [5–7].

An energy equation that describes self-heating is presented in [7]. In the case of flexural vibrations, it takes the form

$$\lambda_{11}\theta_{,xx} + \lambda_{22}\theta_{,yy} - (2\delta/h)\theta + W/h = 0. \tag{1.2}$$

The problem for the sandwich plate is reduced to the following nonlinear system of differential equations for the deflection and self-heating temperature:

$$\frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right] + \frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\right]$$

$$+\frac{\partial^2}{\partial x\partial y}\left[2(1-\nu)D\left(\frac{\partial^2 w}{\partial x\partial y}\right)\right] - (\gamma h)\omega^2 w - p_0(x,y) - \left(\frac{\partial^2 M_0}{\partial x^2} + \frac{\partial^2 M_0}{\partial y^2}\right) = 0,$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{2\alpha}{(\lambda h)}(T - T_{\text{ex}}) + \frac{\omega E''(T)h^3}{24a^2(1-\nu^2)(\lambda h)} \times \left\{\left(\frac{\partial^2 w'}{\partial x^2}\right)^2\right.$$

$$+\left(\frac{\partial^2 w''}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w'}{\partial y^2}\right)^2 + \left(\frac{\partial^2 w''}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w'}{\partial x^2}\frac{\partial^2 w'}{\partial y^2}\right.$$

$$\left.\left.+\frac{\partial^2 w''}{\partial x^2}\frac{\partial^2 w''}{\partial y^2}\right) + 2(1-\nu)\left[\left(\frac{\partial^2 w'}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 w''}{\partial x\partial y}\right)^2\right]\right\} = 0, \tag{1.3}$$

where the notation is the same as in [7].

When both sensors and actuators are used, forced vibrations are additionally damped as described in [7]: the actuator voltage V_{A} is proportional to the rate of variation in the sensor voltage $\dot{V}_{\text{S}}$ or the current:

$$V_{\text{A}} = -G_{\text{S}}\dot{V}_{\text{S}}, \quad V_{\text{A}} = -G_{\text{S}}I, \quad G_{\text{S}} = GG_{\text{e}}, \tag{1.4}$$

where G_{e} is the resistance of the amplifier, and G is selected so that free vibrations are damped the fastest or the amplitude of forced (including resonant) vibrations is decreased maximally. A formula for the charge of the sensor in the case of short-circuited electrodes is presented in [7]:

$$Q = -\gamma_{31}(h_0 + h_1)\iint_{(S_1)}(\kappa_1 + \kappa_2)dxdy. \tag{1.5}$$

If we use the formula

$$V_{\text{A}} = -G_{\text{S}}\frac{dQ}{dt}, \tag{1.6}$$

then

$$V_{\text{A}} = G_{\text{S}}\gamma_{31}(h_0 + h_1)\iint_{(S_1)}(\dot{\kappa}_1 + \dot{\kappa}_2)dxdy. \tag{1.7}$$

It is this relation that will be used below because it has fewer electromechanical parameters.

Substituting (1.7) into (1.3) and representing the deflection as $w = e^{i\omega t}\hat{w}(x, y)$, we arrive at a differential equation for $\hat{w}(x, y)$. In the case of resonant vibrations, it can be solved by the Bubnov–Galerkin method, when $\hat{w} = A\tilde{w}(x, y)$, where A is the complex amplitude. The equation of motion includes the quantity

$$M_0 = m_0 FA, \tag{1.8}$$

where

$$m_0 = G_S \gamma_{31}^2 (h_0 + h_1)^2, \quad F = \iint_{(S_1)} \left(\frac{\partial^2 \tilde{w}}{\partial x^2} + \frac{\partial^2 \tilde{w}}{\partial y^2} \right) dxdy. \tag{1.9}$$

To determine M_0, it is necessary to evaluate the integral F over the areas S_1 of the sensor.

2. Problem-Solving Methods. To analyze the effect of the feedback factor G_S and self-heating temperature on the effectiveness of damping of the forced resonant vibrations with both sensors and actuators, it is first necessary to plot amplitude–frequency characteristics (AFCs) regardless of these factors, then to calculate these characteristics taking into account the self-heating temperature, and to compare the results. The above system of nonlinear differential equations is quite complicated and can be solved by iterative methods in combination with numerical methods (for example, the finite-element or discrete-orthogonalization method [1–4, 11]). As in [5–11], we will make active use of the Bubnov–Galerkin method. We will restrict ourselves to the case where the real and imaginary parts of the shear modulus are linearly dependent on temperature

$$G = G' + G'', \quad G' = G_0' - G_1' T, \quad G'' = G_0'' - G_1'' T. \tag{2.1}$$

The mechanical, physical, and thermal properties of the material (polyethylene) are presented in [12].

Assuming that forced vibrations occur at the first resonant frequency, we will use a standard expression for the deflection of a hinged plate:

$$\hat{w} = \sin k_1 x \sin p_1 y, \quad k_1 = \frac{\pi}{a}, \quad p_1 = \frac{\pi}{b}. \tag{2.2}$$

If we use the concept of complex characteristics [2], the expression for the imaginary part of the shear modulus will change because of the additional damping. To keep it simple, we will proceed as follows: in the solution that disregards the additional damping [8, 10], we will replace the imaginary part of the complex shear modulus with that modified to account for the additional damping. The real part of the complex shear modulus remains the same.

Let us first consider the case where the properties of the passive and piezoactive materials are independent of temperature.

The self-heating temperature was determined in [8, 10] for two cases of thermal boundary conditions disregarding the additional (active) damping, which is due to the use of sensors and actuators. One case is the heat-insulated edges of the plate. The energy equation has the following exact solution:

$$\theta = \theta_0 + \theta_1 \cos 2k_1 x + \theta_2 \cos 2p_1 y + \theta_3 \cos 2k_1 x \cdot \cos 2p_1 y. \tag{2.3}$$

The additional damping will only affect the amplitude of mechanical vibrations. Therefore, the expressions for coefficients (2.3) presented in [8, 10] remain the same.

The other case is a constant temperature T_{ex} prescribed at the edges of the plate. The energy equation has the following approximate solution:

$$\theta = T - T_{ex} = \theta_0 \sin k_1 x \cdot \sin p_1 y. \tag{2.4}$$

The expression for θ_0 was derived in [8, 10] by the Bubnov–Galerkin method and is as follows:

$$\theta_0 = \frac{8\omega W_0^2 a^2}{(\lambda h)\pi^4} \cdot \frac{a_0 + \frac{2}{3}a_1 + \frac{1}{9}a_2}{1+\left(\frac{a}{b}\right)^2 + \frac{2\alpha a^2}{\lambda h \pi^2}}. \tag{2.5}$$

The plate is most heated at the center. Therefore, the maximum temperature $\theta_{max} = \theta_0$ in both cases of thermal boundary conditions. Equating the expressions for θ_0 to the Curie temperature, we obtain the critical load parameter W_{0C}^2 above which the sensors and actuators no longer perform their function:

$$W_{0C}^2 = \frac{4\alpha\theta_C}{a_0\omega}, \qquad W_{0C}^2 = \frac{(\lambda h)\pi^4\theta_C}{8\omega a^2} \frac{1+\left(\frac{a}{b}\right)^2 + \frac{2\alpha a^2}{\lambda h\pi^2}}{a_0 + \frac{2}{3}a_1 + \frac{1}{9}a_2} \tag{2.6}$$

for heat-insulated edges and edges kept at a constant temperature, respectively.

A coupled problem with physical nonlinearity due to the dependence of the material properties on temperature and the nonlinear dependence of the dissipation function on strains and temperature was also solved in [8, 10]. Given T_{ex} at the edge of the plate, the temperature was approximated by (2.4). The deflection was approximated by (2.2). The mechanical problem was solved by the variational and Bubnov–Galerkin methods. The following expression for the complex amplitude of vibrations has been obtained:

$$A = \frac{\Delta_1}{\Delta_2}. \tag{2.7}$$

Expressions for Δ_1 and Δ_2 can be found in [8, 10].

The self-heating temperature follows from the energy equation:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - \frac{2\alpha}{\lambda h}\theta + \frac{\omega}{2(\lambda h)} D'' \frac{|A|^2}{(\lambda h)} f(x, y) = 0, \quad D'' = \frac{G'' h^3}{6(1-\nu)}, \tag{2.8}$$

where

$$f(x, y) = (k_1^4 + p_1^4 + 2\nu k_1^2 p_1^2)\sin^2 k_1 x \cdot \sin^2 p_1 y + 2(1-\nu)k_1^2 p_1^2 \cos^2 k_1 x \cdot \cos^2 p_1 y. \tag{2.9}$$

The following expression for temperature was obtained in [8, 10] based on the variational and Bubnov–Galerkin methods:

$$\theta_0 = \frac{C_1 X}{C_0 + C_1 X}, \qquad X = |A_0|^2. \tag{2.10}$$

The complex amplitude (2.7) was represented in the following form [8, 10]:

$$A_0 = q_0 / (A_1 + iA_2), \tag{2.11}$$

where $q_0 = \frac{16}{\pi^2} p_0$, $A_1 = B_1 - b_1 T_0$, $A_2 = B_2 - b_2 T_0$.

Expressions for $b_1, b_2, B_1, B_2, C_0, C_1, C_2$ can be found in [8, 10].

The squared amplitude of vibrations follows from (2.11):

$$X = \frac{q_0^2}{(B_1^2 + B_2^2) - 2(b_1 B_1 + b_2 B_2)T_0 + (b_1^2 + b_2^2)T_0^2}, \quad T_0 = \theta_0 + T_{ex}. \tag{2.12}$$

Eliminating X from (2.10) and (2.12), we obtain a cubic equation for $y = T_0 / T_{ex}$:

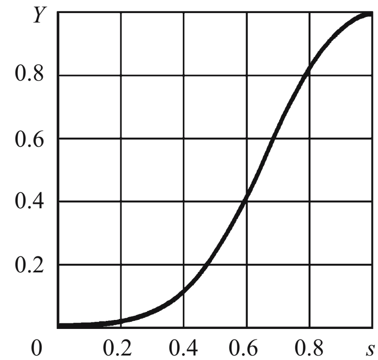

Fig. 1

$$y^3 - e_2 y^2 + e_1 y - e_0 = 0, \tag{2.13}$$

where

$$e_0 = \frac{C_1 q_0 / C_0}{T_{\mathrm{ex}}^3 (b_1^2 + b_2^2)}, \quad e_1 = \frac{B_1^2 + B_2^2 + C_2 q_0^2 / C_0}{T_{\mathrm{ex}}^2 (b_1^2 + b_2^2)}, \quad e_2 = 2\frac{b_1 B_1 + b_2 B_2}{T_{\mathrm{ex}} b_1^2 + b_2^2}. \tag{2.14}$$

Once the self-heating temperature has been determined, the frequency dependence of the amplitude $\sqrt{X}$ can be established by formula (2.12).

For the heat-insulated edges, all the above formulas remain the same change, except that

$$C_0 = 2\alpha a^2, \quad C_1 = \frac{\omega}{2} D_0'' \frac{\pi^4}{a^2}, \quad C_1 = \frac{\omega}{2} D_1'' \frac{\pi^4}{a^2}. \tag{2.15}$$

After the determination of the self-heating temperature, it is possible to analyze its influence on the effectiveness of active damping of resonant vibrations. To this end, it is necessary to substitute this temperature into the above formulas for the sensor, say, formula (1.5) for the its charge. Note that the above expression for the amplitude of vibrations and the self-heating temperature disregard the additional damping due to the joint operation of sensors and actuators. To allow for it, it is necessary to calculate M_0 by formula (1.4)–(1.9), substitute it into Eq. (1.3), solve the thus modified nonlinear problem by a variational method or the Bubnov–Galerkin method, and assess the effect of the additional damping on the amplitude–frequency characteristics taking into account self-heating. As mentioned above, all the above formula will remain the same if B_2 is replaced by

$$\widetilde{B}_2 = B_2 + G_2, \tag{2.16}$$

where the additional damping is characterized by G_2.

For a hinged rectangular plate, we obtain

$$G_2 = 4 G_S \gamma_{31}^2 (h_0 + h_1)^2 \left(\frac{b}{a}\right)^2 \left[1 + \left(\frac{a}{b}\right)^2\right]^2 \left[\sin\frac{\pi}{2}\left(\frac{c}{a}\right) \sin\frac{\pi}{2}\left(\frac{d}{b}\right)\right]^2. \tag{2.17}$$

Thus, in addition to damping caused by hysteresis losses in the material, there is active damping characterized by the parameter G_2 that can be controlled over a wide range. To calculate the self-heating temperature, it is necessary to substitute the expression for the amplitude of vibrations that accounts for this additional damping into the expression for the dissipation function.

To plot AFCs, it is necessary to introduce (2.16) into the above formulas and to calculate new amplitude- and temperature-frequency characteristics. By comparing them, we can assess the effect of the feedback factor and self-heating temperature.

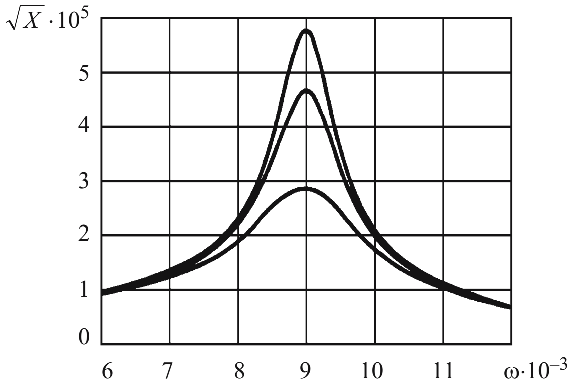

Fig. 2

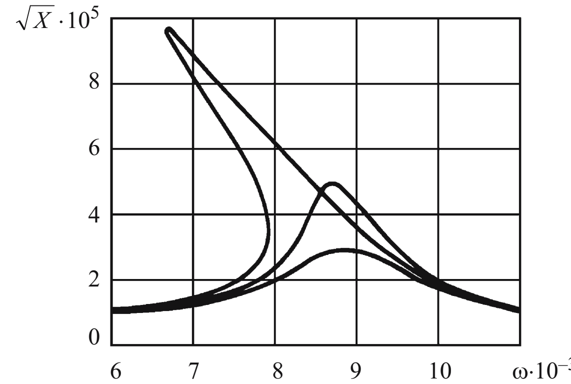

Fig. 3

3. Analysis of the Results. In what follows, we will consider a polyethylene plate. The sensors and actuators are made of $TsTS_TBS$-2 piezoceramics (see [13] for its thermoelectromechanical properties). It follows from (2.17) that the effectiveness of active damping depends on the dimensions of the sensors and actuators:

$$G_2 = 4G_S\gamma_{31}^2(h_0 + h_1)^2\left(\frac{b}{a}\right)^2\left|1+\left(\frac{a}{b}\right)^2\right|^2\left[\sin\frac{\pi}{2}\left(\frac{l}{L}\right)\right]^4, \tag{3.1}$$

where l and L are the diagonals of the piezoelectric elements and plate, respectively. The parameter G_2 is maximum when $l = L$, i.e., the plate is fully covered by piezoinclusions.

Figure 1 shows the dependence of $Y = G_2 / E_1$, where

$$E_1 = \frac{64}{a^2}G_S\gamma_{31}^2(h_0 + h_1)^2\left(\frac{b}{a}\right)\left[1+\left(\frac{a}{b}\right)^2\right]^2$$

on the dimensionless diagonal $s = l / L$.

This parameter tends to zero as the diagonal decreases and tends to one as the diagonal of the piezoinclusion tends to the diagonal of the plate. It can be seen from (3.1) that once the temperature has reached the Curie point T_C, when $\gamma_{31} = 0$, it is no longer possible to control the amplitude of vibrations by active damping because $G_2(T_C) = 0$. Equating the self-heating temperature to T_C, we obtain the maximum amplitude of loading above which the amplitude of vibrations cannot be controlled. For temperature-independent properties, the critical amplitude of mechanical loading is defined by formula (2.6).

We will now analyze the influence of the parameter G_2 on the effectiveness of damping for a square sandwich plate with polyethylene core and $TsTS_TBS$-2 piezoceramic face layers of opposite polarization. The plate is hinged and a constant temperature is maintained at its boundary. The characteristics of the plate are: $\alpha = 0.5$ W/(m$^2\times$°C) is the coefficient of heat transfer to the environment of temperature $T_{ex} = 20$ °C; $p_0 = 2500$ Pa is the amplitude of normal surface pressure. If the plate has side length a = 0.0536 m, then linear-resonance frequency ω_r = 16 432 Hz. Figure 2 shows AFCs in the case of temperature-independent mechanical properties of the passive material for different values of G_2 ($G_2 = 0$ for the upper curve, $G_2 = 0.5B_2$ for the middle curve, and $G_2 = 1.5B_2$ for the lower curve).

Figure 3 shows how the physical nonlinearity due to the temperature dependence of the material properties affects the AFCs.

Figures 4 (properties are independent of temperature) and 5 (properties are dependent on temperature) shows similar characteristics for five-fold greater heat-transfer coefficient ($\alpha = 2.5$ W/(m$^2\times$°C)).

As is seen, such a physical nonlinearity is responsible for the ambiguity of the ADCs and hysteresis.

The temperature–frequency characteristic is similar.

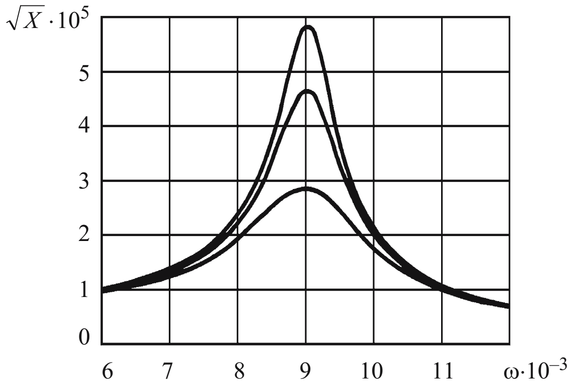

Fig. 4

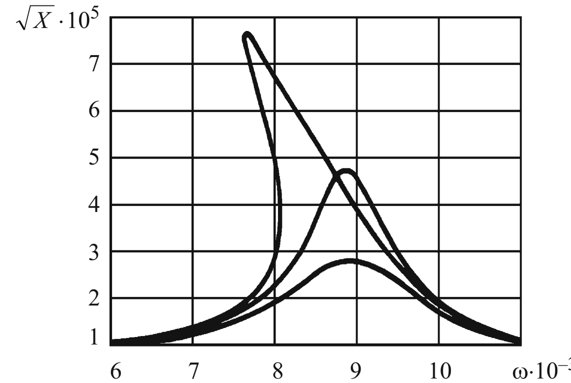

Fig. 5

The figures indicate that the effect of the nonlinearity weakens and the self-heating temperature decreases with increasing feedback factor G_2.

4. Conclusions. We have solved the problem of active damping of the forced flexural resonant vibrations of a thin rectangular viscoelastic plates with piezoelectric sensors and actuators, taking into account self-heating and the temperature dependence of the material properties. The nonlinear problem has been solved by the Bubnov–Galerkin method. The formulas derived and the numerical results obtained show that self-heating has a significant effect on the effectiveness of the active damping of resonant vibrations for both temperature-dependent and temperature-independent properties of the piezoactive and passive materials. For example, after the temperature reaches the Curie point, the element keeps its integrity, but no longer performs its functions, which is a specific type of thermal failure. The maximum mechanical load that causes this type of failure has been determined. The numerical results illustrate the influence of the self-heating temperature, feedback factor, and the dimensions of the sensors and actuators on the effectiveness of the active damping of resonant vibrations of plates.

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