NONLINEAR VIBRATIONS OF CYLINDRICAL SHELLS FILLED WITH A FLUID AND SUBJECTED TO LONGITUDINAL AND TRANSVERSE PERIODIC EXCITATION

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The paper proposes an approach to studying the nonlinear vibrations of thin cylindrical shells filled with a fluid and subjected to a combined transverse–longitudinal load. Methods of nonlinear mechanics are used to find and analyze periodic solutions of the system of equations that describes the dynamic behavior of the shell when the natural frequencies of the shell and the frequencies of both periodic forces are in resonance relations

Keywords: elastic cylindrical shell, ideal incompressible fluid, combined load, amplitude–frequency response, stability

Introduction. The flexural vibrations of thin cylindrical shells filled with a fluid are studied in a geometrically nonlinear formulation in a great many publications. Such studies are reviewed in, e.g., [6, 9–11, etc.]. Most studies deal with dynamic problems for fluid-filled shells subjected to either radial (transverse) or axial (longitudinal) force. However, in real operation conditions, shell structures conveying a fluid (such as segments of pipelines) are quite often subjected to combined loading, i.e., a combination of longitudinal and transverse periodic forces. Therefore, of practical and scientific interest is to study the dynamic behavior of shells filled with a fluid and subjected to a combined vibratory load. The superposition principle fails here because of the nonlinearity of such a problem formulation: the response of the dynamic shell–fluid system to a combination of longitudinal and transverse periodic loads is not the sum of the responses of this system to the individual loads. Here we may expect qualitatively new nonlinear effects that were not observed in the partial problems of forced [7, 10–14, 18] or parametric [2, 4, 11, 16] vibrations of shell–fluid objects.

The present paper sets out to develop a method and to apply it to study the nonlinear deformation of elastic cylindrical shells filled with a fluid and subjected to longitudinal-and-transverse periodic excitation. We will primarily analyze the dynamic behavior of filled shells in the worst (with respect to dynamic stress) case where the natural frequencies of the shell–fluid system are in resonance relations with the frequencies of both external periodic forces. The results obtained in the general case will be compared with those obtained in the partial cases where either only longitudinal or only transverse load acts.

1. Nonlinear Equations of Motion of a Fluid-Filled Shell. For the equations of motion of a shell filled with a fluid, we will use the geometrically nonlinear equations of the theory of shallow shells in mixed form [3, 4]:

$$\frac{D}{h}\nabla^4 w=\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \Phi}{\partial x^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 \Phi}{\partial x\partial y}+\frac{1}{R}\frac{\partial^2 \Phi}{\partial x^2}-\rho\frac{\partial^2 w}{\partial t^2}-\varepsilon\rho\frac{\partial w}{\partial t}+\frac{q}{h}-\frac{P_{\mathrm{h}}}{h},$$

$$\frac{1}{E}\nabla^4\Phi=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\frac{1}{R}\frac{\partial^2 w}{\partial x^2}, \tag{1.1}$$

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where the notation is conventional: w is the transverse deflection (positive toward the center of curvature); R and h are the radius and thickness of the shell; Φ is the stress function on the midsurface; $D = Eh^3/[12(1-\mu^2)]$ is cylindrical stiffness (E is the elastic modulus, μ is Poisson's ratio); ρ is the density of the shell material; ε is the damping factor; P_h is the hydrodynamic pressure exerted by the fluid on the shell; $q = q_0(x, y)\cos\Omega t$ is the external periodic pressure nonuniformly distributed over the lateral surface of the shell ($q_0(x, y)$ is some distribution function, Ω is the frequency of pressure). The longitudinal coordinate x is reckoned from one of the ends of the shell.

In addition to the lateral load q, the shell is subjected to an axial pulsing (with period $T_2 = 2\pi/\nu$) pressure $\tilde{N}_x(t) = N_0 + N_1\cos\nu t$ applied to the ends $x = 0$ and $x = l$ (l is the length of the shell; N_0, N_1 = const, $N_0 > 0$). The boundary conditions are

$$w = 0, \quad \mathrm{v} = 0, \quad M_x = 0, \quad N_x = \tilde{N}_x(t) \quad \text{at} \quad x = 0, l, \tag{1.2}$$

where v is the circumferential displacement of the mid-surface; M_x is the bending moment; and N_x is the axial force.

To study the dynamic behavior of the shell under combined loading, we will restrict ourselves to a trimonial approximation of the deflection w:

$$w = f_1(t)\cos sy \sin\lambda x + f_2(t)\sin sy \sin\lambda x + f_3(t)\sin^4\lambda x, \tag{1.3}$$

where $f_k(t)$ ($k = 1, 2, 3$) are unknown functions of time; $\lambda = m\pi/l$ and $s = n/R$ are wave numbers.

Expansion (1.3) is valid in the case where the shell with fluid has no close and multiple frequencies [10, 14, 15]. Otherwise (if there are internal resonances [1, 7, 15, 20]), the deflection function w should include more modes. The last term in (1.3) accounts for the effect of preferential inward buckling [3, 7], which was discovered experimentally [5] and is characteristic of nonlinear vibrations.

Substituting (1.3) into the second equation in (1.1), we find the stress function Φ in the form

$$\Phi = \Phi_p + \Phi_0, \tag{1.4}$$

where Φ_p is a partial solution of the equation, and

$$\begin{aligned}\Phi_p &= \Phi_1\cos sy \sin\lambda x + \Phi_2\sin sy \sin\lambda x + \Phi_3\cos 2\lambda x + \Phi_4\cos 2sy + \Phi_5\sin 2sy \\ &+ \Phi_6\cos sy \sin 3\lambda x + \Phi_7\sin sy \sin 3\lambda x + \Phi_8\cos 4\lambda x + \Phi_9\cos sy \sin 5\lambda x + \Phi_{10}\sin sy \sin 5\lambda x,\end{aligned} \tag{1.5}$$

Φ_0 is a homogeneous solution of this equation [3],

$$\Phi_0 = -\frac{\tilde{N}_x y^2}{2} - \frac{Kx^2}{2}. \tag{1.6}$$

The functions Φ_k ($k = 1, 2, \ldots, 10$) on the right-hand side of Eq. (1.5) are expressed in a certain manner in terms of the displacement functions f_1, f_2, f_3 in (1.3), the wave numbers, and the physical and geometrical parameters of the shell,

$$\Phi_1 = \frac{E\lambda^2 f_1}{R\Delta(\lambda, s)}(1 - Rs^2 f_3), \quad \Phi_2 = \frac{E\lambda^2 f_2}{R\Delta(\lambda, s)}(1 - Rs^2 f_3) \text{ etc.,} \tag{1.7}$$

where $\Delta(A, B) = (A^2 + B^2)^2$ is an operator.

The unknown function K appearing in (1.6) is determined from the periodicity condition for the circumferential displacement v (the condition that the shell is closed [3]):

$$\int_0^{2\pi R} \frac{\partial \mathrm{v}}{\partial y} dy = \int_0^{2\pi R} \left[\frac{1}{E}\left(\frac{\partial^2 \Phi}{\partial x^2} - \mu \frac{\partial^2 \Phi}{\partial y^2} \right) - \frac{1}{2}\left(\frac{\partial w}{\partial y} \right)^2 + \frac{w}{R} \right] dy = 0. \tag{1.8}$$

As a result, we obtain

$$K = \mu \tilde{N}_x - \frac{Es^2}{8}\left[\left(f_1^2 + f_2^2\right) - \frac{3f_3}{Rs^2}\right]. \tag{1.9}$$

The hydrodynamic pressure can be found from the following approximate formula [4]:

$$P_{\mathrm{h}} = -\rho_0 \left.\frac{\partial \varphi}{\partial t}\right|_{r=R}, \tag{1.10}$$

where ρ_0 is the density of the fluid, which is supposed perfect and incompressible; $\varphi = \varphi(x, y, \Theta)$ is the velocity potential, which can be found by solving the boundary-value problem

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \Theta^2} = 0,$$

$$\left.\frac{\partial \varphi}{\partial r}\right|_{r=R} = -\frac{\partial w}{\partial t}, \quad \left.\frac{\partial \varphi}{\partial r}\right|_{r=0} < \infty, \quad \frac{\partial \varphi}{\partial t} = 0 \quad \text{at} \quad x = 0, \quad x = l, \tag{1.11}$$

where x, r, Θ are cylindrical coordinates ($0 \le x \le l$, $0 \le r \le R$, $0 \le \Theta \le 2\pi$).

The potential φ has the following analytic expression [12]:

$$\varphi = -(\dot{f}_1 \cos sy + \dot{f}_2 \sin sy)\frac{I_n(\lambda r)}{\lambda I_n'(\lambda R)}\sin \lambda_m x - \frac{\dot{f}_3}{2l}\sum_{k=1,3,5\ldots}^{\infty} M_k \frac{I_0(\lambda_k r)}{\lambda_k I_0'(\lambda_k R)}\sin \lambda_k x. \tag{1.12}$$

The parameter M_k is given by:

$$M_k = \frac{192\lambda^4}{\lambda_k(\lambda_k^2 - 4\lambda^2)(\lambda_k^2 - 16\lambda^2)}, \quad \lambda_k = \frac{k\pi}{l} \quad (k = 1, 3, \ldots). \tag{1.13}$$

Without loss of generality, we will assume that the external radial pressure is distributed in one of the modes in (1.3), namely,

$$q_0(x, y) = Q_0 \cos sy \sin \lambda x \quad (Q_0 = \text{const}). \tag{1.14}$$

Substituting (1.10) and (1.14) into the first equation in (1.1) and applying the Bubnov–Galerkin method, we obtain the following system for the unknown functions $f_k(t)$ ($k = 1, 2, 3$):

$$\ddot{f}_1 + (\omega_1^2 - \alpha_1 \cos \nu t)f_1 + \varepsilon_1 \dot{f}_1 + k_1(f_1^2 + f_2^2)f_1 + k_2 f_1 f_3 + k_3 f_1 f_3^2 = Q \cos \Omega t,$$

$$\ddot{f}_2 + (\omega_2^2 - \alpha_2 \cos \nu t)f_2 + \varepsilon_2 \dot{f}_2 + k_1(f_1^2 + f_2^2)f_2 + k_2 f_2 f_3 + k_3 f_2 f_3^2 = 0,$$

$$\ddot{f}_3 + (\omega_3^2 - \alpha_3 \cos \nu t)f_3 + \varepsilon_3 \dot{f}_3 + k_4(f_1^2 + f_2^2) + k_5(f_1^2 + f_2^2)f_3 = Q_1 \tilde{N}_x(t), \tag{1.15}$$

where ω_j ($j = 1, 2, 3$) are the natural frequencies of the shell with the added mass of the fluid,

$$\omega_1^2 = \omega_2^2 = \frac{1}{\rho m_{01}}\left(\frac{D}{h}\Delta(\lambda, s) + \frac{E\lambda^4}{R^2 \Delta(\lambda, s)}\right) - \frac{N_0}{\rho m_{01}}(\lambda^2 + \mu s^2),$$

$$\omega_3^2 = \frac{64}{35\rho m_{03}}\left(\frac{8D\lambda^4}{h} + \frac{35}{64}\frac{E}{R^2}\right) - \frac{16}{7}\frac{N_0 \lambda^2}{\rho m_{03}}, \quad \alpha_1 = \alpha_2 = \frac{N_1}{\rho m_{01}}(\lambda^2 + \mu s^2),$$

$$\alpha_3 = \frac{16}{7}\frac{N_1\lambda^2}{\rho m_{03}}, \quad \varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{m_{01}}, \quad \varepsilon_3 = \frac{\varepsilon}{m_{03}}, \quad m_{01} = 1 + \frac{\rho_0}{\rho}\frac{I_n(\lambda R)}{\lambda h I_n'(\lambda R)},$$

$$m_{03} = 1 + \frac{16}{35}\frac{\rho_0}{\rho h l^2}\sum_{k=1,3,5,\ldots}^{\infty}\frac{I_0(\lambda_k R) M_k^2}{\lambda_k I_0'(\lambda_k R)}, \quad Q = \frac{Q_0}{\rho h m_{01}}, \quad Q_1 = -\frac{48\mu}{35\rho R m_{03}}. \tag{1.16}$$

The constant coefficients $k_1, k_2, \ldots, k_5$ in Eqs. (1.15), which characterize the geometrical nonlinearity of the shell, are given by

$$k_1 = \frac{E}{16\rho m_{01}}(\lambda^4 + 3s^4), \quad k_2 = -\frac{Es^2}{\rho R m_{01}}\left[\frac{5}{8} + \frac{2\lambda^4}{\Delta(\lambda, s)}\right],$$

$$k_3 = \frac{E\lambda^4 s^4}{\rho m_{01}}\left[\frac{1}{\Delta(\lambda, s)} + \frac{4}{\Delta(3\lambda, s)} + \frac{1}{\Delta(5\lambda, s)}\right], \quad k_4 = \frac{16}{35}\frac{m_{01}}{m_{03}}k_2, \quad k_5 = \frac{32}{35}\frac{m_{01}}{m_{03}}k_3. \tag{1.17}$$

Since the resonant modes of vibration of the shell are of great practical interest, we will assume that the resonance relations $\omega_1 \approx \Omega$ and $\omega_1 \approx \nu/2$ hold. They follow from the first approximation of Eqs. (1.15) [1]. We will also somewhat simplify system (1.15) by taking into account the inequalities $f_3 \ll f_1$ and $f_3 \ll f_2$ known from the nonlinear theory of shells [3, 7] and determine the function f_3 from the quasistatic problem. Doing this gives the approximate formula

$$f_3 \approx -\frac{k_4(f_1^2 + f_2^2)}{\omega_3^2}, \tag{1.18}$$

which is valid because the nonlinear terms in (1.15) are small compared with the linear terms. Moreover, it follows from physical considerations [1] that the amplitudes of axial pulsations are also proportional to a small parameter.

Substituting (1.18) into (1.15) and retaining nonlinear terms up to the third power of the generalized coordinates f_k (k = 1, 2), we derive the following system of equations:

$$\ddot{f}_1 + (\omega_1^2 - \alpha_1 \cos\nu t) f_1 + \varepsilon_1 \dot{f}_1 + \gamma(f_1^2 + f_2^2) f_1 = Q\cos\Omega t,$$

$$\ddot{f}_2 + (\omega_2^2 - \alpha_2 \cos\nu t) f_2 + \varepsilon_2 \dot{f}_2 + \gamma(f_1^2 + f_2^2) f_2 = 0, \tag{1.19}$$

where $\gamma = k_1 - k_2 k_4 / \omega_3^2$.

The nonlinear equations (1.19) are a starting point for studying the dynamic deformation of a fluid-filled shell under combined periodic loading.

2. Averaged Equations. To find periodic solutions of system (1.19), we will examine the case $\nu = 2\Omega \approx \omega_1$ that provides preconditions for simultaneous excitation of parametric vibrations in the zone of demultiplication resonance and forced resonant vibrations with frequency Ω. In the first approximation, these solutions are sought in the following form [1]:

$$f_1 = u_1 \cos\Omega t + u_2 \sin\Omega t, \quad f_2 = u_3 \cos\Omega t + u_4 \sin\Omega t, \tag{2.1}$$

where u_i (i = 1, 2, 3, 4) are unknown functions of time that can be found by averaging. Applying the averaging method, we obtain the system of equations

$$\frac{du_1}{d\tau} = -\beta u_1 + M_1 u_2 + T_2 K u_3, \quad \frac{du_2}{d\tau} = -\beta u_2 - M_2 u_1 + T_2 K u_4 + Q,$$

$$\frac{du_3}{d\tau} = -\beta u_3 + M_1 u_4 - T_2 K u_1, \quad \frac{du_4}{d\tau} = -\beta u_4 - M_2 u_3 - T_2 K u_2, \tag{2.2}$$

where

$$\beta=\varepsilon_1\Omega, \quad M_1=M+H, \quad M_2=M-H, \quad M=\Delta+T_1A^2, \quad H=\alpha_1/2,$$

$$\Delta=\omega_1^2-\Omega^2, \quad K=u_1u_4-u_2u_3, \quad A^2=(u_1^2+u_2^2+u_3^2+u_4^2)/2,$$

$$T_1=3\gamma/2, \quad T_2=\gamma/2, \quad \tau=t/(2\Omega). \tag{2.3}$$

To find the stationary solutions $u_k=u_{k0}$ ($k=1, 2, 3, 4$), we will equate, as usual, the right-hand sides of Eqs. (2.2) to zero. If $K_0=u_{10}u_{40}-u_{20}u_{30}\neq 0$, then these solutions will have the following form in the general case ($Q\neq 0$, $H\neq 0$):

$$u_{10}=\frac{QM_{10}S_1\Delta_0}{\Delta_{00}}, \quad u_{20}=\frac{Q\beta S_2\Delta_0}{\Delta_{00}}, \quad u_{30}=-\frac{2Q\beta\Delta_0M_{10}T_2K_0}{\Delta_{00}},$$

$$u_{40}=\frac{QT_2K_0(M_{10}M_{20}S_1-\beta^2S_2)}{\Delta_{00}}, \tag{2.4}$$

where

$$M_{10}=M_0+H, \quad M_{20}=M_0-H, \quad M_0=\Delta+T_1A_0^2, \quad A_0^2=\frac{1}{2}\sum_{k=1}^{4}u_{k0}^2,$$

$$S_1=\Delta_0-T_2^2K_0^2, \quad S_2=\Delta_0+T_2^2K_0^2, \quad \Delta_0=\beta^2+M_{10}M_{20}, \quad \Delta_{00}=\beta^2S_2^2+M_{10}M_{20}S_1^2. \tag{2.5}$$

Equations (2.2) yield the additional relation

$$K_0^2=\frac{-\beta^2-M_{10}(\Delta+H+T_2A_0^2)}{T_2^2}. \tag{2.6}$$

Substituting (2.6) into (2.4) and taking notation (2.5) into account, we derive an equation relating the total amplitude A_0 of steady-state flexural vibrations and the frequency Ω (or ν) of external periodic load. If $K_0\neq 0$, this equation will be as follows:

$$T_2Q^2=2T_2A_0^2[M_0(M_0-5H)+(M_0+H)(\Delta+5T_2A_0^2)]+4(\Delta+H)M_0^2+4\beta^2(M_0-H). \tag{2.7}$$

It is obvious that this equation can have several real roots $A_0^2=A_0^2(\Delta)$. To ascertain whether they are stable or not, it is necessary to analyze the variational equations set up for system (2.2) in view of solutions (2.4). Assuming, for example [8] that

$$u_k=u_{k0}+A_ke^{\lambda_0\tau} \quad (k=1, 2, 3, 4), \tag{2.8}$$

where A_k are small variations, λ_0 is a characteristic number, we derive the following characteristic equation from (2.2):

$$\lambda_0^4+c_1\lambda_0^3+c_2\lambda_0^2+c_3\lambda_0+c_4=0, \tag{2.9}$$

where c_j ($j=1, 2, 3, 4$) are constant coefficients,

$$c_1=4\beta, \quad c_2=6\beta^2+d_1, \quad c_3=2\beta(d_1+2\beta^2), \quad c_4=\beta^2(d_1+\beta^2)+d_2, \tag{2.10}$$

and

$$d_1=d_1(A_0)=2M_0[M_0+(T_1-T_2)A_0^2]+\beta^2+M_0(\Delta+T_2A_0^2)+7H^2+10H(\Delta+2T_2A_0^2),$$

$$d_2=d_2(A_0)=M_0^4+2(T_1-T_2)M_0^3A_0^2-2T_2^2(T_1-T_2)K_0^2M_0A_0^2$$

$$+4T_1T_2(K_0^2-A_0^4)(M_0^2-T_2^2K_0^2)+2T_2(T_1-2T_2)K_0^2M_0^2+T_2^3(3T_2-2T_1)K_0^4-7H^4-8H^3(\Delta-2T_2A_0^2)$$

$$+H^2[2(T_2-T_1)M_0A_0^2-2M_0^2+12(\Delta-2T_2A_0^2)-2T_2^2K_0^2]+8H(M_0^2-T_2^2K_0^2)(\Delta+H+2T_2A_0^2). \tag{2.11}$$

The solutions $A_0^2(\Delta)$ and the associated vibrations of the shell will be stable if the real parts of all the roots of Eqs. (2.9) are nonpositive. The stability criteria are as follows ($\beta > 0$) [8]:

$$2\beta^2+d_1>0, \quad (\beta^2+d_1)\beta^2+d_2>0, \quad (4\beta^2+d_1)^2-4d_2>0. \tag{2.12}$$

3. Special Cases.

3.1. Forced Vibrations of Fluid-Filled Shells. Let $N_1 = 0$ and $Q \neq 0$. In this case, it is necessary to set $H = 0$ in (2.7) and (2.12). Equation (2.7) predicts two qualitatively different responses of the shell to radial pressure. One is such relationship between the parameters u_{k0} that $K_0 = 0$, resulting in the following amplitude–frequency response (AFR):

$$\Delta=-T_1A_0^2\pm\sqrt{\frac{Q^2}{2A_0^2}-\beta^2}, \tag{3.1}$$

which is typical for the theory of forced single-mode vibrations of shells [1, 12]. To establish the stable segments of the AFR, it is necessary to set

$$d_1=2M_0[M_0+(T_1-T_2)A_0^2], \quad d_2=M_0^2[M_0^2+2(T_1-T_2)M_0A_0^2-4T_1T_2A_0^4] \tag{3.2}$$

in the stability criteria (2.12).

If $K_0 \neq 0$, i.e., when $\beta^2 + M_0(\Delta + T_2A_0^2) \neq 0$, then the AFR will be different: the amplitude A_0 and frequency mismatch Δ are related in a more complex manner (compared with (3.1)):

$$e_0A_0^6+e_1A_0^4+e_2A_0^2+e_3=0, \tag{3.3}$$

where

$$e_0=T_1(T_1-T_2)^2, \quad e_1=(3T_1-T_2)(T_1-T_2)\Delta,$$

$$e_2=(3T_1-2T_2)\Delta^2+T_1\beta^2, \quad e_3=(\beta^2+\Delta^2)\Delta-\frac{T_2}{4}Q^2. \tag{3.4}$$

Whether the roots of the last equation are stable ($A_0^2 = A_0^2(\Delta)$) is established using conditions (2.12) where $H = 0$.

3.2. Parametric Vibrations. Let $Q = 0$ and $N_1 \neq 0$ ($H \neq 0$). In this case, the shell is parametrically excited in the zone of the principal parametric resonance $\omega_1 \approx \nu/2$ [1]. The solution for the functions f_1 and f_2 is as follows:

$$f_1=u_1\cos\frac{\nu t}{2}+u_2\sin\frac{\nu t}{2}, \quad f_2=u_3\cos\frac{\nu t}{2}+u_4\sin\frac{\nu t}{2}. \tag{3.5}$$

The AFR is described by the equations

$$\Delta=-T_1A_0^2\pm\sqrt{H^2-\beta^2} \quad \text{if} \quad K_0=0, \tag{3.6}$$

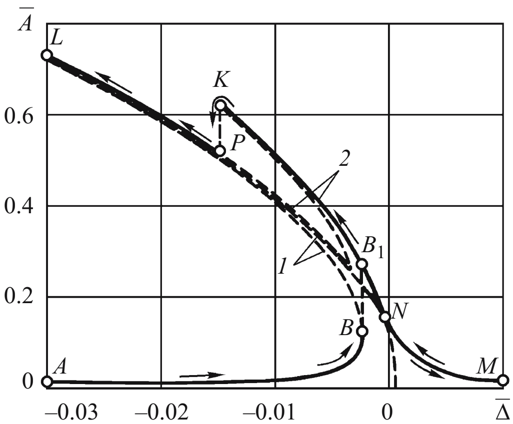
$$\Delta=-\frac{p_1}{2}\pm\sqrt{\frac{p_1^2}{4}-p_2} \quad \text{if} \quad K_0\neq 0. \tag{3.7}$$

Here $K_0^2=(\beta^2+M_{10}M_{20})/T_2^2$ and

$$p_1=H+\frac{1}{2}(3T_1+T_2)A_0^2, \quad \Delta=\omega_1^2-\frac{\nu^2}{4}, \quad \beta=\frac{\varepsilon_1\nu}{2},$$

TABLE 1

ω_1, Hz	n					
	3	4	5	6	7	8
ω_{11}	227.15	157.53	126.41	**124.83**	147.39	187.55
ω_{12}	223.88	148.43	105.89	**89.65**	102.44	138.72



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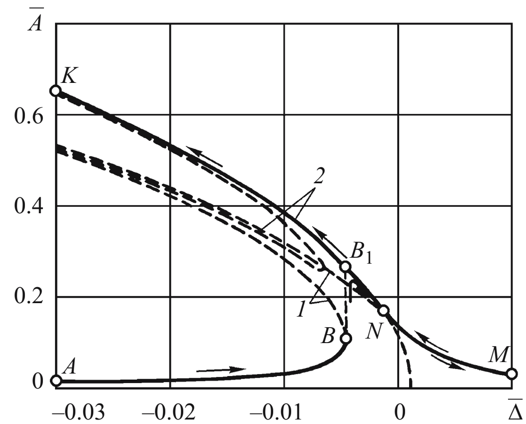


b

Fig. 1

$$p_2 = \beta^2 + \frac{T_1}{2}(T_1 + T_2)A_0^4 + \frac{H}{2}(T_1 + T_2)A_0^2. \tag{3.8}$$

As in the previous case, whether the solutions found from (3.6) and (3.7) are stable is established using criteria (2.12) where $K_0 = 0$ and $K_0 \neq 0$, respectively.

4. Numerical Example. Consider a shell filled with a fluid with $\rho_0 = 10^3$ kg/m^3 and having the following characteristics: $E = 2\cdot10^{11}$ Pa, $\rho = 7.8\cdot10^3$ kg/m^3, $h/R = 4\cdot10^{-3}$, $l/R = 2.5$, $\mu = 0.3$, $R = 0.16$ m.

Table 1 collects the lower natural frequencies ω_1 of the shell deforming in one longitudinal half-wave mode ($m = 1$) under axial load N_x with the following values of the static component N_0: $N_0 = 0$ (ω_{11}) and $N_0 = 3\cdot10^7$ Pa (ω_{12}). It can be seen that the flexural mode at the minimum natural frequency is characterized by the same circumferential wave number $n = 6$ in both cases ($N_0 = 0$ and $N_0 \neq 0$). It is this mode that we will consider below in studying the response of the shell to external periodic loads. This mode is known to require minimum energy to be excited compared with other modes [3].

For the shell undergoing purely forced vibrations ($H = 0$), Fig. 1 illustrates the AFRs plotted from Eqs. (3.1) (curves *1*) and Eqs. (3.3) (curves *2*) for $Q_0 = 1$ Pa, $\varepsilon_1 = 0.64$ sec^{-1}, $\bar{A} = A_0/h$, and $\bar{\Delta} = \Delta/\omega_1^2$. Figures 1*a* and 1*b* correspond to $N_0 = 0$ (zero static force) and $N_0 = 3\cdot10^7$ Pa.

The stable and unstable segments of the curves are established by criteria (2.12) and shown by solid and dashed lines, respectively.

An analysis of the above results reveals a qualitative difference between the cases of $K_0 = 0$ and $K_0 \neq 0$. As follows from (2.4), the former case takes place when $u_{30} = u_{40} = 0$, which corresponds to the steady-state dynamic deflection w (1.3) in the form of a standing wave [12] with the axisymmetric term $w_{11}(x)$:

$$w(x, y) = (u_{10} \cos \Omega t + u_{20} \sin \Omega t) \cos sy \sin \lambda x + w_{11}(x),$$

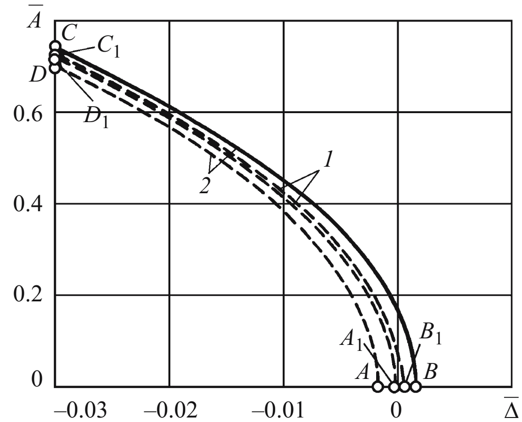
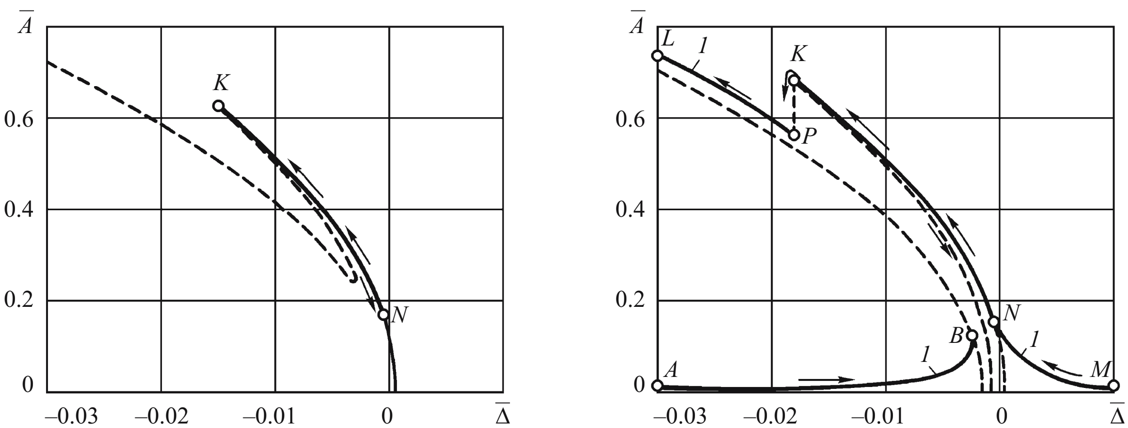


Fig. 2



a *b*

Fig. 3

$$w_{11}(x) = -\frac{k_4}{\omega_3^2}(u_{10}\cos\Omega t + u_{20}\sin\Omega t)^2 \sin^4 \lambda x.$$

The latter case takes place when $u_{10}u_{40} \neq u_{20}u_{30}$ and corresponds to a generalized (with variable amplitude and phase) circumferential traveling wave [13, 19].

Figure 1*a* demonstrates that as the external excitation frequency Ω is slowly varied, the deformation mode of the shell in the resonance zone changes between a standing wave and a traveling wave. The segments *AB*, *MN*, and *LP* of the AFR correspond to the standing wave, while the segment *NK* to the traveling wave. The change from one deformation mode to another is either smooth or abrupt, depending on whether the frequency Ω is increased or decreased. It should also be noted that the longitudinal static compression of the shell not only shifts the AFR toward lower values of Ω, but also strengthens the effect of geometrical nonlinearity (the skeletal curve becomes less steep).

For the shell undergoing parametric vibrations ($Q = 0$), Fig. 2 shows the AFR plotted from Eq. (3.6) with $H = 10^2$ sec^{-2} (curves *1*) and $H = 10^3$ sec^{-2} (curves *2*) for $N_0 = 0$.

As shown in [7], if Eqs. (1.15) neglect the interaction of the modes, then the upper branches *BC* and B_1C_1 will be completely stable. Otherwise, these branches will coincide with one of the instability boundaries, which can be remedied by incorporating small geometrical imperfections of the shell into Eqs. (1.19) [7].

Figure 3 illustrates the evolution of the AFR of the shell under both longitudinal and transverse periodic loading for $K_0 \neq 0$, which is the most interesting case, and for $H = 10^2$ sec^{-2} (Fig. 3*a*) and $H = 10^3$ sec^{-2} (Fig. 3*b*). The traveling waves are stable at excitation frequencies $\Omega = \nu / 2$ at which the AFR has the segment *KN*. Beyond it, we have an ordinary standing wave (single-mode).

When $N_0 \neq 0$, the results will be qualitatively similar to those presented above, with the only difference that the AFR is shifted along the frequency axis and tilted somewhat.

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