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DYNAMICS OF A CHAIN SYSTEM OF RIGID BODIES WITH GRAVITY-FRICTION SEISMIC DAMPERS: FIXED SUPPORTS

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A design model for a chain system of *N* elastically linked rigid bodies with a spheroidal gravity-friction damper is proposed. The Lagrange–Painlevé equations of the first kind are used to construct nonlinear dynamical models of a mechanical system undergoing translational vibrations about the equilibrium position. The conditions under which the system moves in one plane are established. The double nonstationary phase–frequency resonance of a system with $N = 2$ is analyze. After the numerical integration of the systems of differential equations, the phase–frequency surfaces are plotted and examined for several combinations of system parameters under two-frequency loading

Keywords: chain system, rigid body, seismic damper, forced vibrations, double phase–frequency resonance, two-frequency loading, phase–frequency surfaces

Introduction. Seismic isolation of foundations is an effective protective measure for buildings and other structures [2–4, 9, 11, 13, 15, 16, 18, 19, 21, 22]. Special devices of various degree of complexity are built in between the foundation and the superstructure to weaken the coupling between the structure and the ground, which is equivalent to increasing the period of natural vibrations of the superstructure.

Structures equipped with any seismic-isolation mechanism (SIM) are usually modeled by a vibrating system with one degree of freedom. The period of vibration of the rigid body about the foundation plate of the SIM is assumed to be approximately four times greater than the fundamental period of vibration of the ground during an earthquake. If this condition is not satisfied, a structure with an SIM cannot be modeled by a rigid body alone, especially if dry friction dampers are used. A more acceptable model of such a structure is a vertical chain system of several elastically connected rigid bodies with an SIM.

SIMs with dry and Coulomb friction have certain benefits due to the dead zone [17, 19]. Such SIMs make the systems of rigid bodies insensitive to small non-seismic loads such as wind. Dry friction and sliding modes make SIM constraints nonideal; therefore, it is impossible to use the Lagrangian equations of the second kind to adequately model the small vibrations of rigid bodies by analogy with systems with ideal holonomic constraints [14].

This paper continues the studies on the vibrations of systems of rigid bodies with gravity-friction seismic dampers [18, 19] based on dynamic models in the form of the Lagrange equations of the first kind with finite constraints and undetermined multipliers. The nonideality of the constraints is described by Painlevé sliding friction forces [12]. One of the key dynamic parameters of the system of rigid bodies is the dynamic amplification factor μ_i of the ith mass. It depends on many dynamic parameters of the protected object and the kinematic parameters of translational motion. Such systems may enter into double phase-frequency resonance [5, 7, 8, 11, 19], which causes abnormally high values of the factor μ_i. This phenomenon is observed in the system being considered and can be analyzed by plotting phase-resonance surfaces.

1. Description of a Chain System of *N* Elastically Connected Rigid Bodies with Spherical Seismodampers. Figure 1 shows a massive rigid body (plate) M_0 that bears a system of K identical fixed spherical supports of radius r and undergoes prescribed motion in an inertial coordinate system $O\xi\eta\varsigma$. The spherical supports contact with a platform of mass m_1 with many

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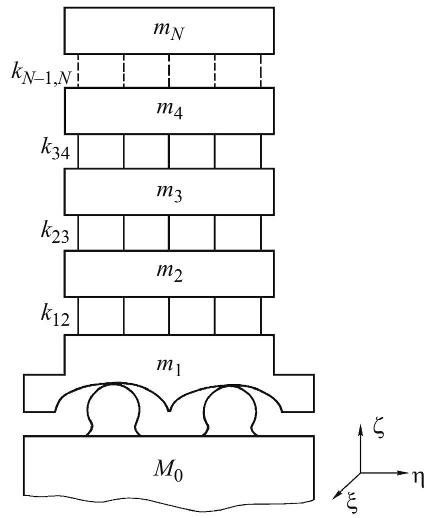

Fig. 1

identical hemispherical depressions of radius R ($R > r$). In equilibrium, the contact points are at the vertices of the depressions. The platform supports a chain system of $N-1$ elastically connected rigid bodies of mass $m_2, m_3, \ldots, m_N$ and can move over the spherical supports. This platform, the plate M_0, and the spherical supports constitute an SIM.

The platform may undergo translational and rotational motion together with the bodies. The small spatial vibrational motion of one body (platform) and $K = 3$, 4 moving supports was studied in [9].

To model the system of N elastically connected rigid bodies, we will make some assumptions. Let the inertia-rigidity parameters of the system are such that the equations of small translational and rotational motions about the equilibrium position are uncoupled [1], which means, according to [9], the following:

(i) all the supports contact with the body m_1 along the periphery of one circle;

(ii) the centers of mass of all bodies lie on the same axis that passes through the center of the circle along which the supports are arranged;

(iii) the principal axes of inertia of all bodies are collinear;

(iv) the bodies are connected by flexible bars whose longitudinal tension and vertical displacement are neglected;

(v) the sliding friction force is considered only at the points of contact between the spherical supports and the hemispherical depressions;

(vi) the rigid bodies move translationally.

2. System of Ordinary Differential Equations of Translational Motion of the Chain System of Rigid Bodies. When the system moves translationally, all points of the bodies move along identical trajectories. Therefore, we will consider one support (Fig. 2).

We choose a Cartesian coordinate system O_1xyz with the origin at the upper point of the spherical support, the Oz-axis directed vertically upwards, the O_1x- and O_1y-axes along the width and length of the carried bodies (let they be parallelepipeds for simplicity). In equilibrium, the centers of mass of the bodies are on the Oz-axis. The coordinate system O_1xyz is noninertial; therefore, the inertial forces due to accelerated translational motion of O_1xyz will be considered external forces. The axes of O_1xyz and $O\xi\eta\varsigma$ are parallel.

Let us use the Lagrangian equations of the fist kind and Painlevé's method of describing nonideal constraints [12]. The position of the system of N rigid bodies is defined by the coordinates of their centers of mass $x_i, y_i, z_i, i = \overline{2, N_0}$. The coordinates x_1, y_1, z_1 define the position of the point A_1 of the depression of the platform m_1. These coordinates are subject to the constraints

$$\Phi_1(x_1, y_1, z_1) = x_1^2 + y_1^2 + (z_1 - \rho)^2 = \rho^2 \quad (\rho = R - r), \tag{2.1}$$

which is the equation of sphere since the point A_1 that undergoes translational motion over (without separation from) the platform describes a sphere of radius ρ. Due to assumption (iv), the coordinates z_i are related by

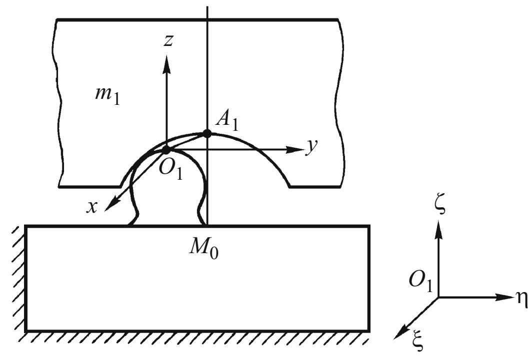

Fig. 2

$$\Phi_i(z_1, z_2, \ldots, z_N) = z_i - z_1 = c_i = \text{const} \qquad (i = \overline{2, N}). \tag{2.2}$$

Considering (2.1) and (2.2), we write the Lagrangian equation of the first kind as

$$m_1\ddot{x}_1 = -m_1\ddot{\xi} + k_{21}^x(x_2 - x_1) + \lambda_1 \frac{\partial \Phi_1}{\partial x_1} + F_{\mathrm{F}x},$$

$$m_1\ddot{y}_1 = -m_1\ddot{\eta} + k_{21}^y(y_2 - y_1) + \lambda_1 \frac{\partial \Phi_1}{\partial y_1} + F_{\mathrm{F}y},$$

$$m_1\ddot{z}_1 = -m_1\ddot{\varsigma} - m_1 g + \sum_{i=1}^{N} \frac{\partial \Phi_i}{\partial z_1}\lambda_i + F_{\mathrm{F}z},$$

$$m_i\ddot{x}_i = -m_i\ddot{\xi} - k_{i,i-1}^x(x_i - x_{i-1}) + k_{i+1,i}^x(x_{i+1} - x_i),$$

$$m_i\ddot{y}_i = -m_i\ddot{\eta} - k_{i,i-1}^y(y_i - y_{i-1}) + k_{i+1,i}^y(y_{i+1} - y_i),$$

$$m_i\ddot{z}_i = -m_i\ddot{\varsigma} + \lambda_i \frac{\partial \Phi_i}{\partial z_i} - m_i g \qquad (i = 2, N), \tag{2.3}$$

where $k_{i,i-1}^x, k_{i,i-1}^y$ are the total stiffnesses of the elastic bars along the Ox- and Oy-axes, respectively, between the $(i-1)$th and ith bodies, $k_{lk}^{\pi} = k_{kl}^{\pi}$ $(\pi = x, y)$, $k_{N+1,N}^x = k_{N+1,N}^y = 0$. The shear stiffness of n equitype bars of length l rigidly fixed in two neighboring masses is defined by $k_{i,i-1} = nEJ/(12l^3)$, where EJ is the flexural stiffness of the bar; λ_i $(i = \overline{1, N})$ are unknown Lagrangian multipliers; $F_{\mathrm{F}x}, F_{\mathrm{F}y}, F_{\mathrm{F}z}$ are the projections of the Coulomb friction force at the contact point onto the surfaces of the platform and spherical support. Let us calculate the coordinates of the vector $\overline{F}_{\mathrm{F}}$ that lies in a plane tangential to the sphere (2.1) [12]. The unit vector tangential to the trajectory of the point A on the surface of constraint (2.1) is defined by

$$\overline{\tau} = \overline{e}_x \frac{\dot{x}}{v} + \overline{e}_y \frac{\dot{y}}{v} + \overline{e}_z \frac{\dot{z}}{v}, \qquad v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}, \tag{2.4}$$

where $\overline{e}_x, \overline{e}_y, \overline{e}_z$ are the unit vectors of O_1xyz.

The force of sliding friction is opposite to the vector $\overline{\tau}$ and equal to the product $\mu|\overline{N}|$, where $|\overline{N}|$ is the normal component of the reaction of nonideal constraint (2.1); μ is the coefficient of sliding friction, which depends on the material of the contacting bodies.

According to the mechanical sense of the terms with the factor λ_1 in Eq. (2.3), we have

$$|\bar{N}| = \left|\lambda_1 \sqrt{\left(\frac{\partial \Phi_1}{\partial x_1}\right)^2 + \left(\frac{\partial \Phi_1}{\partial y_1}\right)^2 + \left(\frac{\partial \Phi_1}{\partial z_1}\right)^2}\right| = 2|\lambda_1|\rho = 2\rho\lambda_1 \mathrm{sign}\lambda_1. \tag{2.5}$$

Thus, the components of the force of dry sliding friction are given by

$$F_{\mathrm{F}x} = -2\lambda_1 \mathrm{sign}\lambda_1 \rho\tau_x, \quad F_{\mathrm{F}y} = -2\lambda_1 \mathrm{sign}\lambda_1 \rho\tau_y, \quad F_{\mathrm{F}z} = -2\lambda_1 \mathrm{sign}\lambda_1 \rho\tau_z,$$

where $\tau_x = \dot{x}/v$, $\tau_y = \dot{y}/v$, and $\tau_z = \dot{z}/v$ are the direction cosines between the axes of O_1xyz and the velocity vector of the point A_1 on the surface of sphere (2.1). All the right-hand sides of the system of equations (2.3) are defined, and $3N$ equations together with N constraints are enough to find $4N$ variables x_i, y_i, z_i, λ_i $(i = \overline{1, N})$.

3. Eliminating Lagrangian Multipliers and Deriving a Matrix System of 2*N* Differential Equations of the Second Order. According to (2.2), we have

$$\frac{\partial \Phi_i}{\partial z_i} = 1, \quad \frac{\partial \Phi_1}{\partial z_1} = -1, \quad \ddot{z}_i = \ddot{z}_1 \quad (i = 2, \ldots, N). \tag{3.1}$$

Let us sum the equations for $\ddot{z}_i$ $(i = \overline{1, N})$ in (2.3) to obtain a system of equations without λ_i $(i = \overline{2, N})$. The multiplier λ_1 appears only in the first three equations of (2.3). Let us rearrange them as

$$m_1(\ddot{x}_1 + \ddot{\xi}) - k_{21}^x (x_2 - x_1) = 2\lambda_1 \left(x_1 - \rho\mu \,\mathrm{sign}\, \lambda_1 \frac{\dot{x}_1}{v} \right),$$

$$m_1(\ddot{y}_1 + \ddot{\eta}) - k_{21}^y (y_2 - y_1) = 2\lambda_1 \left(y_1 - \rho\mu \,\mathrm{sign}\, \lambda_1 \frac{\dot{y}_1}{v} \right),$$

$$m_\Sigma(\ddot{z}_1 + \ddot{\varsigma} + g) = 2\lambda_1 \left[(z_1 - \rho) - \rho\mu \,\mathrm{sign}\, \lambda_1 \frac{\dot{z}_1}{v} \right],$$

$$m_\Sigma = \sum_{i=1}^{N} m_i. \tag{3.2}$$

Since $z_1 = z_1(t)$, we find $\ddot{z}_i(t) = \ddot{z}_1(t)$ from Eq. (2.2) and the multipliers $\lambda_i(t)$ from Eq. (2.3), which contain z_i $(i = \overline{2, N})$, i.e., $\lambda_i = m_i(\ddot{z}_1(t) + \ddot{\varsigma}(t) + g)$.

Thus, the task is to find $2N$ unknowns $x_i(t)$, $y_i(t)$ $(i = 1, N)$, the variable $z_1(t)$ and $\lambda_1(t)$; i.e., $2N + 2$ unknowns total. To this end, we use $2N + 1$ differential equations plus the constraint equation (2.1).

We use the constraint equations (2.1) to eliminate z_1 and its derivatives $\dot{z}_1$, $\ddot{z}_1$ from Eq. (3.2). The multiplier λ_1 is found from the third equation in (3.2) and substituted into the first two equations. To simplify the expressions, we introduce additional notation:

$$\Delta(x_1, y_1) = -\sqrt{\rho^2 - x_1^2 - y_1^2} < 0, \tag{3.3}$$

$$z_1 - \rho = \Delta(x_1, y_1), \quad \dot{z}_1 = -(\dot{x}_1 P(x_1, y_1) + \dot{y}_1 Q(x_1, y_1)),$$

$$\ddot{z}_1 = -(\ddot{x}_1 P(x_1, y_1) + \ddot{y}_1 Q(x_1, y_1) + W(x_1, y_1, \dot{x}_1, \dot{y}_1)), \tag{3.4}$$

$$P(x_1, y_1) = x_1 / \Delta(x_1, y_1), \quad Q(x_1, y_1) = y_1 / \Delta(x_1, y_1),$$

$$W(x_1, y_1, \dot{x}_1, \dot{y}_1) = (\dot{x}_1^2 + \dot{y}_1^2) / \Delta(x_1, y_1) + (x_1, \dot{x}_1 + y_1, \dot{y}_1)^2 / \Delta^3(x_1, y_1). \tag{3.5}$$

Hereafter, we will omit arguments in the notation of these quantities: $P(x_1, y_1) = P$, $Q(x_1, y_1) = Q$, $W(x_1, y_1, \dot{x}_1, \dot{y}_1) = W$.

Eliminating the variables z_1, $\dot{z}_1$, $\ddot{z}_1$, and λ_1 yields the two equations

$$(m_1(\ddot{x}_1+\ddot{\xi})-k_{21}^x(x_2-x_1))(\Delta(x_1,y_1)+\mu\rho\,\mathrm{sign}\,\lambda_1(\dot{x}_1P+\dot{y}_1Q)/\bar{v})$$

$$+m_\Sigma(\ddot{x}_1P+\ddot{y}_1Q+W-\ddot{\varsigma}-g)(x_1-\mu\rho\,\mathrm{sign}\,\lambda_1\dot{x}_1/\bar{v})=0, \tag{3.6}$$

$$(m_1(\ddot{y}_1+\ddot{\eta})-k_{21}^y(y_2-y_1))(\Delta(x_1,y_1)+\mu\rho\,\mathrm{sign}\,\lambda_1(\dot{x}_1P+\dot{y}_1Q)/\bar{v})$$

$$+m_\Sigma(\ddot{x}_1P+\ddot{y}_1Q+W-\ddot{\varsigma}-g)(y_1-\mu\rho\,\mathrm{sign}\,\lambda_1\dot{y}_1/\bar{v})=0, \tag{3.7}$$

where

$$\bar{v}=\sqrt{\dot{x}_1^2+\dot{y}_1^2+(\dot{x}_1P+\dot{y}_1Q)^2}\,. \tag{3.8}$$

As is seen, Eqs. (3.6) and (3.7) depend only on the sign of λ_1. With $\lambda_1=\pm1$, Eqs. (3.6) and (3.7) can describe stable or unstable motions. Let us select the sign of λ_1 from the static solution of the third equation in (3.2), i.e., $\dot{z}_1=\ddot{z}_1=0$, $\ddot{\varsigma}=0$, $x_1=y_1=z_1=0$. In this case $\lambda_1=-0.5m_\Sigma g/\rho$; therefore, sign $\lambda_1=-1$. Next, we will use Eqs. (3.6) and (3.7) with negative λ_1.

After the elimination of z_1, $\dot{z}_1$, $\ddot{z}_1$, and λ_1, system (2.3) becomes

$$M\ddot{q}+H\dot{q}+Cq=E, \tag{3.9}$$

where $q^{\mathrm{T}}=(x_1,y_1,x_2,y_2,\ldots,x_N,y_N)=(q_1,q_2,\ldots,q_{2N-1},q_{2N})$ is the $2N$-dimensional vector of variables x_i, y_i $(i=\overline{1,N})$; "T" denotes transposition, M, H, and C are $2N\times2N$ matrices with elements m_{ij}, h_{ij}, c_{ij} $(i,j=1,2N)$, which are functions of the variables $x_1, y_1, \dot{x}_1, \dot{y}_1$; $E=E(t,x_1,y_1,x_1,\dot{y}_1,x_2,y_2,\ddot{\xi},\ddot{\eta})$ is a $2N$-dimensional vector;

$$m_{11}=m_1(\Delta(x_1,y_1)-\mu\rho(\dot{x}_1P+\dot{y}_1Q)\bar{v}^{-1})+m_\Sigma P(x_1+\mu\rho\dot{x}_1/\bar{v}),$$

$$m_{12}=m_\Sigma Q(x_1+\mu\rho\dot{x}_1/\bar{v}),\qquad M_{1l}=0\quad(l=\overline{3,2N}),$$

$$m_{21}=m_\Sigma P(y_1+\mu\rho\dot{y}_1/\bar{v}),$$

$$m_{22}=m_1(\Delta(x_1,y_1)-\mu\rho(\dot{x}_1P+\dot{y}_1Q)\bar{v}^{-1})+m_\Sigma Q(y_1+\mu\rho\dot{y}_1/\bar{v}),$$

$$m_{2l}=0\quad(l=\overline{3,2N}),\qquad m_{2i-1,j}=m_i\delta_{2i-1,j},$$

$$m_{2i,j}=m_i\delta_{2i,j}\quad(i=2,3,\ldots,N), \tag{3.10}$$

δ_{kl} is the Kronecker delta.

If $\mu\neq0$, the matrix M is nonsymmetric, i.e., the nonideality of constraints disturbs the symmetry of Lagrangian systems, which is one of their fundamental characteristics. If $\mu=0$, we have $Q(x_1,y_1)x_1=P(x_1,y_1)y_1$; therefore, $m_{12}=m_{21}$, which makes the inertia matrix symmetric.

Let us now calculate the elements of the damping and stiffness matrices:

$$h_{11}=-m_\Sigma(g+\ddot{\varsigma}-w)\frac{\mu\rho}{\bar{v}}-m_1\ddot{\xi}P\mu\rho/\bar{v},\qquad h_{12}=-m_1\ddot{\xi}Q\mu\rho/\bar{v},$$

$$h_{22}=-m_\Sigma(g+\ddot{\varsigma}-w)\frac{\mu\rho}{\bar{v}}-m_1\ddot{\eta}Q\mu\rho/\bar{v},\qquad h_{21}=-m_1\ddot{\eta}P\mu\rho/\bar{v},$$

$$h_{1l}=h_{2l}=0\quad(l=\overline{3,2N}),\qquad h_{ik}=0\quad(i=\overline{3,2N},k=\overline{1,2N}). \tag{3.11}$$

The stiffness matrix C has five diagonals. The leading diagonal has the following elements:

$$c_{2i-1,2i-1}=k_{i,i-1}^x+k_{i+1,i}^x\quad\text{and}\quad c_{2i,2i}=k_{i,i-1}^y+k_{i+1,i}^y.$$

The two neighboring (lower and upper) quazidiagonals have zero elements:

$$c_{2i,2i-1} = c_{2i,2i+1} = c_{2i-1,\,2i-2} = c_{2i-1,\,2i} = 0.$$

The next two quazidiagonals have the following elements:

$$c_{2i-1,2i-3} = -k^x_{i,i-1}, \quad c_{2i-1,2i+1} = -k^x_{i+1,i}, \quad c_{2i,2i-2} = -k^y_{i,i-1}, \quad c_{2i,2i+2} = -k^y_{i+1,i} \quad (i = \overline{2,N}).$$

The first two rows of the matrix C contain the follows elements:

$$c_{11} = k^x_{21}\Delta(x_1, y_1) - m_\Sigma(\ddot{\varsigma} + g - w), \quad c_{12} = 0, \quad c_{13} = -k^x_{21}\Delta(x, y),$$

$$c_{1e} = 0 \quad (e = \overline{4,N}), \quad c_{21} = 0, \quad c_{22} = k^y_{21}\Delta(x_1, y_1) - m_\Sigma(\ddot{\varsigma} + g - w),$$

$$c_{23} = 0, \quad c_{24} = -k^x_{21}\Delta(x_1, y_1), \quad c_{2e} = 0 \quad (e = \overline{5,N}). \tag{3.12}$$

The elements c_{11} and c_{22} include the inertia-mass term $-m_\Sigma(\ddot{\varsigma} + g - w)$ introduced formally, by analogy with the equation that describes the vibrations of a pendulum with moving suspension center.

The coordinates of the vector E are the following:

$$E_1 = -m_1\Delta(x_1, y_1)\ddot{\xi} - \mu\rho(\dot{x}_1 P + \dot{y}_1 Q)/v \cdot k^x_{21}(x_2 - x_1),$$

$$E_2 = -m_1\Delta(x_1, y_1)\ddot{\eta} - \mu\rho(\dot{x}_1 P + \dot{y}_1 Q)/v \cdot k^y_{21}(y_2 - y_1),$$

$$E_{2i-1} = -m_i\ddot{\xi}, \quad E_{2i} = -m_i\ddot{\eta} \quad (i = \overline{2,N}). \tag{3.13}$$

Formulas (3.11) for nonlinear elements of the dry friction dissipation matrix H can easily be transformed into formulas for the elements of the viscous friction dissipation matrix to describe the case where the surfaces of the spherical support and depression are lubricated with a viscous fluid. It is sufficient that the parameter μ be not constant but linearly dependent on the magnitude of the relative velocity of the contacting surfaces:

$$\mu = f_c\bar{v}. \tag{3.14}$$

Substituting (3.14) into (3.11) yields

$$h_{11} = -\left(m_\Sigma(g + \ddot{\varsigma} - w) + m_1\ddot{\xi}P\right)f_c\rho = \bar{h}_{11},$$

$$h_{22} = -\left(m_\Sigma(g + \ddot{\varsigma} - w) + m_1\ddot{\xi}Q\right)f_c\rho = \bar{h}_{22},$$

$$h_{12} = -m_1\ddot{\xi}Qf_c = \bar{h}_{12}, \quad h_{21} = -m_1\ddot{\eta}Pf_c = \bar{h}_{11},$$

$$\bar{h}_{1l} = \bar{h}_{2l} = 0 \quad (l = \overline{3, 2N}), \quad \bar{h}_{ij} = 0 \quad (i = \overline{3, 2N}, j = \overline{1, 2N}). \tag{3.15}$$

As is seen, the elements of the matrix H depend on the dynamic pressure exerted by the rigid bodies on the spherical seismic damper, and $h_{12} \neq h_{21}$ if $Q\ddot{\xi} \neq P\ddot{\eta}$.

4. Equations of Small Vibrations of the System of Bodies. With (3.10)–(3.13) and $\lambda_1 < 0$, the nonlinear equations of motion (3.9) are exact for any x_1, y_1 $(i = \overline{1,N})$ that satisfy the inequality

$$x_1^2 + y_1^2 < \rho^2. \tag{4.1}$$

As is seen, only the first two equations in (3.9) are nonlinear. Of practical value are equations that describe small vibrations, i.e.,

$$x_1^2 + y_1^2 << \rho^2 \quad (i = \overline{1,N}). \tag{4.2}$$

In the first two equations, we can discard the terms proportional to $x_1^n + y_1^n, \dot{x}_1^n + \dot{y}_1^n$ for $n >> 2$ and to $\ddot{x}_1 x_1, \ddot{x}_1 y_1, \ddot{y}_1 x_1$, $\ddot{y}_1 y_1$. The accelerations $\ddot{\xi}, \ddot{\eta}, \ddot{\varsigma}$ are assumed considerable; therefore, the products $\ddot{\eta} x_1, \ddot{\eta} y_1, \ddot{\xi} x_1, \ddot{\xi} y_1$ are kept.

According to (3.3) and (3.4), we have $\Delta(x_1, y_1) \approx -\rho, z_1 \approx 0, \dot{z}_1 \approx 0, \ddot{z}_1 \approx \ddot{x}_1 x_1/\rho + \ddot{y}_1 y_1/\rho \approx 0, P \approx -x_1/\rho, Q \approx -y_1/\rho$, $W \approx 0$ if x_1 and y_1 are small.

Thus, Eqs. (3.6), (3.7), and (3.8) become

$$\ddot{x}_1 + \mu\left[\frac{m_\Sigma}{m_1}(\ddot{\varsigma}+g) - \frac{x_1}{\rho}\ddot{\xi}\right]\frac{\dot{x}_1}{\sqrt{\dot{x}_1^2+\dot{y}_1^2}} - \mu\frac{y_1}{\rho}\ddot{\xi}\frac{\dot{y}_1}{\sqrt{\dot{x}_1^2+\dot{y}_1^2}} + \frac{m_\Sigma}{m_1}(\ddot{\varsigma}+g)\frac{x_1}{\rho} + \frac{k_{21}^x}{m_1}x_1 - \frac{k_{21}^x}{m_1}x_2 = -\ddot{\xi}(t),$$

$$\ddot{y}_1 + \mu\left[\frac{m_\Sigma}{m_1}(\ddot{\varsigma}+g) - \frac{y_1}{\rho}\ddot{\eta}\right]\frac{\dot{y}_1}{\sqrt{\dot{x}_1^2+\dot{y}_1^2}} - \mu\frac{x_1}{\rho}\ddot{\eta}\frac{\dot{x}_1}{\sqrt{\dot{x}_1^2+\dot{y}_1^2}} + \frac{m_\Sigma}{m_1}(\ddot{\varsigma}+g)\frac{y_1}{\rho} + \frac{k_{21}^y}{m_1}y_1 - \frac{k_{21}^y}{m_1}y_2 = -\ddot{\eta}(t). \quad (4.3)$$

Equations (4.3) should be supplemented with the equations from (2.3) that do not contain the multipliers $\lambda_i, i = \overline{2, N}$,

$$m_i \ddot{x}_i = -m_i \ddot{\xi} - k_{i,i-1}^x (x_i - x_{i-1}) + k_{i+1,i}^x (x_{i+1} - x_i),$$

$$m_i \ddot{y}_i = -m_i \ddot{\eta} - k_{i,i-1}^y (y_i - y_{i-1}) + k_{i+1,i}^y (y_{i+1} - y_i),$$

$$(i = \overline{2, N}), \quad k_{N+1,N}^x = k_{N+1,N}^y = 0. \quad (4.4)$$

The system of equations (4.3), (4.4) models the small forced vibrations of the chain system of N elastically connected rigid bodies with a spheroidal gravity-friction seismic damper.

This system (4.3) is true for nonzero initial velocities:

$$\dot{x}_1(0) \neq 0, \quad (\dot{y}_1(0) \neq 0) \quad \text{or} \quad v(0) \neq 0. \quad (4.5)$$

As in [19], two cases should be considered when $v(t_0) = 0$,

$$|F_\text{F}| \geq |F_3|, \quad |F_\text{F}| < |F_3|, \quad (4.6)$$

where

$$|F_\text{F}| = \sqrt{(F_\text{F}^x)^2 + (F_\text{F}^y)^2}, \quad |F_3| = \sqrt{(F_3^x)^2 + (F_3^y)^2},$$

$$F_\text{F}^x = \mu\left(\frac{m_\Sigma}{m_1}(\ddot{\varsigma}+g) - \frac{x_1\ddot{\xi}(t)}{\rho} - \frac{y_1\ddot{\xi}(t)}{\rho}\right), \quad F_\text{F}^y = \mu\left(\frac{m_\Sigma}{m_1}(\ddot{\varsigma}+g) - \frac{y_1\ddot{\eta}(t)}{\rho} - \frac{x_1\ddot{\eta}(t)}{\rho}\right),$$

$$F_3^x = -\left(\ddot{\xi}(t) - \frac{m_\Sigma}{m_1}(\ddot{\varsigma}+g)\frac{x_1}{\rho} - \frac{k_{21}^x}{m_1}(x_1 - x_2)\right), \quad F_3^y = -\left(\ddot{\eta}(t) - \frac{m_\Sigma}{m_1}(\ddot{\varsigma}+g)\frac{y_1}{\rho} - \frac{k_{21}^y}{m_1}(y_1 - y_2)\right),$$

$|F_\text{F}|$ is the force of friction; $|F_3|$ is the exciting force acting on the mass m_1.

In the first case, the platform does not move and

$$\ddot{x}_1 = 0, \quad x_1 = \text{const}, \quad \dot{x}_1 = 0,$$

$$\ddot{y}_1 = 0, \quad y_1 = \text{const}, \quad \dot{y}_1 = 0$$

instead of Eqs. (4.3).

In the second case, according to (4.6), the force of friction is less than the exciting force. Therefore, at the first step of motion, the force of friction is opposite to the exciting force. To obtain $\overline{v}(t_0) \neq 0$, we should replace the direction cosines of the velocity vector $\dot{x}_1 / v$, $\dot{y}_1 / v$ in Eq. (4.3) with $F_3^x/|F_3|$, $F_3^y/|F_3|$. Next, we use Eqs. (4.3) and (4.4) with $\bar{v} \neq 0$. Thus, conditions (4.6) generate a three-structural model of the system of rigid bodies with a gravity-friction damper.

If $k_{21}^x = k_{21}^y = 0$, $m_\Sigma = m_1$ in Eq. (4.3), then we will obtain the equation of small vibrations of one rigid body on a spheroidal gravity-friction damper. These exact equations differ by quantities of the second order of smallness in elements of the matrix H from the approximate equations derived in [19] based on the Lagrangian method of the second kinds for a one-degree-of-freedom system.

For $\ddot{\xi}(t) = \beta\ddot{\eta}(t)$, $|\beta| = \text{const}$, $k_{ij}^x = k_{ij}^y \neq 0$, the system of nonlinear differential equations (4.3), (4.4) has the following solution:

$$x_i(t) = \beta y_i(t) \quad (i = \overline{1,N}). \tag{4.7}$$

This solution makes it possible to halve the number of equations in system (4.3), (4.4) by introducing new variables u_i and v_i:

$$u_i = x_i + \beta y_i, \quad v_i = x_i - \beta y_i \quad (i = \overline{1,N}).$$

Then the system of equations (4.3), (4.4) has the solution $v_i = 0, u_i = u_i(t)$ for $v_i(0) = 0, u_i(0) \neq 0$, which describes the motion of the bodies only in the plane that crosses the plane Oxy along the straight line $x + \beta y = 0$.

Let us determine the natural frequencies and modes of small undamped vibrations of the system of N elastically connected rigid bodies with a spherical seismic damper. The linearized inertia and stiffness matrices of system (4.3), (4.4) are denoted by M_0 and C_0. These matrices follow from the matrices M and C when $\ddot{\xi}(t) = \ddot{\eta}(t) = \ddot{\zeta}(t) = 0, \mu = 0$ and $\approx x_1^2, y_1^2, x_1\dot{x}_1, y_1\dot{y}_1$ are rejected. Obviously, these matrices will have constant coefficients and become symmetric and positive definite. Then the frequency equation

$$\det|-M_0\omega^2 + C_0| = 0 \tag{4.8}$$

will have real solutions.

Denote the roots of Eq. (4.8) by $\omega_i\ (\overline{1,N})$.

Since Eqs. (4.3) and (4.4) are not coupled in the case of small undamped vibrations along the *Ox*- and *Oy*-axes, the frequency equation (4.8) becomes

$$\det|-M_0^x\omega_x^2 + C_0^x| \det|M_0^y\omega_y^2 + C_0^y| = 0,$$

where M_0^x and C_0^x (M_0^y and C_0^y) are the $N \times N$ matrices composed of odd (even) columns and rows of the matrices M_0 and C_0.

Thus, the procedures of determining the natural frequencies and modes of the chain system of rigid bodies are identical along the *Ox*- and *Oy*-axes, and the set of natural frequencies ω_i can be represented as ω_{ix}, ω_{iy} $(i = 1, N)$, with $\det|-M_0^x\omega_{ix}^2 + C_0^x| = 0$, $\det|-M_0^y\omega_{jx}^2 + C_0^y| = 0$.

Assume that there are no multiple frequencies among $\omega_{ix}\,(\omega_{iy})\ (i = \overline{1,N})$. Denote by q_x, q_y the N-dimensional vectors $q_x^{\mathrm{T}} = (x_1, x_2, \ldots, x_N)$, $q_y^{\mathrm{T}} = (y_1, y_2, \ldots, y_N)$.

Then the solution of the autonomous linearized system of equations (4.3), (4.4) for $\mu = 0$ (which corresponds to frequencies ω_{ix}) takes the form

$$q_{xk} = \sum_{i=1}^{N} X_{ik} \cos(\omega_{ix} t + \varphi_{ix}), \quad \varphi_{ix} = \text{const} \quad (k = \overline{1,N}), \tag{4.9}$$

where X_{ik} are the components of the vector $X_i^{\mathrm{T}} = (X_{1i}, X_{2i}, \ldots, X_{Ni})$, which is a nontrivial solution of the equation

$$(-M_0^x \omega^{2}{}_{ix} + C_0^x)X_i = 0 \quad (i = \overline{1, N}). \tag{4.10}$$

The solution describing the small vibrations of the system along the *Oy*-axis has a similar form:

$$q_{yk} = \sum_{j=1}^{N} Y_{kj} \cos(\omega_{jy} t + \varphi_{jy}), \qquad (-M_0^y \omega_{jy}^2 + C_0^y) Y_j = 0,$$

$$Y_j^{\mathrm{T}} = (Y_{1j}, Y_{2j}, \ldots, Y_{Nj}), \qquad \varphi_{jy} = \text{const} \quad (k = \overline{1, N}). \tag{4.11}$$

5. Double Phase-Frequency Resonance of the System of Two Elastically Linked Rigid Bodies with a Spherical Friction Damper. A numerical analysis reveals abnormally high values of the amplification factor for a linear system subjected to multifrequency perturbation. This is because, according to [7, 8], the seismic spectrum includes harmonics with initial phases differing by π and similar frequencies and amplitudes. This initiated a research of the double nonstationary phase–frequency resonance in vibrating systems [11, 12]. Quantitative characteristic of this resonance are phase–frequency surfaces. Their configuration gives an indication of the existence domain for the empirical probability density function of the main parameters of the majorant curve [6] that approximates the curve $\mu(T_c)$; T_c is the natural period of vibration. This curve is the base curve of the resonant-oscillation theory of seismic stability of structures.

Let us plot the phase-resonance surfaces for each body of the system of N elastically connected bodies. The amplification factor for the ith mass is given by

$$\mu_i = \mu_i(T_{cj}) = \sqrt{\frac{\max\limits_t \left[(\ddot{x}_i(t) + \ddot{\xi}(t))^2 + (\ddot{y}_i(t) + \ddot{\eta}(t))^2 + (\ddot{z}_i(t) + \ddot{\varsigma}(t))^2 \right]}{\max\limits_t [\ddot{\xi}^2(t) + \ddot{\eta}^2(t) + \ddot{\varsigma}^2(t)]}}, \tag{5.1}$$

where T_{cj} is the jth natural period of vibration of the system.

For practical purposes, we need to know the phase-resonance surfaces for the horizontal two-frequency damped perturbation of the system of rigid bodies. In this case, the external perturbation is defined by

$$\ddot{\xi}(t) = u_1(t), \qquad \ddot{\eta}(t) = u_2(t), \qquad \ddot{\varsigma}(t) = 0, \tag{5.2}$$

$$u_k(t) = \sum_{i=1}^{2} a_{ki} \exp(-\lambda_i t) \cos(v_i t + \alpha_i) \quad (k = 1, 2). \tag{5.3}$$

The distribution of amplitudes of the two-frequency perturbation along the O_1x- and O_1y-axes is characterized by the matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{5.4}$$

We have discussed some results on double phase–frequency resonance in the case of small vibrations under condition (4.2) and the numerical integration of Eqs. (4.3) and (4.4). In plotting phase-resonance surfaces, all the parameters of the system are assumed constant, except for the initial phases α_1, α_2 of perturbation. The frequencies of perturbation harmonics should be close to resonant. Let us plot, as an example, the surfaces ($i = 1, 2$) for a two-mass chain of bodies ($N = 2$) with the following parameters:

$$v_1 = 2\pi / T_1, \quad v_2 = 2\pi / T_2, \quad T_1 = T_c(1 - \varepsilon), \quad T_2 = T_c(1 + \varepsilon), \quad \varepsilon = 0.1,$$

$$\rho = g \frac{T_c^2}{4\pi^2}, \quad T_c = 1, \quad \lambda_1 = 0.3 / T_1, \quad \lambda_2 = 0.3 / T_2, \quad \mu = 0.1, \quad k_{21}^{\gamma} / m_2 = \left(\frac{2\pi}{T_3^{\gamma}} \right)^2,$$

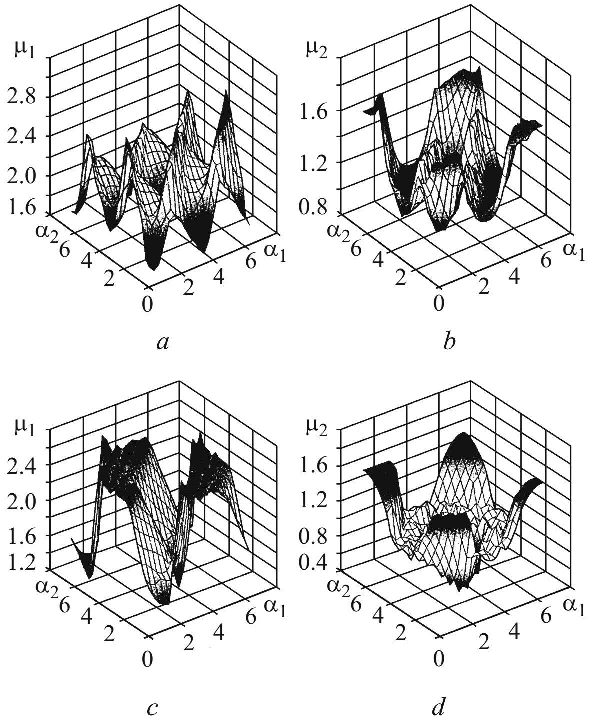

Fig. 3

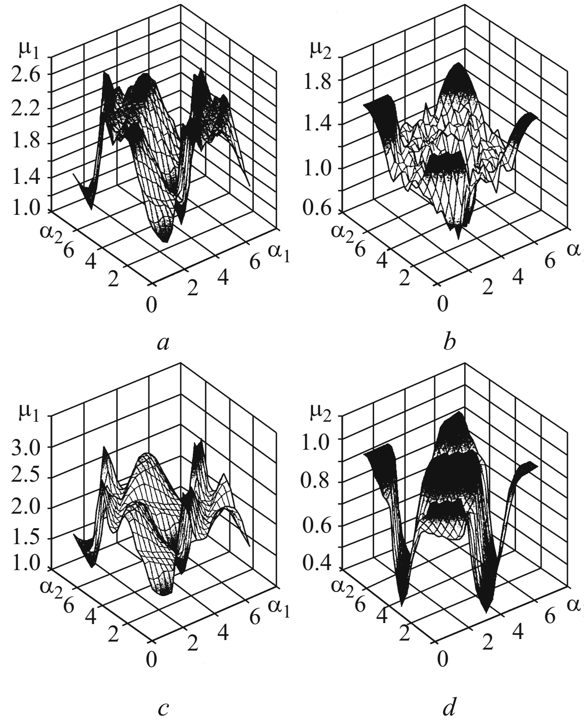

Fig. 4

$$\gamma = x, y, \quad T_3^x = 1.5, \quad T_3^y = 1.2, \quad m_\Sigma / m_1 = 2, \quad g = 9.81 \text{ m/sec}^2.$$

Let us consider two cases of distribution of external harmonics along the *Ox*- and *Oy*-axes:

I. $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$—each harmonic acts along one coordinate axis;

II. $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$—both harmonics act along each coordinate axis with equal intensities.

Figures 3*a* and 3*b* show the surfaces $\mu_1(\alpha_1, \alpha_2)$, $\mu_2(\alpha_1, \alpha_2)$ for the first case.

As is seen, the first mass vibrates more intensively than the second one. Its surface has four resonance humps separated by three valleys that represent antiresonance. This phase-resonance surface is similar to the surface of a one-mass system obtained in [11]. For the second mass, the phase-resonance surface also has several humps and valleys of more complex configuration. For this case, we have $x_2(t) \neq y_2(t)$.

Figures 3*c* and 3*d* represent the phase-resonance surfaces for case II where the perturbations along the *Ox*- and *Oy*-axes are identical. As is seen, the surface for the first mass has two humps separated by a valley. This is typical of a system with one degree of freedom [20]. The phase-resonance surface for the second mass has peaks near the corners of the square domain α_1, $\alpha_2 \in [0, 2\pi]$. It is similar to the phase-resonance surface of a system with one degree of freedom subject to nonresonant two-frequency perturbation [9]. All the surfaces are not smooth.

Figures 4*a* and 4*b* show the phase-resonance surfaces for the case where the system (4.3), (4.4) has solution (5.7) $(i = \overline{1, N})\, x_i = y_i$ with $T_3^x = T_3^y = 1.5$ and a degenerate matrix *a* such that the perturbations along the coordinate axes are identical: $O\varsigma = O_1$, $\ddot{\xi}(t) = \ddot{\eta}(t)$. As is seen, these phase-resonance surfaces are qualitatively similar to the surfaces in the previous case. Figures 4*c* and 4*d* represent the case of viscous friction $\mu = f_c \sqrt{\dot{x}_1^2 + \dot{y}_1^2}$. This friction makes the phase-resonance surfaces smooth, which is typical of linear vibrating systems [20].

Conclusions. We have modeled a chain system of *N* elastically connected rigid bodies with a spheroidal gravity-friction seismic damper. We used the Painlevé–Lagrange equations of the first kind as a nonlinear dynamic model of translational vibrations of the mechanical system about the equilibrium position. Existence conditions for the motion of the system of bodies in one plane have been established. The double nonstationary phase–frequency resonance in the system with $N = 2$ has been analyzed. After numerical integration of systems of differential equations, we have plotted the phase–frequency surfaces for several combinations of parameters of the dynamic system subject to two-frequency loading.

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