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VIBRATIONS OF A RIGID BODY WITH A CONTROLLED FRICTIONAL ELECTROMAGNETIC SEISMIC DAMPER: NONLINEAR MODEL

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A nonlinear mathematical model of a gravitational vibratory system with a controlled electromagnetic seismic damper is developed. The dependence of the force of attraction of ferromagnetic bodies by the solenoid of the frictional device on the solenoid current is established for a specific solenoid design. The analytic expression for this force is derived by the least-squares method using a system of continuous piecewise-linear functions. It is used to describe the pressure of the solenoid on the friction surface. For multifrequency inertial excitation, the acceleration amplification factor is evaluated depending on the time constant and gain for two cases of control. The possibility of damping vibrations by controlling the absolute velocity and acceleration is established

Keywords: gravitational vibratory system, electromagnetic seismic damper, nonlinear model, feedback circuit, least-squares method, multifrequency inertial excitation

Introduction. Seismic isolation mechanisms (SIMs) have been used for earthquake protection since the second half of the 20th century [5, 6, 11, 14–17, 20–23]. Such mechanisms are designed to weaken the coupling between protected structures and ground and to dissipate the energy of seismic disturbances. Controlled semiactive seismic dampers have come to be used in recent decades [13–17, 22]. Semiactive control systems fall into the class of systems in which control is used to change the physical and mechanical properties of shock-absorbing elements of damping devices. Such devices may include electro- and magnetorheological liquid materials whose parameters are strongly dependent on the electric and magnetic fields applied to them. Such controlled devices are quite difficult to manufacture and operate. However, controlled damping can easily be produced by using assemblies with controlled pressure between frictional elements that contain solenoids with ferromagnetic cores, and there is no need for any liquid rheological materials. The cores slide over ferromagnetic surfaces, overcoming the Coulomb force proportional to the contact pressure and dependent on the ampere-turns of the solenoid [3]. To determine the contact force of attraction between arbitrary electromagnets, use is made of a theoretical-and-experimental method for each design of electromagnetic damping device.

Here we use a theoretic-and-experimental method to find the contact force of interaction between a solenoid core and a ferromagnetic body. We will model the forced vibrations of a system with a controlled electromagnetic frictional damper of a one-degree-of-freedom object for two different control laws for the solenoid current.

1. Mathematical Model of a Vibrating System with a Feedback-Controlled Electromagnetic Frictional Damper. Figure 1 shows a vibratory system consisting of a rigid body of radius R having semispherical hollows and resting on spherical supports of radius $r < R$ that allow horizontal and vertical movements of the body over the horizontal ferromagnetic surface of the platform. The body has a solenoid with a ferromagnetic core freely sliding along the vertical guide and contacting with the rough horizontal surface.

If there is a current I in the winding of the solenoid, its shoe is pressed to the friction surface by a force $N(I)$. This generates a sliding friction force F_{fr} opposite to the horizontal velocity of the moving mass,

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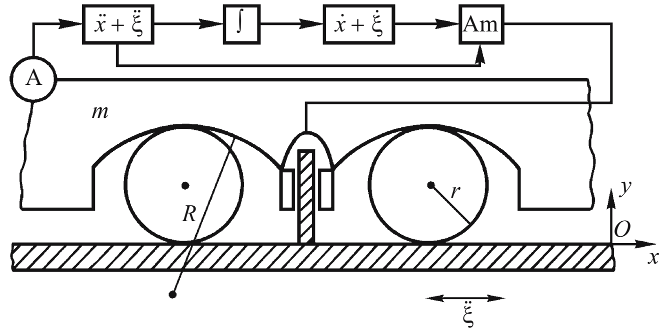

Fig. 1

$$F_{\text{fr}} = fN(I)\operatorname{sign}\dot{x}, \tag{1.1}$$

where f is the coefficient of sliding friction; $N(I) = N(-I)$ is a continuous even function of I; $\dot{x} = \dfrac{dx}{dt}$ is the velocity of the body. A feedback control system [9, 18, 19] that includes an accelerometer, an integrator, and an amplifier can be used to set the solenoid current proportional to the absolute acceleration or velocity of the body. To this end, it is necessary to measure the absolute acceleration and integrate it once. The feedback circuit is generally inertial; therefore, to obtain transparent results, we assume it to be a first-order lag [9] with a perfect integrator. Then the solenoid current and the measured parameter can be related by

$$T\frac{dI}{dt} + I = \begin{cases} K_{\text{v}}\dfrac{d}{dt}(x+\xi(t)), \\ K_{\text{a}}\dfrac{d^2}{dt^2}(x+\xi(t)), \end{cases} \tag{1.2}$$

where T, K_{a} (K_{v}) are the time constants and gain of the current control system; $\xi(t)$ is the law of motion of the platform. The first relation in (1.2) represents the velocity control, while the second relation the acceleration control. It is obvious that a signal proportional to the absolute velocity of the body can only be found by integrating the absolute acceleration measured by the accelerometer.

Let us supplement relations (1.2) with the differential equation of horizontal motion of the system of bodies:

$$m_1\frac{d^2x}{dt^2} + fN(I)\operatorname{sign}\frac{dx}{dt} + cx = -m_1\frac{d^2\xi}{dt^2}, \tag{1.3}$$

where m_1 is the mass of all moving elements referred to the horizontal axis, $c = \dfrac{gm}{4(R-r)}$ is the gravitational stiffness of the body (see Addendum); m is the mass of the body; $g = 9.81$ m/sec^2. With $T \neq 0$, the system of equations (1.2), (1.3) can be reduced to Cauchy form. It will permit numerical integration if the function $N(I)$ satisfies the Lipschitz property. A piecewise-linear approximation of $N(I)$ makes the system of equations multistructural, each structure being a system of linear differential equations with constant coefficients, which allows analytic integration by joining the solutions at the points of discontinuity of the function $N(I)$. The system of equations in Cauchy form has discontinuous right-hand sides and, if f is small, can be analytically integrated by methods intended for such systems [1, 4, 8].

In the case of acceleration control, system (1.2), (1.3), if $T = 0$, is reduced to a second-order equation not resolved for the highest derivative:

$$\frac{d^2x}{dt^2} + \bar{f}N\left(K_{\text{a}}\left(\frac{d^2x}{dt^2} + \frac{d^2\xi(t)}{dt^2}\right)\right)\operatorname{sign}\frac{dx}{dt} + \omega^2 x = -\frac{d^2\xi}{dt^2} \quad (\omega^2 = c/m_1,\ \bar{f} = f/m_1). \tag{1.4}$$

Since it is difficult to solve such equations, the system can be reduced, under certain conditions, to Cauchy form and integrated if the function $N(I)$ is even and piecewise-linear and the parameter $\bar{f} << 1$ is small. Indeed, we set $N(I) = n_1|I|$ and take into account the Coulomb force $p_{\text{s}} f \operatorname{sign}\dot{x}$ generated by the weight p_{s} of the dead solenoid. Then Eq. (1.4) becomes

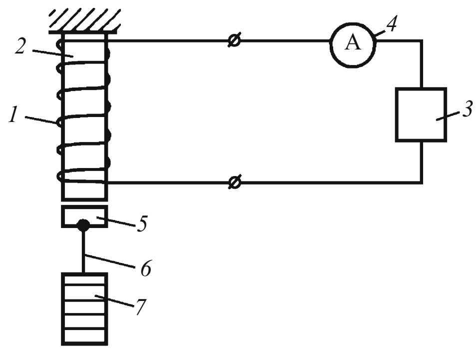

Fig. 2

$$(\ddot{x}+\ddot{\xi}(t))(1+\lambda_a \operatorname{sign}(\dot{x}(\ddot{x}+\ddot{\xi}(t))))+\lambda_0 \operatorname{sign}\dot{x}+\omega^2 x=0$$

$$(\lambda_a = K_a fn_1 / m_1, \lambda_0 = fp_s / m_1). \tag{1.5}$$

Where the obvious identity $\operatorname{sign}(ab)=\operatorname{sign}a\cdot\operatorname{sign}b$ is used. This equation can be resolved for $\ddot{x}$ only if $\lambda_a < 1$. In this case, sign $(\ddot{x}+\ddot{\xi}) = -\operatorname{sign}E(x,\dot{x})$, where $E(x,\dot{x})=\omega^2 x+\lambda_0 \operatorname{sign}\dot{x}$.

Finally, we obtain the following system of equations in Cauchy form with discontinuous right-hand side:

$$\dot{x}=y, \qquad \dot{y}=-\left(\frac{E(x,y)}{1-\lambda_a \operatorname{sign}(yE(x,y))}+\ddot{\xi}\right). \tag{1.6}$$

Equations (1.6) and (1.5) are valid unless the coordinate x is in the stagnation zone $|\omega^2 x+\ddot{\xi}(t)|>\lambda_0$, $y=0$. If λ_0, λ_a is small, Eq. (1.6) has small discontinuities and can approximately be integrated by the asymptotic or averaging method [1, 4, 9] or numericaly. For $\xi(t)\equiv 0$, Eq. (1.5) can be analytically integrated by joining the solutions.

Here the question arises, which of the two control laws (1.2) more effectively damps the vibrations of the body. The answer is relatively simple in the case of natural vibrations ($\dot{\xi}(t)=\ddot{\xi}(t)=0$). The force of friction vanishes only at $\dot{x}=0$ in the former case and at $\dot{x}=0$ and $\ddot{x}=0$ in the latter case. Hence, the latter law is less effective. However, this issue should specially be examined in the case of forced multifrequency vibrations ($\dddot{\xi}(t)\neq 0$).

Thus, to analyze the vibrations of the body, it is necessary, according to Eqs. (1.2) and (1.3), to set the force $N(I)$ of contact interaction between the solenoid and the platform.

2. Contact Force of Attraction of the Solenoid by the Ferromagnetic Body and Its Experimental Determination. Assume that the force $N(I)$ is a continuous function of the variable I. An analytic expression for $N(I)$ should be selected from the condition for the magnetic-flux density in the solenoid core taking into account the nature of magnetic moments and forces [10, pp. 331, 332, 346]. According to the theory of electromagnetic phenomena in a homogeneous magnetic field, only a couple acts on the solenoid. The solenoid tends to align with the field. If, however, the magnetic field is inhomogeneous, the solenoid is subjected not only to a couple but also to a force that causes it to translate. The theory of permanent magnets predicts that the attractive force is proportional to the magnetic moment p, magnetic-field strength gradient $\Delta H/\Delta x$, and the angle of the magnetic core. It is the inhomogeneous magnetic field that is responsible for the attraction or repulsion of magnets, including solenoids.

The elevating force of an electromagnet can be determined following the engineering approach [3, p. 263]:

$$F=-\frac{\partial W}{\partial h}, \tag{2.1}$$

where $W=0.5B^2 sh/\mu_0$ is the magnetic energy in the gap of the magnetic circuit consisting of a winding with a horseshoe core, an armature, and a thin air gap between the poles and the armature; B is the magnetic-flux density; μ_0 is the magnetic permeability of air; h is the thickness of the gap; s is the surface area of the pole portion under which the magnetic field is

TABLE 1

N, N	0	0.5	1.0	1.5	2.0	3.0	4.0
I, mA	0	18.2	31.1	36.8	43.8	57.1	63.5

TABLE 2

N, N	4.6	5.1	5.4	5.6	5.8
I, mA	70	80	90	100	110

homogeneous. The minus sign in (2.1) indicates that the force tends to reduce the distance between the poles. The elevating force of an electromagnet with a mobile armature is the force needed to separate the armature from the poles.

Formula (2.1) structurally coincides with the definition of a generalized force with a potential, though magnetic fields are of vortical rather than potential nature. Formula (2.1) yields

$$F = KI^2, \quad (2.2)$$

where $K = \left(\frac{n}{R_m}\right)^2 \Big/ (\mu_0 s)$ is a design factor; n is the number of turns of the electromagnet; I is the current; R_m is the reluctance of the magnetic circuit.

Assume that the force $N(I)$ (that presses the solenoid to the ferromagnetic material) is expressed by some even function of I close to a parabola for currents that do not cause magnetic saturation of the core. Due to the contradiction between these two approaches, the characteristic $N(I)$ for a specific solenoid is expedient to obtain from measurements of the force separating ferromagnetic weights from the solenoid core.

To illustrate the situation described above, we will experimentally plot the force $N(I)$ attracting ferromagnetic bodies depending on the solenoid current. To this end, the force of separation of the solenoid core and simple ferromagnetic bodies with flat contact surfac is determined. The experimental setup is shown in Fig. 2.

An electromagnet (solenoid) consisting of coil *1* and magnetically soft steel core *2* was vertically fixed in a holder. The coil was connected to a B5-47 dc power supply unit *3*; the current strength in the coil was measured with ShCh 4313 digital device *4*. The coil had 10,000 turns of a 0.16 PÉL copper wire with a resistance of 4 kΩ. The core dimensions: outer diameter 22.5 mm, inner diameter 10 mm, length 66 mm. The weight of the solenoid was 1 N.

A fixed core with a diameter of 10 mm was set coaxially with the coil and had a thickening with a diameter of 13 mm and a thickness of 2.5 mm at the lower end. Test weights *7* were fixed with threads *6* to a steel disk (initial weight) with a diameter of 13 mm, which is in direct contact with the core of solenoid *5*. The following weights were one by one held by the electromagnet: 0.5, 1.0, 1.5, 2.0, 3.0, 4.0 N.

At the beginning of each test, a current that surely kept the weight in contact with the solenoid core was passed through the coil. Then the current was gradually decreased to the level I at which the weight dropped. Seven tests were conducted for each value of N. The results were processed by rejecting the minimum and maximum values and averaging the remaining data. The error of measurement did not exceed 10%. Tables 1 and 2 measure the weight in newtons and the current in milliampers.

For technical reasons, experimental data were only obtained for $I < 63.5$ mA (Fig. 3). The curve $N(I)$ was extrapolated to $I > 60$ mA, assuming that the bending point of curve $N(I)$ takes place for $I < 65$ mA. These data correspond to Table 2.

3. Approximation of the Contact Force of Attraction as a Function of the Current. The contact force exerted by a solenoid with a magnetically soft core to attract ferromagnetic bodies will be approximated by a piecewise-linear even function using the least-squares method. For coordinate functions, we will use even rhombic harmonics of multiple argument. Rhombic functions are a special case of periodic functions defined on closed lines of $L_4$4PC symmetry class. According to [7], rhombic harmonics are expressed as

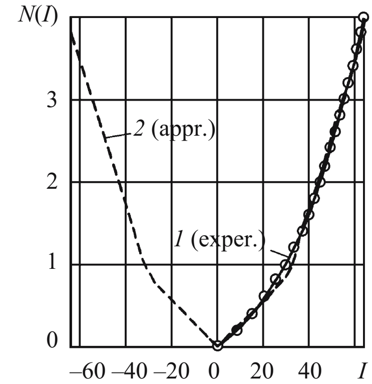

Fig. 3

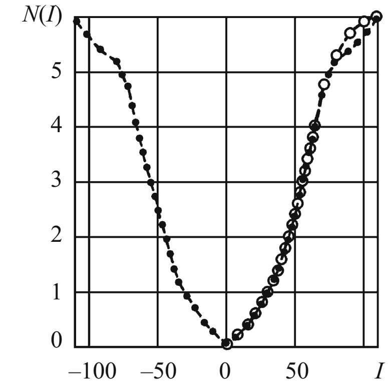

Fig. 4

$$\operatorname{cor}\sigma=\begin{cases}1-|\bar\sigma|/q, & |\bar\sigma|\le 2q,\\ |\bar\sigma|/q-3, & 2q<|\sigma|\le 4q,\end{cases}$$

$$\operatorname{sir}\sigma=\begin{cases}\bar\sigma/q, & 0\le|\bar\sigma|\le q,\\ 2\operatorname{sign}\bar\sigma-\bar\sigma/q, & q<|\bar\sigma|\le 3q,\\ \bar\sigma/q-4\operatorname{sign}\bar\sigma, & 3q<|\bar\sigma|\le 4q,\end{cases} \tag{3.1}$$

$q=\sqrt{2}, \bar\sigma=\sigma-E(\sigma/4/q)4q, E(x)$ is the integer part of x; σ is a real number. Assuming that the function $N(I)$ is even in the range $|I|\le I_{\max}$, we expand it into a series of even rhombic harmonics:

$$N(I)=b_0-\sum_{l=1}^{n} b_l \operatorname{cor}(l\sigma(I)), \tag{3.2}$$

where the minus sign makes the steepness of $N(I)$ positive for $I=0$,

$$\sigma(I)=2qI/I_{\max}. \tag{3.3}$$

According to formula (3.3), as the current I varies in the range $-I_{\max}\le I\le I_{\max}$, the dimensionless argument of the function $\operatorname{cor}\sigma(I)$ changes within its period: $-2q\le\sigma\le 2q$.

The obvious physical condition $N(0)=0$ must be satisfied. It yields

$$b_0-\sum_{l=1}^{n} b_l=0. \tag{3.4}$$

Finding b_0 from (3.4) and substituting it into (3.2), we obtain the approximating formula

$$N(I)=\sum_{l=1}^{n} b_l(1-\operatorname{cor} l\sigma(I)). \tag{3.5}$$

It vanishes at $I=0$.

For a curve of sign-definite curvature such as a parabola, this formula can be restricted to two terms containing b_1 and b_2. This approximation is called birhomboparaboloidal. Figure 4 shows the birhomboparaboloidal approximation of the contact force for $I_{\max}=63.5$ mA ($b_1=1.9$, $b_2=-0.5$). The salient point indicates the beginning of intensive orientation of molecular currents of the solenoid core along the vector of magnetic-field strength.

Let us determine the tangents of the angles of segments of the birhomboparaboloidally approximated curve $N=N(I)$. According to formulas (3.1) and (3.5), for the initial segments of $N(I)$ for $0\le|\sigma(I)|\le q$, we have $N(I)=(b_1+2b_2)|\sigma(I)|/q$, whence

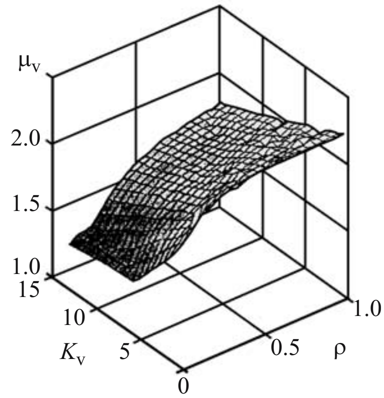

Fig. 5

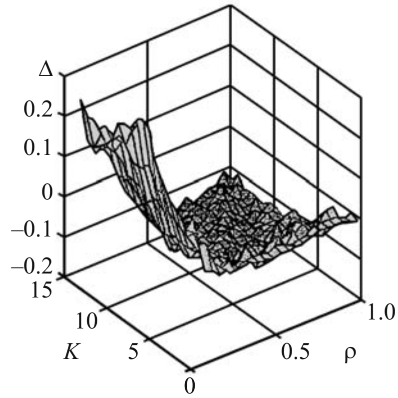

Fig. 6

$$\tan\alpha_1(I) = \frac{dP}{dI} = (b_1 + 2b_2)\frac{2}{I_{\max}}\operatorname{sign} I \quad (0 \le I \le I_{\max}/2).$$

For $0 < |\sigma(I)| \le 2q$, according to the same formulas, we have $N(I) = 4b_2 + (b_1 - 2b_2)\frac{|\sigma(I)|}{q}$.

The tangent of the second segment is calculated by the formula

$$\tan\alpha_2(I) = (b_1 - 2b_2)\frac{2}{I_{\max}}\operatorname{sign} I \quad (I_{\max}/2 \le I \le I_{\max}).$$

Figures 3 and 4 show experimental results by circles (curve *1*) and approximating curve (dashed curve *2*).

The data in Tables 1 and 2 for $0 \le I \le 110$ mA are approximated by expression (3.5), where $n = 6$. The six-dimensional vector of parameters b obtained by the least-squares method is as follows:

$$b = (3.6 \quad 1.1\cdot 10^{-1} \quad -6.7\cdot 10^{-1} \quad 6.4\cdot 10^{-2} \quad -1.3\cdot 10^{-2} \quad -8.4\cdot 10^{-2}). \tag{3.6}$$

The approximating curve $N(I)$ is shown in Fig. 3. This curve has three almost linear segments corresponding to the magnetization due to ampere-turns of the solenoid and the molecular currents in its core.

The numerical integration of the nonlinear equations (1.2), (1.3) gives the acceleration amplification factor

$$\mu = \max_t |\ddot{x}(t) + \ddot{\xi}(t)| / \max_t |\ddot{\xi}(t)|.$$

The reference-frame acceleration is represented by superposition of damped harmonics:

$$\ddot{\xi}(t) = -\sum_{k=1}^{M} a_k e^{-\lambda_k t}[\lambda_k \cos(\nu_k t + \alpha_k) + \nu_k \sin(\nu_k t + \alpha_k)].$$

The curve $N(I)$ was obtained for $n = 6$ and b presented above for Tables 1 and 2. The other parameters are as follows: $f/m_1 = 0.5$, $a_k = a = 1$, $\alpha_k = \frac{2\pi}{M}(k-1)$, $\lambda_k = \lambda = 0.05$, $\omega = 1$, $\nu_k = \frac{\omega}{M}k\ (k = \overline{1, M})$, $M = 4$, $K_v \in [0.5, 15]$, $T = \frac{2\pi}{\omega}\rho\ (\rho \in [0.1, 1])$. The initial conditions are zero: $x(0) = 0$, $\dot{x}(0) = 0$, $I(0) = 0$.

Figure 5 shows the surface $\mu_v = \mu(K_v, \rho)$. As is seen, the factor μ_v decreases as the gain K_v increases and the time constant $T(\rho)$ decreases.

In the case of acceleration control of the damper, we calculate the function $\mu = \mu_a(K_a, \rho)$ for the same values of ρ and $K_a = K_v = K$ and set up the difference $\mu_v(K_v, \rho) - \mu_a(K_a, \rho) = \Delta(K, \rho)$.

Figure 6 shows the surface $\Delta(K, \rho)$ characterizing the quantitative difference of the effects of control laws (1.2) on the damping of forced vibrations. From Fig. 6 it is seen that the difference Δ is alternating-sign. This means that if the time constant is small, velocity control is more effective. As the constant T increases, acceleration control becomes more preferable.

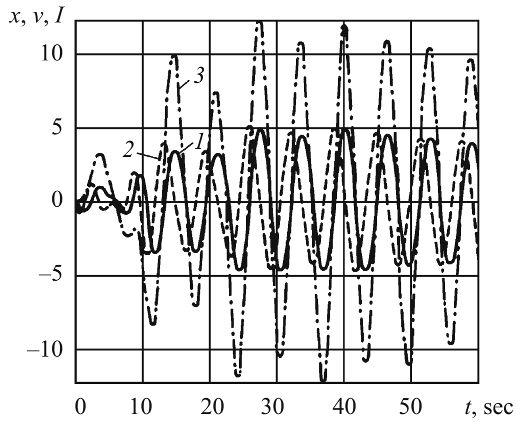

Fig. 7

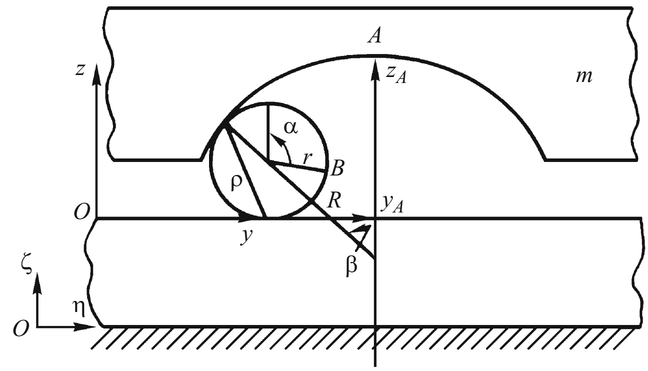

Fig. 8

Figure 7 shows the following processes: $x(t)$ (curve *1*), $\dot{x}(t) = v(t)$ (curve *2*), $I(t)$ (curve *3*) for $K_v = 15, \rho = 1$; they show that current amplitude $I_A = 15$ mA in the solenoid is such that $I_A < I_{max}$.

Addendum. Calculation of the Gravitational Stiffness of a Rigid Body on Spherical Supports. Following [21], we will calculate the horizontal stiffness of the suspension of a rigid body undergoing small translational deviations from the equilibrium position. Figure 8 shows the body in a nonequilibrium position.

Denote by y_A and z_A the coordinates of the highest point *A* of the semispherical hollow in Cartesian coordinate system *Oyz* with the origin at the point of contact of the sphere and the flat rolling surface in equilibrium position. When the rigid body is in equilibrium, the points *A* and *B* coincide. The body of mass *m* undergoes translational vibrations; therefore, its arbitrary point moves in the same way as the point *A* of the hollow. The coordinates of the point *A* (y_A, z_A) are related by the integrable kinematic constraint

$$\dot{y}_A^2 + \dot{z}_A^2 = v_A^2, \tag{A.1}$$

$$z_A = (R - r)(1 - \cos \chi\alpha) + 2r, \qquad v_A = \dot{\alpha}\rho(\alpha), \tag{A.2}$$

where α is the angle between the vertical and the radius *r* drawn from the center of the sphere to the point *B* on its surface, which coincides with the point *A* if the system of bodies is in equilibrium position ($\alpha = 0$); $\rho(\alpha)$ is the instantaneous radius of rotation that connects the contact points of the sphere and the body and the horizontal plane in a nonequilibrium position:

$$\rho(\alpha) = r\sqrt{2(1 + \cos \chi\alpha)}, \qquad \chi = r/(R - r), \tag{A.3}$$

where $\chi\alpha = \beta$ is the angle between the vertical and the radius *R* of the hollow drawn to the contact point of the sphere and the body. Let α be the Lagrangian generalized coordinate of the rigid body. Using formulas (A.1)–(A.3), we get

$$\dot{z}_A = \dot{\alpha} r \sin \chi\alpha, \qquad \dot{y}_A = r\dot{\alpha}(1 + \cos \chi\alpha). \tag{A.4}$$

According to these formulas, the Lagrangian function has the form $L = T - \Pi$, where $T = 0.5m_1(\dot{y}_A^2 + \dot{z}_A^2)$, $\Pi = mgz_A$, $m_1 = m + J/(4r^2)$; *J* is the moment of inertia of the homogeneous sphere about an axis tangential to its surface. Using formulas (A.1) and (A.4), we set up a Lagrangian equation of the second kind for the generalized coordinate α:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\alpha}} - \frac{\partial T}{\partial \alpha} = -\frac{\partial \Pi}{\partial \alpha}.$$

If α is small, we have $\dot{y}_A = 2r\dot{\alpha}$, $z_A = 2r + \frac{1}{2}\frac{r^2}{(R - r)}\alpha^2$. Taking into account these relations and using (A.5), we obtain the linear equation

$$4m_1 r^2 \ddot{\alpha} + \frac{mgr^2 \alpha}{(R-r)} = 0.$$

Considering that $y_A = 2r\alpha$, we find

$$m_1 \ddot{y}_A + \frac{mg}{4(R-r)} y_A = 0.$$

The coefficient of y_A is the gravitational stiffness $c = \frac{mg}{4(R-r)}$ referred to the *Oy*-axis of the body undergoing small deviations from the equilibrium position.

Conclusions. A nonlinear mathematical model of a gravitational vibratory system with a controlled electromagnetic seismic damper has been developed. We have experimentally established the dependence of the force of attraction of ferromagnetic bodies by a solenoid of specific design on the current. An analytic expression of the force has been obtained using the least-squares method and a system of continuous piecewise-linear functions of special form. It was used as a mathematical model of the force pressing the solenoid to the surface of the frictional element of the damper. In the case of multifrequency inertial excitation, we have calculated the acceleration amplification factor depending on the time constant and gain of the feedback circuit for two control laws for the solenoid current. It has been established that it is possible to damp by controlling absolute velocity and acceleration.

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