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RESONANT EFFECTS OF LOCAL LOADS ON CIRCULAR SANDWICH PLATES ON AN ELASTIC FOUNDATION

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The axisymmetric forced vibrations of a circular sandwich plate on an elastic foundation are studied. The plate is subjected to axisymmetric surface and mechanical loads with frequency equal to one of the natural frequencies of the plate. The foundation reaction is described by the Winkler model. To describe the kinematics of an asymmetric sandwich, the hypothesis of broken normal is used. The core is assumed to be light. The analytical solution of the problem is obtained and numerical results are analyzed

Keywords: circular sandwich plate, axisymmetric forced vibrations, Winkler elastic foundation, axisymmetric surface, mechanical load

Introduction. Analytic and numerical results on the vibrations of circular sandwich plates not bonded to an elastic foundation were obtained in [1–3]. The nonlinear vibrations of layered plates were studied in [4]. The quasistatic deformation of sandwich structures on an elastic foundation was studied in [5]. The natural vibrations of a sandwich rod bonded to a Winkler foundation were studied in [6].

We will consider an asymmetric (throughout the thickness) circular elastic sandwich plate bonded to an elastic foundation and undergoing small axisymmetric transverse vibrations excited by surface and mechanical resonant loads, i.e., loads with frequency equal to one of the natural frequencies of the plate.

1. Problem Formulation. We will use a cylindrical coordinate system r, φ, z (Fig. 1) fixed to the midsurface of the core. For isotropic face layers of thickness h_1 and h_2, the Kirchhoff hypotheses are accepted. The incompressible core ($h_3 = 2c$) is light, i.e., the work of the tangential stresses σ_{rz} in the tangential direction can be neglected. The deformed normal of the core remains straight, but turns through some additional angle ψ. The displacements at the boundaries of the layers are continuous. There is a rigid diaphragm at the edge of the plate to prevent the relative movement of the layers. The external vertical load does not depend on the coordinate φ: $q = q(r, t)$. The outside surface of the second face layer sustains the reaction q_R of the elastic foundation.

For symmetry reasons, the tangential displacements of the layers are zero, and the deflection w of the plate, the relative shear ψ in the core, and the radial displacement u of the coordinate surface do not depend on the coordinate φ, i.e., $u(r, t)$, $\psi(r, t)$, $w(r, t)$. Hereafter these functions are unknown. The thickness and density of the kth layer are denoted by h_k and ρ_k, respectively.

The relationship between the reaction and the deflection is described by the Winkler model ($q_R = \kappa_0 w$, κ_0 is the foundation modulus (modulus of subgrade reaction)).

The system of partial differential equations describing the forced transverse vibrations of a circular sandwich plate not bonded to an elastic inertialess foundation and disregarding the reduction and rotary inertia of the normal in the layers is obtained from Lagrange's variational principle taking into account the variation of the work of inertial forces [1, 3]. In this case, the third equation includes the reaction of the elastic foundation. This system has the form

$$L_2(a_1 u + a_2 \psi - a_3 w,_r) = 0, \qquad L_2(a_2 u + a_4 \psi - a_5 w,_r) = 0,$$

$$L_3(a_3 u + a_5 \psi - a_6 w,_r) - M_0 \ddot{w} - \kappa_0 w = -q, \tag{1.1}$$

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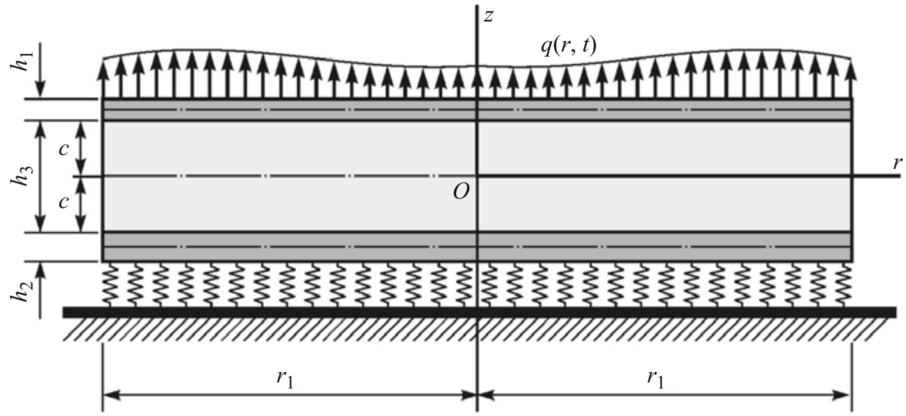

Fig. 1

where $M_0 = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$. The coefficients a_i and differential operators L_2 and L_3 are defined by

$$a_1 = \sum_{k=1}^{3} h_k K_k^+, \quad a_2 = c(h_1 K_1^+ - h_2 K_2^+), \quad K_k^+ \equiv K_k + \frac{4}{3} G_k,$$

$$a_3 = h_1\left(c + \frac{1}{2} h_1\right) K_1^+ - h_2\left(c + \frac{1}{2} h_2\right) K_2^+, \quad a_4 = c^2\left(h_1 K_1^+ + h_2 K_2^+ + \frac{2}{3} c K_3^+\right),$$

$$a_5 = c\left[h_1\left(c + \frac{1}{2} h_1\right) K_1^+ + h_2\left(c + \frac{1}{2} h_2\right) K_2^+ + \frac{2}{3} c^2 K_3^+\right],$$

$$a_6 = h_1\left(c^2 + c h_1 + \frac{1}{3} h_1^2\right) K_1^+ + h_2\left(c^2 + c h_2 + \frac{1}{3} h_2^2\right) K_2^+ + \frac{2}{3} c^3 K_3^+,$$

$$L_2(g) \equiv \left(\frac{1}{r}(rg),_r\right),_r \equiv g,_{rr} + \frac{g,_r}{r} - \frac{g}{r^2}, \qquad L_3(g) \equiv \frac{1}{r}(r L_2(g)),_r \equiv g,_{rrr} + \frac{2g,_{rr}}{r} - \frac{g,_r}{r^2} + \frac{g}{r^3}.$$

The problem of finding the functions $u(r, t)$, $\psi(r, t)$, and $w(r, t)$ is closed by adding boundary and initial to (1.1), (1.2).

2. Natural Vibrations. Let us consider a homogeneous system of differential equations describing the natural vibrations of a circular sandwich plate on an elastic inertialess foundation. It follows from (1.1) when $q = 0$ and is reduced to the following form after some transformations:

$$u = b_1 w,_r + C_1 r + C_2 / r, \qquad \psi = b_2 w,_r + C_3 r + C_4 / r,$$

$$L_3(w,_r) + \kappa^4 w + M^4 \ddot{w} = 0, \tag{2.1}$$

where $b_1 = \dfrac{a_3 a_4 - a_2 a_5}{a_1 a_4 - a_2^2}$, $b_2 = \dfrac{a_1 a_5 - a_2 a_3}{a_1 a_4 - a_2^2}$, $\kappa^4 = \kappa_0 D$, $M^4 = M_0 D$, $D = \dfrac{a_1(a_1 a_4 - a_2^2)}{(a_1 a_6 - a_3^2)(a_1 a_4 - a_2^2) - (a_1 a_5 - a_2 a_3)^2}$.

Since the desired solution is bounded, it is necessary to set $C_2 = C_4 = 0$ at the origin of coordinates of solid plates.

In the case of free vibrations, the unknown deflection is expressed as

$$w(r, t) = v(r)(A\cos(\omega t) + B\sin(\omega t)), \tag{2.2}$$

where $v(r)$ is an unknown coordinate function; ω is the natural frequency of the plate; A and B are constants of integration determined from the initial conditions (2).

Substituting expression (2.2) into the last equation in (2.1), we obtain an equation to determine the coordinate function $v(r)$:

$$L_3(v,_r) - \lambda^4 v = 0, \tag{2.3}$$

where

$$\lambda^4 = \beta^4 - \kappa^4, \quad \beta^4 = M^4 \omega^2. \tag{2.4}$$

The solution of Eq. (2.3) can be represented in the following form [7]:

$$v(\lambda r) = C_5 J_0(\lambda r) + C_6 I_0(\lambda r) + C_7 Y_0(\lambda r) + C_8 K_0(\lambda r), \tag{2.5}$$

where J_0 and Y_0 are the zero-order Bessel functions of the first and second kinds (Neumann function), respectively; I_0 and K_0 are the zero-order modified Bessel functions of the first and second kinds; $C_5, \ldots, C_8$ are constants of integration.

Omitting the description of these functions, we just point out that $Y_0(\lambda r)$ and $K_0(\lambda r)$ have a logarithmic singularity at the origin of coordinates [8], i.e., at the center of the plate. Therefore, it is necessary to set $C_7 = C_8 = 0$ in (2.5).

If the edge is clamped, the transcendental equation for finding the eigenvalues λ_n of a circular sandwich plate bonded to an elastic inertialess foundation follows from the corresponding boundary conditions:

$$I_1(\lambda r_1) J_0(\lambda r_1) + J_1(\lambda r_1) I_0(\lambda r_1) = 0. \tag{2.6}$$

If the edge is hinged, then this equation is

$$J_0(\lambda r_1)\left[a_7\left(\lambda I_0(\lambda r_1) - \frac{I_1(\lambda r_1)}{r_1}\right) + \frac{a_8}{r_1} I_1(\lambda r_1)\right]$$

$$+ I_0(\lambda r_1)\left[a_7\left(\lambda J_0(\lambda r_1) - \frac{J_1(\lambda r_1)}{r_1}\right) + \frac{a_8}{r_1} J_1(\lambda r_1)\right] = 0, \tag{2.7}$$

where

$$a_7 = a_6 - a_3 b_1 - a_5 b_2, \quad a_8 = a_{60} + a_3 b_1 + a_5 b_2,$$

$$a_{60} = h_1\left(c^2 + ch_1 + \frac{1}{3}h_1^2\right)K_1^- + h_2\left(c^2 + ch_2 + \frac{1}{3}h_2^2\right)K_2^- + \frac{2}{3}c^3 K_3^-, \quad K_k^- \equiv K_k - \frac{2}{3}G_k.$$

After numerical solution of Eqs. (2.6) and (2.7), the parameters β_n and the natural frequencies ω_n can be determined in terms of the eigenvalues λ_n from formulas (2.4).

In the general case, a system of orthonormalized eigenfunctions $v_n \equiv v(\lambda_n, r)$ is introduced to describe the deflection of the plate undergoing free transverse vibrations:

$$v_n \equiv \frac{1}{d_n}\left[J_0(\lambda_n r) - \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)} I_0(\lambda_n r)\right], \tag{2.8}$$

where $C_6 = -C_5 J_0(\lambda r_1) / I_0(\lambda r_1)$, which follows from the boundary condition $w = 0$. The constants d_n are determined from the normalization conditions for functions (2.8):

$$d_n^2 = \int_0^{r_1}\left[J_0(\lambda_n r) - \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)} I_0(\lambda_n r)\right]^2 r dr.$$

Finally, the unknown dynamic deflection of a circular sandwich plate on an elastic inertialess foundation is expanded into a series of fundamental orthonormalized eigenfunctions (2.8):

$$w(r,t) = \sum_{n=0}^{\infty} v_n (A_n \cos(\omega_n t) + B_n \sin(\omega_n t)). \tag{2.9}$$

The radial displacement and relative shear follow from the first two equations in (2.1) and the boundary conditions $\psi(r_1, t) = u(r_1, t) = 0$:

$$u(r,t) = b_1 \sum_{n=0}^{\infty} \varphi_n (A_n \cos(\omega_n t) + B_n \sin(\omega_n t)),$$

$$\psi(r,t) = b_2 \sum_{n=0}^{\infty} \varphi_n (A_n \cos(\omega_n t) + B_n \sin(\omega_n t)), \tag{2.10}$$

where the system of functions $\varphi_n \equiv \varphi_n(\lambda_n, r)$ is given by

$$\varphi_n(\lambda_n, r) = \frac{\lambda_n}{d_n}\left[J_1(\lambda_n r_1) r - J_1(\lambda_n r) + \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)}(I_1(\lambda_n r_1) r - I_1(\lambda_n r))\right].$$

The coefficients A_n and B_n in formulas (2.9) and (2.10) are determined from the initial conditions of motion.

3. Analytic Solution. To describe the forced vibrations of the plate, the external load $q(r, t)$ and the unknown displacements $u(r, t)$, $\psi(r, t)$, and $w(r, t)$ are expanded into the following series:

$$q(r,t) = M_0 \sum_{n=0}^{\infty} v_n q_n(t), \qquad u(r,t) = b_1 \sum_{n=0}^{\infty} \varphi_n T_n(t),$$

$$\psi(r,t) = b_2 \sum_{n=0}^{\infty} \varphi_n T_n(t), \qquad w(r,t) = \sum_{n=0}^{\infty} v_n T_n(t). \tag{3.1}$$

The coefficients of the series $q_n(t)$ are obtained by multiplying the first relation in (3.1) by v_n and integrating it over the area of the plate.

The equation for the unknown function $T_n(t)$ follows from the third equation in (1.1) after substitution of expressions (3.1):

$$\ddot{T}_n + \omega_n^2 T_n = q_n. \tag{3.2}$$

The general solution of Eq. (3.2) is as follows:

$$T_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{1}{\omega_n}\int_0^t \sin \omega_n (t - \tau) q_n(\tau) d\tau.$$

The coefficients A_n and B_n are determined from the initial conditions.

Problem 1. We will solve three problems. Let us consider elastic circular sandwich plates undergoing transverse vibrations under a local resonant surface harmonic load uniformly distributed over a circle of radius $b \le r_1$:

$$q(r,t) = q_0 H_0(b - r)(D\cos(\omega_k t) + E\sin(\omega_k t)) \quad (q_0, D, E, k = \text{const}), \tag{3.3}$$

where $H_0(r)$ is the Heaviside function of order zero (vanishes if the argument is negative and equals unity otherwise). The frequency ω_k of external force is equal to one of the natural frequencies ω_n of the circular sandwich plate.

The solution of the corresponding initial–boundary-value problem is sought in the form (3.1), i.e., is expanded into a series of orthonormalized eigenfunctions (2.8). The parameters of vibrations $q_n(t)$ corresponding to the nth harmonic of (3.3) are obtained in the form v_n,

$$q_n(t) = \frac{1}{M_0}\int_0^{r_1} q(r,t) v_n r dr = D_n \cos(\omega_k t) + E_n \sin(\omega_k t), \tag{3.4}$$

$$D_n = \frac{Dq_0}{M_0 d_n \lambda_n}\left[J_1(\lambda_n r_1) - \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)} I_1(\lambda_n r_1)\right],$$

$$E_n = \frac{Eq_0}{M_0 d_n \lambda_n}\left[J_1(\lambda_n r_1) - \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)} I_1(\lambda_n r_1)\right].$$

The differential equation (3.2) for the function $T_n(t)$ takes the following form in view of (3.4):

$$\ddot{T}_n(t) + \omega_n^2 T_n(t) = D_n \cos(\omega_k t) + E_n \sin(\omega_k t). \tag{3.5}$$

Its solution can be represented in the form

$$T_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t) + y_n(t), \tag{3.6}$$

$$y_n(t) = \begin{cases} \dfrac{D_n}{\omega_n^2 - \omega_k^2}\cos(\omega_k t) + \dfrac{E_n}{\omega_n^2 - \omega_k^2}\sin(\omega_k t), & n \ne k, \\ -\dfrac{E_k}{2\omega_k} t\cos(\omega_k t) + \dfrac{D_k}{2\omega_k} t\sin(\omega_k t), & n = k, \end{cases}$$

$y_n(t)$ is a partial solution of Eq. (3.5).

Without loss of generality, we can use zero initial conditions of motion: $w(r,0) = \dot{w}(r,0) = 0$. Substituting expression (3.1) into them and taking into account function (3.6) and the fact that system v_n (2.8) is orthonormalized, we obtain the constants of integration:

$$A_n = -\begin{cases} \dfrac{D_n}{\omega_n^2 - \omega_k^2}, & n \ne k, \\ 0, & n = k, \end{cases} \qquad B_n = \frac{1}{\omega_n}\left[\begin{cases} \dfrac{\omega_k E_n}{\omega_n^2 - \omega_k^2}, & n \ne k, \\ -\dfrac{E_k}{2\omega_k}, & n = k \end{cases}\right]. \tag{3.7}$$

Thus, the transverse vibrations of the circular sandwich plate under resonant load (3.3) are described by expressions (3.1) where the function $T_n(t)$ and the constants of integration are defined by (3.6) and (3.7). If $b = r_1$, we obtain a solution for a plate under a resonant load distributed over the whole outside surface of the core layer.

Problem 2. Let a uniformly distributed resonant surface load act over a ring $a \le r \le b$ on the surface of the plate. It can be expressed as a difference of two surface loads (3.3):

$$q(r,t) = q_0(H_0(b-r) - H_0(r-a))(D\cos(\omega_k t) + E\sin(\omega_k t)). \tag{3.8}$$

The parameters of load (3.8) expanded into a series of orthonormalized eigenfunctions (1.2) are given by

$$D_n = \frac{q_0 D}{M_0 d_n \lambda_n}\left(bJ_1(\lambda_n b) - aJ_1(\lambda_n a) - \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)}(bI_1(\lambda_n b) - aI_1(\lambda_n a))\right),$$

$$E_n = \frac{q_0 E}{M_0 d_n \lambda_n}\left(bJ_1(\lambda_n b) - aJ_1(\lambda_n a) - \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)}(bI_1(\lambda_n b) - aI_1(\lambda_n a))\right). \tag{3.9}$$

After that, the function $T_n(t)$ follows from expression (3.6) where the constants of integration A_n and B_n are defined by formulas (3.7) with (3.9).

Problem 3. Consider a circular sandwich plate undergoing forced vibrations under a linear force $Q(r, t)$ applied along a circle of radius a:

$$Q(r,t) = Q_0 H_0(r-a) H_0(a-r)(D\cos\omega_k t + E\sin\omega_k t).$$

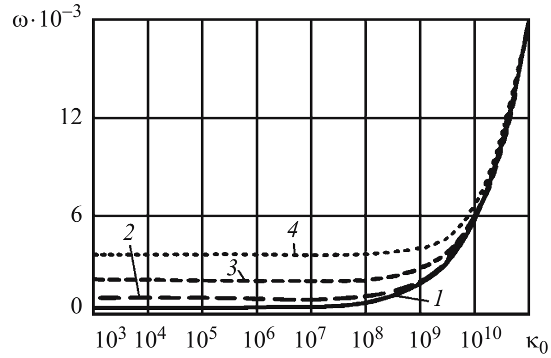

Fig. 2

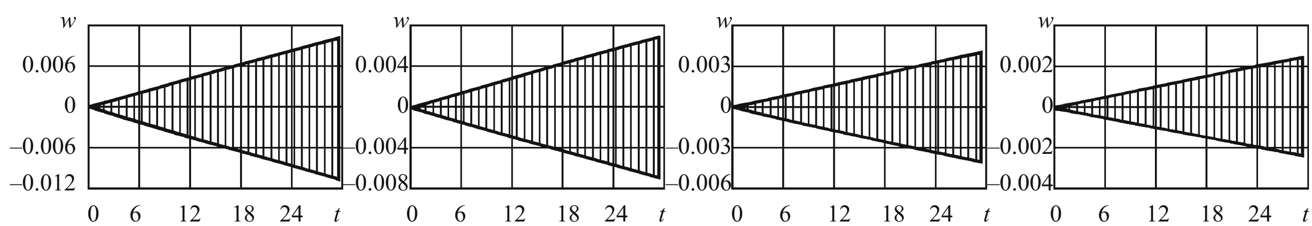

Fig. 3

To solve the problem, we will use coefficients (3.9) in the case of a surface load distributed over a ring $[a-\xi, a+\xi]$. Let $q_0 = Q_0/(2\xi)$ and tend ξ to zero. Then

$$D_n = \frac{Q_0 D}{M_0 d_n \lambda_n} \lim_{\xi \to 0} \left[\frac{1}{2\xi} \left((a+\xi) J_1(\lambda_n (a+\xi)) - (a+\xi) \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)} I_1(\lambda_n (a+\xi)) \right.\right.$$

$$\left.\left. -(a-\xi) J_1(\lambda_n (a-\xi)) + (a-\xi) \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)} I_1(\lambda_n (a-\xi)) \right) \right] = \frac{Q_0 a D}{M_0 d_n} \left(J_0(\lambda_n a) - \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)} I_0(\lambda_n a) \right),$$

$$E_n = \frac{Q_0 a E}{M_0 d_n} \left(J_0(\lambda_n a) - \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)} I_0(\lambda_n a) \right). \tag{3.10}$$

The function $T_n(t)$ follows from expression (3.6) where the constants of integration A_n and B_n are defined by formulas (3.7) with (3.10).

Assume that the resultant of the linear force is independent of the radius of the circle. In this case, the coefficients of its expansion into series of eigenfunctions follow from expressions (3.10) if $Q_0 = Q_1/(2\pi a)$. Then we have the formulas

$$D_n = \frac{Q_1 D}{2\pi M_0 d_n} \left(J_0(\lambda_n a) - \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)} I_0(\lambda_n a) \right),$$

$$E_n = \frac{Q_1 E}{2\pi M_0 d_n} \left(J_0(\lambda_n a) - \frac{J_0(\lambda_n r_1)}{I_0(\lambda_n r_1)} I_0(\lambda_n a) \right). \tag{3.11}$$

For the function $T_n(t)$, expression (3.6) with (3.7) and (3.11) remains valid.

4. Numerical Results. First, we numerically analyzed Eq. (2.6). The first 15 roots were calculated to 0.001. They numerically coincide with the eigenvalues for the plate without elastic foundation [1].

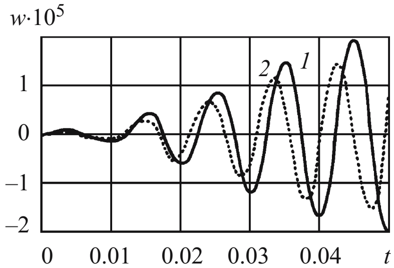

Fig. 4

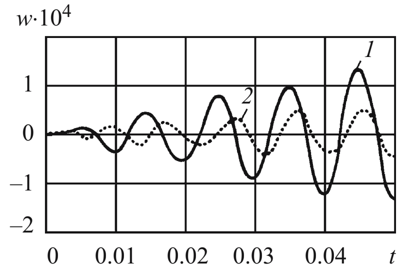

Fig. 5

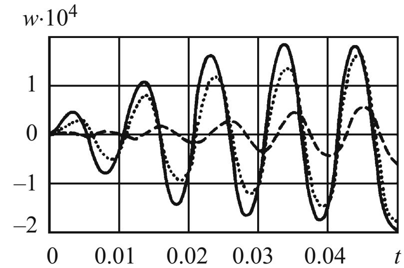

Fig. 6

Numerical results are presented for a clamped plate with layers made of D16T and PTFE materials. The natural frequencies ω_n are calculated from formula (2.4) using the eigenvalues and the parameters of the layers $h_1 = h_2 = 0.01$, $c = 0.05$. Only the first six terms of series (3.1) are summed because the contribution of the subsequent (to 100) terms is less than 0.1%.

Figure 2 shows the first four natural frequencies ω_n (ω_0 (curve *1*), ω_1 (curve *2*), ω_2 (curve *3*), ω_3 (curve *4*)) of a clamped plate depending on the stiffness of the elastic foundation κ_0 (Pa/m). The frequencies are almost constant if the foundation has low stiffness ($\kappa_0 < 10^7$). As the stiffness of the foundation increases to medium magnitudes ($10^7 < \kappa_0 < 10^9$), the fundamental frequency ω_0 increases by a factor of 6.3. If the stiffness of the foundation is high ($10^9 < \kappa_0 < 10^{11}$), the fundamental frequency increases by a factor of 9.5.

Figure 3 shows how the amplitude of resonant vibrations of a plate on an elastic foundation ($\kappa_0 = 10^8$ Pa/m, $q_0 = 50$ Pa) increases under a load distributed over the whole surface ($\omega_k = \omega_0 = 617$ Hz (Fig. 3*a*); $\omega_k = \omega_1 = 712$ Hz (Fig. 3*b*); $\omega_k = \omega_2 = 1025$ Hz (Fig. 3*c*); $\omega_k = \omega_3 = 1585$ Hz (Fig. 3*d*)). The ratios of the deflection (Fig. 3*a*) to the maximum deflections (Fig. 3*b*, *c*, *d*) is approximately 1.5, 2.6, and 4.6, respectively. The abrupt decrease in the amplitude and in the rate of increase of vibrations is due to the presence of the elastic foundation. It makes the system stiffer, increases the first three natural frequencies, and makes them close to each other.

The resonant frequency of the plate is very high; this why the process is graphically indistinguishable on the time interval usedin Fig. 3. This shortcoming is eliminated in Fig. 4 ($\kappa_0 = 10^8$ Pa/m) by using a time interval 1/600 as long as that in Fig. 3 ($\omega_k = \omega_0$ (curve *1*) and $\omega_k = \omega_1$ (curve *2*)). It can be seen that the amplitude of resonant vibrations increases. The frequencies and amplitudes approach each other, but the lowest frequency remains predominant.

Figure 5 shows the time-dependent deflection at the center of a plate on an elastic foundation with $\kappa_0 = 10^8$ Pa/m under a resonant load ($\omega_k = \omega_0$, $q_0 = 1000$ Pa) for various positions of the force ring of width $b - a = 0.25$: $a = 0.25$ (curve *1*) and $a = 0.75$ (curve *2*).

Shifting the force ring toward the edge of the plate increases the force resultant (effective area), but decreases the bending moment, which leads to a substabtial decrease in the deflections.

Figure 6 illustrates the variation in the deflection at the center of a plate on an elastic foundation ($\kappa_0 = 10^8$ Pa/m). The resonant ($\omega_k = \omega_0$) linear load has a constant resultant $Q_1 = 10^4$ N. Formulas (3.11) were used to plot the figure. Curve *1* corresponds to $a = 0$, curve *2* to $a = 0.25$, and curve *3* to $a = 0.75$. The deflection is maximum when a concentrated force is applied at the center of the plate. As the force ring shifts toward the edge, the deflection and shear decrease in absolute magnitude.

Conclusions. We have proposed a procedure for studying the forced vibrations of circular sandwich plates bonded to an elastic foundation and subjected to resonant surface and linear axisymmetric loads. Analytic and numerical solutions to some initial–boundary-value problems for plates with a light core have been obtained.

Note that these solutions are valid for small elastic strains. Such a model describes resonant phenomena only at the initial stage of increase in the amplitude when there are yet no finite strains or plastic deformation and delamination do not yet occur.

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