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USING DISCRETE FOURIER SERIES TO SOLVE BOUNDARY-VALUE STRESS PROBLEMS FOR ELASTIC BODIES WITH COMPLEX GEOMETRY AND STRUCTURE

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The paper deals with some approaches to solving linear and nonlinear boundary-value stress problems for elastic bodies with complex geometry and structure. The problems are described by partial differential equations solved using discrete Fourier series. The results obtained are presented in the form of plots and tables

Keywords: elastic body, boundary-value problem, partial differential equation, stress state, complex geometry, discrete Fourier series

Introduction. The solutions of two-dimensional boundary-value stress problems for plates, shells, and space bodies described by partial differential equations are often represented as Fourier series in powers of one coordinate. This makes it possible to reduce the dimension of problems, making them one-dimensional ones, which could be solved by approximate analytic or numerical methods if the differential equations and boundary conditions would permit separation of variables. In some classes of problems, however, the configuration of the domain, thickness variation, mechanical and other factors are such that it appears impossible to separate variables and reduce the problem to a system of ordinary differential equations [3, 5, 7–10, 22].

The present paper outlines a nontraditional approach to this class of problems. It employs discrete Fourier series, i.e., Fourier series of functions defined on a discrete set of points. Modern computers are capable of calculating series with a sufficiently great number of terms to solve problems with high accuracy. We will solve linear and nonlinear problems for thin and space bodies with complex geometry.

1. Discrete Fourier Series Approach to Solving Boundary-Value Problems for Partial Differential Equations. Let us discuss an approach to solving two-dimensional boundary-value stress–strain problems for elastic bodies under various loading and certain boundary conditions. The problem is reduced to partial differential equations with coefficients dependent on two coordinates. The approach employs discrete Fourier series and makes problems one-dimensional [12–14, 32, 33, 42].

Let the stress–strain state of an elastic body be described by the system of partial differential equations

$$\frac{\partial Z_i}{\partial \alpha} = \Phi_i\left(\alpha, \beta, \frac{\partial^k Z_j}{\partial \beta^k}\right) + f_i(\alpha, \beta) \qquad (i, j, k = \overline{1, l}), \tag{1.1}$$

where $Z_i = Z_i(\alpha, \beta)\,(\alpha_1 \le \alpha \le \alpha_2, \beta_1 \le \beta \le \beta_2)$ are the unknown functions; Φ_i are linear functions; $f_i(\alpha, \beta)$ are the right-hand sides; and $\alpha O \beta$ is an orthogonal curvilinear coordinate system.

For open elastic bodies, this system of equations is supplemented with boundary conditions on $\alpha = \text{const}$ and $\beta = \text{const}$. In the case of bodies closed in one coordinate direction, the boundary conditions for this direction are replaced by periodicity conditions.

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The periodicity in the boundary-value problem for the system of equations (1.1) for elastic bodies closed in, say, the $O\beta$-direction makes it possible to represent solutions for all unknown functions as Fourier series in powers of the coordinate β. It is, however, necessary that no terms in the equations hinder the separation of variables with respect to this coordinate. In simpler problems, variables can be separated by expanding all the functions into Fourier series.

However, the system of differential equations (1.1) often has terms with coefficients that include geometrical and mechanical parameters, which makes it impossible to separate variables and expand the unknown functions into Fourier series. To overcome these difficulties, subsidiary functions expressed in terms of the unknown functions and their derivatives are introduced. Then the governing system of equations becomes

$$\frac{\partial Z_i}{\partial \alpha} = F_i\left(\alpha, \beta, \frac{\partial^k Z_j}{\partial \beta^k}, \varphi_r^p\right) + f_i(\alpha, \beta) \quad (i, j, k = \overline{1, l}, r = \overline{1, R}, p = \overline{1, P}), \tag{1.2}$$

where $\varphi_r^p = \varphi_r^p\left(\alpha, \beta, \frac{\partial^s Z_i}{\partial \alpha^s}, \frac{\partial^t Z_i}{\partial \beta^t}\right) \; (s, t \le l)$.

Since the system of differential equations (1.2) includes both subsidiary (φ_r^p) and unknown (Z_i) functions, the total number of unknown functions exceeds the number of equations. This should be taken into account in solving the boundary-value problem.

To solve the original boundary-value problem, we expand all the functions in (1.2) into Fourier series in powers of β:

$$\widetilde{X}(\alpha, \beta) = \sum_{m=0}^{M} \widetilde{X}_m(\alpha) \cos \lambda_m \beta, \quad \widetilde{Y}(\alpha, \beta) = \sum_{m=0}^{M} \widetilde{Y}_m(\alpha) \sin \lambda_m \beta, \quad \lambda_m = 2\pi m / T, \tag{1.3}$$

where $\widetilde{X}$ and $\widetilde{Y}$ are the unknown and subsidiary functions; T is the period.

Substituting series (1.3) into the system of equations (1.2), separating variables, and performing some transformations, we obtain a coupled system of ordinary differential equations for the amplitudes of series (1.3):

$$\frac{dZ_{im}}{d\alpha} = F_{im}(\alpha, Z_{im}, \varphi_{rm}^p) + f_{im}(\alpha) \quad (i = \overline{1, l}, m = \overline{0, M}, r = \overline{1, R}, p = \overline{1, P}). \tag{1.4}$$

Similar transformations lead to boundary conditions for the amplitudes of the functions Z_{im} at the ends of the interval $\alpha_1 \le \alpha \le \alpha_2$. To solve the boundary-value problem for the system of equations (1.4), we will use the stable discrete-orthogonalization method [2, 6, 8]. The system of equations (1.4) in many cases appears to be stiff because of the inhomogeneity of the mechanical and geometrical properties and the load.

The method used to solve the one-dimensional problem orthogonalizes the vector solutions of the Cauchy problems for a finite number of values of the argument. Besides the amplitudes of the unknown functions, the system of equations (1.4) contains the amplitudes of the subsidiary functions, which should be determined individually. In integrating system (1.4), we calculate the amplitudes of the subsidiary functions from the amplitudes of the unknown functions at some points of the interval α for all harmonics simultaneously at each step of discrete orthogonalization, a value of $\beta_1 \le \beta \le \beta_2$ being constant. The functions defined on a discrete set of points are expanded into Fourier series [23, 29, 30]. As the number of points at which the subsidiary functions are calculated is increased, the discrete Fourier series tends to the exact Fourier series, which is a way to improve the accuracy of the results. We determine the coefficients of these series with Runge's method, substitute them into the system of equations (1.4), and continue its integration, satisfying the boundary conditions at the ends of the interval $\alpha_1 \le \alpha \le \alpha_2$.

Note that the above procedure depends on the form of the right-hand sides in (1.1).

In solving applied problems, use in most cases is made of only few first terms of the discrete Fourier series because the Fourier coefficients rapidly decrease, weakening the effect of higher harmonics. The accuracy of approximations in calculating Fourier series is known to be greatly dependent on the rate of decrease in the Fourier coefficients, which in turn is associated with the differential properties of a function extended to $(-\infty, \infty)$. There are also approximate approaches that make it possible to compare values of a coefficient in a discrete Fourier series with the exact values of the same or other Fourier coefficients for the same function defined analytically [29].

Let us discuss results obtained with this approach.

Let a function $y(x)$ be specified on a set of points, i.e., $y(x_i) = y_i$, $x_i = i2\pi / k$ $(i = 0, 1, 2, \dots, k-1)$. Let us expand the function $f(x)$ defined on a discrete set of points x_i $(i = \overline{0, k-1})$ into a Fourier series:

$$y(x) = a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx) \quad (n \le k/2), \tag{1.5}$$

where the coefficients a_0, a_m, and, b_m are defined by

$$a_0 = \frac{1}{k}\sum_{i=0}^{k-1} y_i, \quad a_m = \frac{2}{k}\sum_{i=0}^{k-1} y_i \cos m\frac{2\pi i}{k}, \quad b_m = \frac{2}{k}\sum_{i=1}^{k-1} y_1 \sin m\frac{2\pi i}{k} \quad (m \le k/2). \tag{1.6}$$

Let us establish the relationship among the approximate and exact values of the Fourier coefficients.

Consider a function $y = f(x)$ that is analytically defined on the interval $[0, 2\pi]$ and doubly differentiable. Its exact Fourier series is

$$y(x) = A_0 + \sum_{j=1}^{\infty} A_j \cos jx + \sum_{j=1}^{\infty} B_j \sin jx, \tag{1.7}$$

where the capital letters denote the exact values of the coefficients. The values $y_i = y(x_i)$ appearing in (1.6) can be calculated by setting $x_i = i2\pi / k$ $(i = 0, 1, 2, \dots, k-1)$. Substituting these values into (1.6) and carrying out some transformations, we obtain

$$a_m = A_m + A_{k-m} + A_{k+m} + A_{2k-m} + \dots,$$

$$b_m = B_m - B_{k-m} + B_{k+m} - B_{2k-m} + \dots \quad (m \le k/2).$$

For example,

$$a_0 = A_0 + A_{12} + \dots, \; a_1 = A_1 + A_{11} + \dots, \; a_2 = A_2 + A_{10} + \dots, \; a_3 = A_3 + A_9 + \dots$$

for $k = 12$ and

$$a_0 = A_0 + A_{24} + \dots, \; a_1 = A_1 + A_{23} + \dots, \; a_2 = A_2 + A_{22} + \dots, \; a_3 = A_3 + A_{21} + \dots,$$

$$a_4 = A_4 + A_{20} + \dots, \; a_5 = A_5 + A_{19} + \dots, \; a_6 = A_6 + A_{18} + \dots, \; a_7 = A_7 + A_{17} + \dots,$$

$$a_8 = A_8 + A_{16} + \dots, \text{ etc.}$$

for $k = 24$.

It can be seen that for adequate accuracy, only two to three harmonics may be retained when $k = 12$ and seven to eight harmonics when $k = 24$.

Discrete-Orthogonalization Method. Consider the linear boundary-value problem

$$\frac{d\bar{g}}{dt} = A(t)\bar{g}(t) + \bar{f}(t) \quad (a \le t \le b) \tag{1.8}$$

with the boundary conditions

$$B_1\bar{g}(a) = \bar{b}_1, \tag{1.9}$$

$$B_2\bar{g}(b) = \bar{b}_2, \tag{1.10}$$

where $\bar{g} = \{g_1, g_2, \dots, g_n\}^{\mathrm{T}}$ is a column vector, $\bar{f}$ is the column vector of the right-hand side, $A(t)$ is a given $n{\times}n$ matrix, B_1 and B_2 are given $k{\times}n$ and $(n-k){\times}n$ matrices ($k < n$), and $\bar{b}_1$ and $\bar{b}_2$ are given vectors.

The essence of the method is as follows. We will seek the solution of the boundary-value problem (1.8)–(1.10) in the form

$$\bar{g}(t)=\sum_{j=1}^{m} C_j \bar{g}_j(t)+\bar{g}_{m+1}(t), \tag{1.11}$$

where $m=\min\{k, n-k\}$ (let $m=n-k$ for definiteness), $\bar{g}_j$ are the solutions of the Cauchy problem for the system of equations (1.8) for $\bar{f}=0$ with initial conditions that satisfy the boundary conditions at the left end of the interval (1.9) for $\bar{b}_1=0$, $\bar{g}_{m+1}$ is the solution of the Cauchy problem for (1.8) with initial conditions that satisfy the boundary conditions (1.9), m is the number of boundary conditions at the right end of the interval of integration.

The discrete-orthogonalization method ensures a stable computational process by orthogonalizing the vector solutions of the Cauchy problem at a finite number of points within the interval of variation in the argument. Let integration points t_s ($s=0$, 1, …, N) divide the segment $[a, b]$ so that $t_0=a$ and $t_N=b$. Of these points, we choose orthogonalizaton points T_i ($i=0$, 1, …, M). The selection of these points is usually determined by the accuracy required, but is arbitrary in other respects.

Let the Cauchy problems have been solved at the point T_i by using, say, the Runge–Kutta method. Denote the solutions by $\bar{u}_r(T_i)$ $(r=1, 2, \dots, m+1)$.

Thus, prior to orthogonalization, we have the following vectors at the point T_i: $\bar{u}_1(T_i), \bar{u}_2(T_i), \dots, \bar{u}_m(T_i), \bar{u}_{m+1}(T_i)$. Let us orthonormalize the vectors $\bar{u}_j(T_j)$ $(j=1, 2, \dots, m)$ at the point T_i: $\bar{z}_1(T_i), \bar{z}_2(T_i), \dots, \bar{z}_m(T_i)$.

The vectors $\bar{z}_i$ can be expressed in terms of the vectors $\bar{u}_i$ as follows:

$$\bar{z}_r=\frac{1}{w_{rr}}\left(\bar{u}_r-\sum_{j=1}^{r-1} w_{rj}\bar{z}_j\right) \qquad (r=1, 2, \dots, m),$$

where $w_{rj}=(\bar{u}_r, \bar{z}_j)$, $w_{rr}=\sqrt{(\bar{u}_r, \bar{z}_r)-\sum_{j=1}^{r-1} w_{rj}^2}$, $j<r$.

The vector $\bar{z}_{m+1}$ is not normalized and is given by

$$\bar{z}_{m+1}=\bar{u}_{m+1}-\sum_{j=1}^{m} w_{m+1,j}\ \bar{z}_j.$$

After transformations, we obtain the matrix equality

$$\begin{vmatrix} \bar{u}_1(T_i) \\ \bar{u}_2(T_i) \\ \cdot \\ \cdot \\ \cdot \\ \bar{u}_m(T_i) \\ \bar{u}_{m+1}(T_i) \end{vmatrix} = \Omega_i \begin{vmatrix} \bar{z}_1(T_i) \\ \bar{z}_2(T_i) \\ \cdot \\ \cdot \\ \cdot \\ \bar{z}_m(T_i) \\ \bar{z}_{m+1}(T_i) \end{vmatrix}, \tag{1.12}$$

where

$$\Omega_i=\Omega(T_i)=\begin{vmatrix} w_{11}(T_i) & 0 & 0 & \cdots & 0 \\ w_{21}(T_i) & w_{22}(T_i) & 0 & \cdots & 0 \\ w_{31}(T_i) & w_{32}(T_i) & w_{33}(T_i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{m1}(T_i) & w_{m2}(T_i) & w_{m_3}(T_i) & \cdots & 0 \\ w_{m+1,1}(T_i) & w_{m+1,2}(T_i) & w_{m+1,3}(T_i) & \cdots & 1 \end{vmatrix}.$$

The vectors $\bar{z}_r(T_i)$ are the initial values of the Cauchy problems for the homogeneous (r = 1, 2,…, m) and inhomogeneous ($r = m+1$) systems of differential equations (1.8) for $T_i \le t \le T_{i+1}$.

After the integration over $T_{M-1} \le t \le T_M$ and the orthogonalization at T_M, we have

$$\bar{g}(T_M) = \sum_{j=1}^{m} C_j^{(M)} \bar{z}_j(T_M) + \bar{z}_{m+1}(T_M). \tag{1.13}$$

Satisfying the boundary conditions at the right-hand end of the integration interval, we obtain a system of m linear algebraic equations for $C_j^{(M)}$ $(j = 1, 2, \dots, m)$. After determining $C_j^{(M)}$, the solution of the boundary-value problem (1.8)–(1.10) at the point $t = T_M$ is defined by (1.13). This completes the forward procedure of the method.

The backward procedure uses the constants $C_j^{(i)}$ $(j = 1, 2, \dots, m)$ to determine $C_j^{(i-1)}$ beginning with $i = M$. Then

$$\Omega'_i \bar{C}^{(i-1)} = \bar{C}^{(i)} \quad (i = 1, 2, \dots, M) \quad \text{or} \quad \bar{C}^{(i-1)} = [\Omega'_i]^{-1} \bar{C}^{(i)}, \tag{1.14}$$

where Ω'_i is a transposed matrix, $\bar{C}^{(i)}$ is a column vector with components $C_1^{(i)}, C_2^{(i)}, \dots, C_m^{(i)}, 1$.

Thus, we can use (1.14) to determine $C_j^{(i)}$ at all the points beginning with $i = M$.

This algorithm requires storing information on the matrices Ω_i and vectors $\bar{z}_r$ $(r = 1, 2, \dots, m+1)$.

All the information obtained at the orthogonalization points is not usually needed in practice; it is sufficient to use the values of the unknown functions at the so-called output points, which are far fewer than the orthogonalization points. In this connection, the following trick may be used to highly reduce the amount of information to be stored.

Let T_{i-1} and T_{i+p} be the output points. Then, Eq. (1.14) leads to

$$\bar{C}^{(i-1)} = \left[\left(\prod_{j=0}^{p} \Omega_{i+j} \right)' \right]^{-1} \bar{C}^{(i+p)}.$$

Thus, to determine the vectors $\bar{C}^{(i-1)}$, it is necessary to store information on the product of matrices $\prod_{j=0}^{p} \Omega_{i+j}$, which greatly saves computer memory.

In solving specific problems, the following inductive procedures may be used to evaluate the accuracy of the solution [8]: (i) increasing the number of orthogonalization points, which greatly reduces the computational error due to the stiffness of the system of equations and (ii) solving the boundary-value problem from left to right and from right to left. Since these schemes are very different, coincidence of all significant figures would allow regarding all digits except the last one as exact.

The sections below discuss the solutions for various classes of elastic bodies found by the above approach.

2. Rectangular Plates. Consider a hinged square plate of varying thickness bent by a transverse load $q = q_0 \sin(\pi x / a) \sin(\pi y / a)$ [11, 14, 42]. The midsurface of the plate is described in a rectangular orthogonal coordinate system Oxy and occupies the domain $0 \le x \le a$, $0 \le y \le a$. The thickness of the plate varies in one coordinate direction as

$$h(x) = h_0 [1 + \alpha(1 - 6x + 6x^2)] \quad (-1 < \alpha < 1), \quad h_0 = \text{const}. \tag{2.1}$$

The stress–strain state of the plate is described by a system of partial differential equations with variable coefficients [42]:

$$\frac{\partial \hat{Q}_y}{\partial y} = \frac{\partial^2}{\partial x^2} \left[D_M (1 - \nu^2) \frac{\partial^2 w}{\partial x^2} - \nu M_y \right] - q,$$

$$\frac{\partial M_y}{\partial y} = \hat{Q}_y - 2 \frac{\partial}{\partial x} \left[D_M (1 - \nu) \frac{\partial \vartheta_y}{\partial x} \right],$$

$$\frac{\partial w}{\partial y} = -\vartheta_y, \quad \frac{\partial \vartheta_y}{\partial y} = \frac{1}{D_M} M_y + \nu \frac{\partial^2 w}{\partial x^2}, \tag{2.2}$$

where $D_M = \frac{Eh^3}{12(1-\nu^2)}$ is the bending stiffness; $\hat{Q}_y = Q_y + \frac{\partial H}{\partial x}$ is the reduced transverse force; H is the torque; M_y, w, and ϑ_y are the bending moment, deflection, and angle of rotation at $y = \text{const}$, respectively; E is the elastic modulus, and ν is Poisson's ratio.

The boundary conditions are

$$w = 0, \quad M_x \equiv \nu M_y - \frac{Eh^3}{12} \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, a, \tag{2.3}$$

$$w = 0, \quad M_y \equiv \nu M_x - \frac{Eh^3}{12} \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = 0, a. \tag{2.4}$$

This problem can be solved by two methods. Since the thickness varies only in one direction (along the x-axis), expanding the unknown functions into Fourier series in powers of the coordinate y and separating variables yields a system of ordinary differential equations, which can be solved by a numerical method for the variable x. The solution may be considered exact. This is one method.

The other method employs discrete Fourier series. To this end, the terms in the governing system of differential equations (2.2) that hinder the separation of variables with respect to x are replaced by subsidiary functions. This yields the system of differential equations

$$\frac{\partial \hat{Q}_y}{\partial y} = \frac{E}{12} \frac{\partial^2 \varphi_1}{\partial x^2} - \nu \frac{\partial^2 M_y}{\partial x^2} - q, \quad \frac{\partial M_y}{\partial y} = \hat{Q}_y - \frac{E}{6(1+\nu)} \frac{\partial \varphi_2}{\partial x},$$

$$\frac{\partial w}{\partial y} = -\vartheta_y, \quad \frac{\partial \vartheta_y}{\partial y} = \frac{12(1-\nu^2)}{E} \varphi_3 + \nu \frac{\partial^2 w}{\partial x^2}, \tag{2.5}$$

where

$$\varphi_1(x, y) = h^3 \frac{\partial^2 w}{\partial x^2}, \quad \varphi_2(x, y) = h^3 \frac{\partial \vartheta_y}{\partial x}, \quad \varphi_3(x, y) = \frac{M_y}{h^3}. \tag{2.6}$$

Formally, the coefficients of system (2.5) are independent of the coordinate x, though the subsidiary functions are dependent. The solution of the boundary-value problem for (2.5) that satisfies the boundary conditions at $x = 0$ and $x = a$ and the functions φ_j $(j = 1, 2, 3)$ on the right-hand sides are expanded into series:

$$\tilde{Y}(x, y) = \sum_{m=1}^{M} \tilde{Y}_m(y) \sin \lambda_m x, \quad \varphi_2(x, y) = \sum_{m=1}^{M} \varphi_{2,m}(y) \cos \lambda_m x, \tag{2.7}$$

where

$$\tilde{Y} = \{\hat{Q}_y, M_y, w, \vartheta_y, \varphi_1, \varphi_3, q\}, \quad \lambda_m = \pi m / a.$$

Substituting (2.7) into (2.5) and (2.4), we get a system of ordinary differential equations for the amplitudes in these series:

$$\frac{d\hat{Q}_{y,m}}{dy} = -\lambda_m^2 \left(\frac{E}{12} \varphi_{1,m} - \nu M_{y,m} \right) - q_m, \quad \frac{dM_{y,m}}{dy} = \hat{Q}_{y,m} + \frac{E}{6(1+\nu)} \lambda_m \varphi_{2,m},$$

$$\frac{dw_m}{dy} = -\vartheta_{y,m}, \quad \frac{d\vartheta_{y,m}}{dy} = \frac{12(1-\nu^2)}{E}\varphi_{3,m} - \nu\lambda_m^2 w_m \quad (m = \overline{1, M}), \tag{2.8}$$

where $q_1 \neq 0, q_2 = \ldots = q_M = 0$.

The boundary conditions are

$$w_m = 0, \quad M_{y,m} = 0 \quad \text{at} \quad y = 0, a. \tag{2.9}$$

The equations in (2.8) are integrated for all harmonics simultaneously. During the integration, we calculate the amplitudes of the subsidiary functions (2.6) at each iteration. To this end, the following quantities are calculated at a number of points x_i $(i = \overline{1, R})$ of the segment [0, a] from the current amplitudes of the unknown functions for a fixed value of y_k:

$$h(x_i) = h_0[1 + \alpha(1 - 6x_i + 6x_i^2)], \quad \varphi_1^i = \varphi_1(x_i, y_k) = -h^3(x_i)\sum_{m=1}^{M}\lambda_m^2 w_m(y_k)\sin\lambda_m x_i,$$

$$\varphi_2^i = \varphi_2(x_i, y_k) = h^3(x_i)\sum_{m=1}^{M}\lambda_m \vartheta_{y,m}(y_k)\cos\lambda_m x_i,$$

$$\varphi_3^i = \varphi_3(x_i, y_k) = \frac{1}{h^3(x_i)}\sum_{m=1}^{M} M_{y,m}(y_k)\sin\lambda_m x_i. \tag{2.10}$$

Next, for a fixed value of y_k, we oddly extend the functions φ_1 and φ_3 and evenly extend the function φ_2 to the segment [a, 2a] and calculate $\varphi_{j,m}(y_k)$ using a standard procedure for determining the coefficients of a tabulated Fourier function of variable x. After that, $\varphi_{j,m}(y_k)$ are substituted into the system of equations (2.8) to take the next step of integration, going from y_k to y_{k+1}. Prior to the integration, initial values of the unknown functions are specified considering the boundary conditions.

Let us examine the influence of the number R of points at which the subsidiary functions are calculated and the number M of points used to solve the problem for a bent square plate on the convergence of the solution obtained using discrete Fourier series to the exact solution.

The input data: $a = 1$, $\nu = 0.3$, $h_0 = 0.1$, $\alpha = 0.3$, $R = 38, 40, 60, 80, 100$, $M = 6, 8, 10, 15$. Table 1 collects the values of the deflection w, bending moment M_x, and angle ϑ_x in the section y = 0.5 versus certain values of the coordinate x. The last row contains the exact solution obtained by the former method.

Table 1 demonstrates that as R is increased and M is kept constant, the approximate solutions for w, ϑ_x (an analog of the first derivative with respect to w), and M_x (an analog of the second derivative with respect to w) converge to the exact solution. For example, the solution for w at x = 0.5 obtained using discrete Fourier series differs from the exact solution by $1 \cdot 10^{-4}$. The errors are similar for ϑ_x and M_x. This is because we solve a system of equations that includes all these functions.

The stress–strain state of rectangular plates having thickness varying in two coordinate directions and being subjected to a normal load was analyzed in [34] considering certain boundary conditions. The effect of variation in the thickness at constant weight on the deformation and strength of the plates was also examined.

For example, the thickness was considered to vary in two coordinate directions as follows:

$$h(x, y) = h_0\left(1 + \beta\cos\frac{2\pi x}{a}\right)\left[1 - \alpha\left(\frac{2y}{b} - 1\right)^2\right] \quad \left(-\frac{a}{2} \leq x \leq \frac{a}{2}, \quad 0 \leq y \leq b\right). \tag{2.11}$$

The weight of the plate is independent of the values of α and β.

Let us consider a plate having the following boundary conditions:

$$w = 0, \quad M_x = \nu M_y - \frac{Eh^3}{12}\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = \pm\frac{a}{2}, \tag{2.12}$$

$$w = 0, \quad \vartheta_y = 0 \quad \text{at} \quad y = 0, \quad y = b \tag{2.13}$$

TABLE 1

R	M	Ew/q_0			$-E\vartheta_x/q_0$			$M_x/10^{-2}q_0$		
		x								
		0.1	0.3	0.5	0	0.2	0.4	0.1	0.3	0.5
38	6	8.327	24.026	31.381	83.167	80.277	38.552	0.365	1.992	2.813
40		8.333	24.039	31.396	83.232	80.308	38.564	0.368	1.992	2.813
38	8	8.324	24.023	31.386	83.081	80.286	38.657	0.355	1.994	2.815
40		8.330	24.036	31.401	83.148	80.318	38.666	0.359	1.996	2.816
38	10	8.324	24.022	31.384	83.048	80.249	38.637	0.355	1.994	2.812
40		8.330	24.035	31.399	83.116	80.282	38.646	0.359	1.994	2.814
40	15	8.331	24.035	31.399	83.086	80.283	38.642	0.369	1.994	2.815
60		8.345	24.665	31.434	83.251	80.159	38.666	0.378	1.998	2.816
80		8.350	24.076	31.446	83.308	80.387	38.674	0.377	1.999	2.816
100		8.352	24.081	31.452	83.334	80.399	38.677	0.378	1.999	2.817
Exact solution		8.357	24.091	31.463	83.377	80.424	38.687	0.379	2.000	2.817

and being subjected to the following distributed transverse load:

$$q = q_0 \sin \frac{\pi}{a}\left(\frac{a}{2} + x\right). \tag{2.14}$$

Though the opposite edges $x = \pm a/2$ are hinged, it is impossible to separate variables with respect to x because the thickness varies along the x-axis. To overcome these difficulties, we introduce subsidiary functions into the governing system of equations. To be calculated, these functions are expanded into Fourier series.

Hence, we write the original system of equations (2.2) in the form (2.5) that contains three subsidiary functions (2.6). The solution of the boundary-value problem for (2.5) with (2.12), (2.13) and, (2.14) is sought in the form

$$R(x, y) = \sum_{n=1}^{N} R_n(y) \sin \lambda_n \left(\frac{a}{2} + x\right),$$

$$\varphi_2(x, y) = \sum_{n=1}^{N} \varphi_{2,n}(y) \cos \lambda_n \left(\frac{a}{2} + x\right), \quad \lambda_n = \frac{\pi n}{a},$$

$$R = \{\hat{Q}_y, M_y, w, \vartheta_y, \varphi_1, \varphi_3, q\}. \tag{2.15}$$

Substituting (2.15) into Eqs. (2.5), (2.9), (2.12), (2.13), and (2.14), we get a system of ordinary differential equations for the amplitudes of these series:

$$\frac{d\hat{Q}_{y,n}}{dy} = -\lambda_n^2\left(\frac{E}{12}\varphi_{1,n} - \nu M_{y,n}\right) - q_n,$$

$$\frac{dM_{y,n}}{dy} = \hat{Q}_{y,n} + \frac{E}{6(1+\nu)}\lambda_n \varphi_{2,n},$$

$$\frac{dw_n}{dy} = -\vartheta_{y,n}, \qquad \frac{d\vartheta_{y,n}}{dy} = \frac{12(1-\nu^2)}{E}\varphi_{3,n} - \nu\lambda_n^2 w_n, \tag{2.16}$$

where $q_1 \neq 0, q_2 = \ldots = q_N = 0\ (n = \overline{1,N})$.

The boundary conditions (2.12) at $x = \pm a/2$ are satisfied automatically and the boundary conditions at $y = 0, y = b$ become

$$w_n = 0, \quad \vartheta_{y,n} = 0 \quad \text{at} \quad y = 0, \quad y = b \quad (n = \overline{1,N}). \tag{2.17}$$

With (2.6), the amplitudes of the subsidiary functions appearing in the governing system of equations are expressed as

$$\varphi_{1,n} = \varphi_{1,n}(y, w_m), \quad \varphi_{2,n} = \varphi_{2,n}(y, \vartheta_{y,m}), \quad \varphi_{3,n} = \varphi_{3,n}(y, M_{y,m}) \quad (m = \overline{1,N}), \tag{2.18}$$

which determine the coupling of all the 4N equations in (2.16).

Besides the unknown functions, the system of equations (2.16) includes subsidiary functions, and the number of unknowns exceeds the number of equations. This necessitates calculating functions (2.18) during the integration of Eqs. (2.16). To this end, we can expand these functions into discrete Fourier series in powers of x.

To find, while integrating Eqs. (2.16), the values of the functions $\varphi_{j,n}(x, y)$ $(j = 1, 2, 3, n = \overline{1,N})$ from the current amplitudes of the unknown functions at a fixed value $y = y_k\ (k = \overline{0,K})$ of the segment [0; b], we calculate the following quantities at a number of points $x_i\ (i = \overline{1,M})$ of the segment [–a/2, a/2]:

$$h_i = h(x_i, y_k),$$

$$\varphi_1^i = \varphi_1(x_i, y_k) = -h_i^3 \sum_{n=1}^{N} \lambda_n^2 w_n(y_k) \sin\lambda_n\left(\frac{a}{2} + x_i\right),$$

$$\varphi_2^i = \varphi_2(x_i, y_k) = h_i^3 \sum_{n=1}^{N} \lambda_n \vartheta_{y,n}(y_k) \cos\lambda_n\left(\frac{a}{2} + x_i\right),$$

$$\varphi_3^i = \varphi_3(x_i, y_k) = \frac{1}{h_i^3} \sum_{n=1}^{N} M_{y,n}(y_k) \sin\lambda_n\left(\frac{a}{2} + x_i\right). \tag{2.19}$$

As a result, we obtain the values of the functions $\varphi_j(x_i, y_k)\ (j = 1, 2, 3)$ at the points $x_i\ (i = \overline{1,M})$. Now these functions can be expanded into discrete Fourier series similar to (2.15) whose coefficients are the missing amplitudes of the subsidiary functions for system (2.16).

Thus, we oddly extend the functions φ_1^i and φ_3^i and evenly extend the function φ_2^i to the segment $[a/2,\ 3a/2]$. Next, we calculate $\varphi_{j,n}\ (j = 1, 2, 3, n = \overline{1,N})$ using a standard procedure for determining the Fourier coefficients of a tabulated function [29]. After that, we substitute $\varphi_{j,n}(y_k)$ into the original system of equations (2.16) and continue the integration over y, going from y_k to y_{k+1}.

To solve the boundary-value problem for the system of equations (2.16) with the boundary conditions (2.17), we apply the stable discrete-orthogonalization method. To this end, we first formulate, based on (2.17), initial conditions at $y = y_0 = 0$ for $2N + 1$ Cauchy problems:

If $q_n = 0$, then

1) $$\hat{Q}_{y,1} = 1, \ \hat{Q}_{y,2} = 0, \ \ldots, \ \hat{Q}_{y,N} = 0, \ M_{y,1} = 0, \ldots, \ M_{y,N} = 0,$$

$$w_1 = 0, \ \ldots, \ w_N = 0, \ \vartheta_{y,1} = 0, \ldots, \ \vartheta_{y,N} = 0,$$

2) $$\hat{Q}_{y,1} = 0, \ \hat{Q}_{y,2} = 1, \ \hat{Q}_{y,3} = 0, \ \ldots, \ \hat{Q}_{y,N} = 0, \ M_{y,1} = 0, \ \ldots, \ M_{y,N} = 0,$$

$$w_1 = 0, \ \ldots, \ w_N = 0, \ \vartheta_{y,1} = 0, \ \ldots, \ \vartheta_{y,N} = 0,$$

…………………………………………………………………..

m) $$\hat{Q}_{y,1} = 0, \ \ldots, \ \hat{Q}_{y,m-1} = 0, \ \hat{Q}_{y,m} = 1, \ \hat{Q}_{y,m+1} = 0, \ \ldots, \ \hat{Q}_{y,N} = 0, \ M_{y,1} = 0, \ \ldots, \ M_{y,N} = 0,$$

$$w_1 = 0, \ \ldots, \ w_N = 0, \ \vartheta_{y,1} = 0, \ \ldots, \ \vartheta_{y,N} = 0,$$

……………………………………………………………………………….

N) $$\hat{Q}_{y,1} = 0, \ \ldots, \ \hat{Q}_{y,N-1} = 0, \ \hat{Q}_{y,N} = 1, \ M_{y,1} = 0, \ \ldots, \ M_{y,N} = 0,$$

$$w_1 = 0, \ \ldots, \ w_N = 0, \ \vartheta_{y,1} = 0, \ \ldots, \ \vartheta_{y,N} = 0,$$

N + 1) $$\hat{Q}_{y,1} = 0, \ \ldots, \ \hat{Q}_{y,N} = 0, \ M_{y,1} = 1, \ M_{y,2} = 0, \ \ldots, \ M_{y,N} = 0,$$

$$w_1 = 0, \ \ldots, \ w_N = 0, \ \vartheta_{y,1} = 0, \ \ldots, \ \vartheta_{y,N} = 0,$$

N + 2) $$\hat{Q}_{y,1} = 0, \ \ldots, \ \hat{Q}_{y,N} = 0, \ M_{y,1} = 0, \ M_{y,2} = 1, \ M_{y,3} = 0, \ \ldots, \ M_{y,N} = 0,$$

$$w_1 = 0, \ \ldots, \ w_N = 0, \ \vartheta_{y,1} = 0, \ \ldots, \ \vartheta_{y,N} = 0,$$

…………………………………………………………………………………..

N + *m*) $$\hat{Q}_{y,1} = 0, \ \ldots, \ \hat{Q}_{y,N} = 0, \ M_{y,1} = 0, \ \ldots, \ M_{y,m-1} = 0, \ M_{y,m} = 1, \ M_{y,m+1} = 0, \ \ldots, \ M_{y,N} = 0,$$

$$w_1 = 0, \ \ldots, \ w_N = 0, \ \vartheta_{y,1} = 0, \ \ldots, \ \vartheta_{y,N} = 0,$$

…………………………………………………………………………………….

2*N*) $$\hat{Q}_{y,1} = 0, \ \ldots, \ \hat{Q}_{y,N} = 0, \ M_{y,1} = 0, \ \ldots, \ M_{y,N-1} = 0, \ M_{y,N} = 1,$$

$$w_1 = 0, \ \ldots, \ w_N = 0, \ \vartheta_{y,1} = 0, \ \ldots, \ \vartheta_{y,N} = 0. \qquad (2.20)$$

If $q_n \neq 0$, then we specify zero initial conditions for the $(2N + 1)$th Cauchy problem:

$$\hat{Q}_{y,1} = 0, \ \ldots, \ \hat{Q}_{y,N} = 0, \ M_{y,1} = 0, \ldots, \ M_{y,N} = 0,$$

$$w_1 = 0, \ \ldots, \ w_N = 0, \ \vartheta_{y,1} = 0, \ \ldots, \ \vartheta_{y,N} = 0. \qquad (2.21)$$

Using (2.20) and (2.21) for each of the $2N + 1$ Cauchy problems, we calculate the functions $\varphi_{j,n}$ $(j = 1, 2, 3, n = \overline{1, N})$ by formulas (2.19) for $y = 0$. Performing the procedure outlined above, we determine the amplitudes of the unknown functions for $y = y_1$, which are in turn used to determine the amplitudes of the missing subsidiary functions $\varphi_{j,n}(y_1)$ $(j = 1, 2, 3)$ needed at the next step of integration. Thus, we perform the integration over y with step $\Delta y_k = y_{k+1} - y_k$ $(k = \overline{0, K-1})$ and the orthogonalization at given points of the interval [0, *b*].

Let us follow this approach to analyze the stress–strain state of plates whose thickness varies but weight remains constant.

Tables 2 and 3 summarize the values of the deflection w and bending moment M_x for a plate with thickness varying in two coordinate directions as (2.11) and weight remaining constant.

Table 2 demonstrates how the deflection depends on the parameters α and β varying along the OY- and OX-axes. The parameter β has a greater effect on the deflection than the parameter α does. It should be noted that the maximum deflection differs a little from that of the plate with thickness constant along the OY-axis. Table 3 shows how the bending moment depends on α and β. For example, as the parameter β increases, the maximum moment at $x = 0$ and $y = 1$ increases more intensively at $\alpha = 0.3$ than at $\alpha = 0$ and $\alpha = -0.3$. If $\beta = -0.3$ and $\beta = -0.2$, the moment is distributed along the OX-axis not smoothly but similarly to the moment in the plate with thickness constant along the OY-axis.

Thus, choosing an appropriate law of variation in the thickness of a plate at constant weight may result in the most rational stress–strain relationship.

3. Complex-Shaped Plates with a Hole. This section outlines an approach to solving two-dimensional bending problems for complex-shaped plates with a hole. The approach can be applied to a wide class of variable-thickness plates with configuration described by orthogonal curvilinear coordinates [15].

Consider an isotropic plate with a doubly connected midplane. We choose a curvilinear orthogonal coordinate system (α_1, α_2) such that the two coordinate lines $\alpha_1 = \alpha_1'$ and $\alpha_1 = \alpha_1''$ coincide with the boundaries of the plate.

The problem is reduced to a governing system of four partial differential equations with variable coefficients and appropriate boundary conditions:

$$\frac{\partial Z_i(\alpha_1, \alpha_2)}{\partial \alpha_1} = f_i\left(\alpha_1, \alpha_2, \frac{\partial^k Z_j}{\partial \alpha_2^k}\right) \quad (i, j = \overline{1,4}, k = \overline{0,4}), \tag{3.1}$$

where

$$Z_1 = \hat{Q}_1, \quad Z_2 = M_1, \quad Z_3 = w, \quad Z_4 = \theta_1, \quad f_1 = -\frac{1}{A_2}\left(\frac{\partial A_2}{\partial \alpha_1}\hat{Q}_1 + \frac{\partial A_1}{\partial \alpha_2}Q_2^*\right) - A_1\left(\frac{1'}{A_2}\frac{\partial Q_2^*}{\partial \alpha_2} + q\right),$$

$$f_2 = -\frac{1}{A_2}\left[\frac{\partial A_2}{\partial \alpha_1}(M_2 - M_1) - 2\frac{\partial A_1}{\partial \alpha_2}H\right] + A_1\left(\hat{Q}_1 - \frac{2}{A_2}\frac{\partial H}{\partial \alpha_2}\right),$$

$$f_3 = -A_1\theta_1, \quad f_4 = A_1\kappa_1 - \frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\theta_2, \quad \kappa_1 = D^{-1}M_1 - \nu\kappa_2, \quad M_2 = D(\kappa_2 + \nu\kappa_1),$$

$$\kappa_2 = -\frac{1}{A_2}\left(\frac{\partial \theta_2}{\partial \alpha_2} + \frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1}\theta_1\right), \quad \theta_2 = -\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2}, \quad H = \frac{D(1-\nu)}{A_2}\left(\frac{\partial \theta_1}{\partial \alpha_2} - \frac{1}{A_2}\frac{\partial A_2}{\partial \alpha_1}\theta_2\right),$$

$$Q_2^* = \frac{1}{A_2}\frac{DM_2}{\partial \alpha_2} + \frac{1}{A_1 A_2}\left[\frac{\partial A_1}{\partial \alpha_2}(M_2 - M_1) + 2\frac{\partial A_2}{\partial \alpha_1}H\right], \tag{3.2}$$

$\hat{Q}_1, M_1, w$, and θ_1 are the reduced shear force, bending moment, deflection, and the angle of rotation of the normal [8], respectively; $A_1(\alpha_1, \alpha_2)$ and $A_2(\alpha_1, \alpha_2)$ are the Lamé parameters; $D = \dfrac{Eh^3}{12(1-\nu^2)}$, $h = (\alpha_1, \alpha_2)$ is the thickness of the plate; E and ν are the elastic modulus and Poisson's ratio; and $q(\alpha_1, \alpha_2)$ is the distributed transverse load.

Let us expand all the functions in (3.1) into series:

$$X(\alpha_1, \alpha_2) = \sum_{n=0}^{N} X_n(\alpha_1)\cos n\alpha_2, \quad Y(\alpha_1, \alpha_2) = \sum_{n=0}^{N} Y_n(\alpha_1)\sin n\alpha_2, \tag{3.3}$$

TABLE 2

α	β	$w \cdot 10^2 E / q_0$			
		$y = 0.5$		$y = 1.0$	
		$x = 0$	$x = 0.4$	$x = 0$	$x = 0.4$
–0.3	–0.3	1.4230	1.0062	2.5134	1.7772
	0	1.1152	0.7858	2.0507	1.4501
	0.3	0.9129	0.6456	1.6973	1.2001
0	–0.3	1.5668	1.1079	2.4970	1.7656
	0	1.2097	0.8554	2.0180	1.4269
	0.3	0.9754	0.6896	1.6469	1.1646
0.3	–0.3	1.7974	1.2709	2.5529	1.8052
	0	1.3491	0.9539	2.0133	1.4236
	0.3	1.0617	0.7507	1.6041	1.1343

TABLE 3

α	β	$M_x / 10^2 q_0$			
		$y = 0.5$		$y = 1.0$	
		$x = 0$	$x = 0.4$	$x = 0$	$x = 0.4$
–0.3	–0.3	2.0290	3.4171	3.4917	5.2875
	0	4.0349	2.8532	7.2581	5.1323
	0.3	6.7177	2.4330	11.5766	4.7182
0	–0.3	2.6773	4.2316	4.3067	6.7722
	0	5.1244	3.6235	9.0585	6.4053
	0.3	8.1625	3.1194	14.4720	5.7546
0.3	–0.3	3.5669	5.4348	5.5853	9.1763
	0	6.5959	4.6640	11.7398	8.3013
	0.3	10.0729	3.9883	18.6417	7.1981

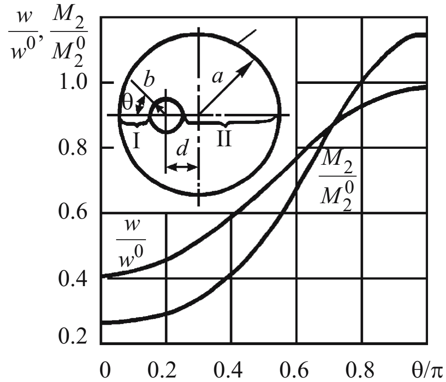

Fig. 1

where $X = \{\hat{Q}_1, M_1, w, \theta_1, f_1, f_2, f_3, f_4, \kappa_1, M_2, \kappa_2, A_1, A_2, h, q\}$, $Y = \{\theta_2, H, Q_2^*\}$.

Substituting (3.3) into (3.1), we obtain a system of ordinary differential equations:

$$\frac{dZ_{in}(\alpha_1)}{d\alpha_1} = f_{in}(\alpha_1, Z_{jm}) \quad (i, j = \overline{1,4},\ n, m = \overline{0,N}), \tag{3.4}$$

Its order depends on the number of terms retained in the series.

The boundary-value problem for (3.4) is solved by discrete orthogonalization, which is a stable numerical method. To calculate the right-hand sides f_{in} of the system during the integration of the Cauchy problems, use is made of expressions (3.3) and (3.2) and a procedure of numerical expansion of a tabulated function into a trigonometric series.

Comparing results obtained with a different number N of terms in series (3.3), we can analyze the solution for convergence and accuracy.

Let us consider, as an example, a circular plate with constant thickness h under a uniform transverse load q. The plate has a noncentral circular hole (Fig. 1). The outer edge is clamped, while the inner edge is free.

To describe the plate, we choose bipolar coordinates [23] related to Cartesian coordinates as

$$x = \frac{m \sinh \alpha_1}{\cosh \alpha_1 + \cos \alpha_2}, \quad y = \frac{m \sinh \alpha_2}{\cosh \alpha_1 + \cos \alpha_2} \quad (-\infty < \alpha_1 < \infty,\ 0 \le \alpha_2 \le 2\pi). \tag{3.5}$$

The first quadratic form is

$$dS^2 = A_1^2 d\alpha_1^2 + A_2^2 d\alpha_2^2. \tag{3.6}$$

The coordinate lines are mutually orthogonal nonconcentric circles. The Lamé parameters are

$$A_1 = A_2 = \frac{m}{\cosh \alpha_1 + \cos \alpha_2}. \tag{3.7}$$

The parameter m and the ends of the integration interval $[\alpha_1', \alpha_1'']$ are related to the dimensions of the plate as

$$m = \frac{a^2 - b^2 - d^2}{2d}, \quad \alpha_1' = \operatorname{Arsinh} \frac{m}{a}, \quad \alpha_1'' = \operatorname{Arsinh} \frac{m}{b}. \tag{3.8}$$

We will use the approach outlined in Sec. 1 and the following input data: $a = 50$, $b = 10$, $d = 20$, $h = 1$, $\nu = 0.3$, $N = 10$.

Figure 1 shows the circumferential bending moment M_2 and deflection w at the inner edge of the plate divided by the moment M_2^0 and deflection w^0 at the inner edge of the same plate but with a central hole.

Table 4 summarizes the bending moments M_1 and M_2 and the deflection w in sections I and II of the plate depending on the horizontal distance ξ from the hole edge. Here $l_1 = a - b - d$ and $l_2 = a - b + d$ are the lengths of sections I and II, respectively.

TABLE 4

$\frac{\xi}{l_1}$	$\frac{M_1}{q}$	$\frac{M_2}{q}$	$w\frac{E\cdot 10^{-5}}{q}$	$\frac{\xi}{l_2}$	$\frac{M_1}{q}$	$\frac{M_2}{q}$	$w\frac{E\cdot 10^{-5}}{q}$
Section I				Section II			
0	0	74.521	4.444	0	0	319.009	10.743
0.210	–56.995	57.999	3.054	0.204	166.230	203.666	10.528
0.392	–101.067	30.489	1.970	0.375	155.480	178.343	9.086
0.611	–161.673	–7.740	0.901	0.653	23.111	96.911	4.402
0.802	–221.934	–44.393	0.270	0.863	–157.514	–6.794	0.863
1.0	–297.253	–89.176	0	1.0	–305.971	–91.791	0

4. Flexible Plates of Complex Geometry. Consider multilayer anisotropic doubly connected plates with complex geometry and smooth contour undergoing nonlinear deformation [4, 5, 17]. The problem is solved in two stages. First, the domain is parametrized. To this end, we numerically construct an orthogonal curvilinear mesh that is closed in one of the directions and is a mapping of a uniform mesh $\omega=\{(ih_1, jh_2), h_1=1/N, h_2=2\pi/M, i=\overline{0,N}, j=\overline{0,M}\}$ defined in a rectangle Ω of the plane (x, y) (provided that the lines $\alpha_1=0$ and $\alpha_2=1$ coincide with the boundaries of D) onto the domain D of the plane (x, y).

The task is to determine the mesh functions x_{ij} and y_{ij} that satisfy the periodicity conditions in the coordinate α_2

$$x_{i0}=x_{iM}, \quad y_{i0}=y_{iM} \quad (i=\overline{0,N}), \tag{4.1}$$

the equations

$$\left(\frac{1}{A_1^2}\frac{\partial^2 x}{\partial\alpha_1^2}+\frac{1}{A_2^2}\frac{\partial^2 x}{\partial\alpha_2^2}\right)_{ij}=0, \quad \left(\frac{1}{A_1^2}\frac{\partial^2 y}{\partial\alpha_1^2}+\frac{1}{A_2^2}\frac{\partial^2 y}{\partial\alpha_2^2}\right)_{ij}=0 \quad (i=\overline{1,N-1}, j=\overline{0,M-1}), \tag{4.2}$$

$$A_1^2=\left(\frac{\partial x}{\partial\alpha_1}\right)^2+\left(\frac{\partial y}{\partial\alpha_1}\right)^2, \quad A_2^2=\left(\frac{\partial x}{\partial\alpha_2}\right)^2+\left(\frac{\partial y}{\partial\alpha_2}\right)^2$$

and the boundary conditions

$$\varphi_i(x_{ij}, y_{ij})=0, \quad \left(\frac{\partial x}{\partial\alpha_1}\frac{\partial x}{\partial\alpha_2}+\frac{\partial y}{\partial\alpha_1}\frac{\partial y}{\partial\alpha_2}\right)_{ij}=0 \quad (i=0,N, j=\overline{0,M-1}), \tag{4.3}$$

where $\varphi_i(x, y)=0$ are the equations describing the boundaries of the plate.

Replacing partial derivatives by second-order differences, we get a system of $2M(N+1)$ nonlinear algebraic equations that can be solved by the method of successive approximations. After the linearization, we solve, at each iteration, two separate systems of linear algebraic equations with tridiagonal matrices for x_{ij} and y_{ij}, respectively. Calculations show that with a properly chosen initial approximation, the iterative process converges quite rapidly. Figure 2 shows an example of orthogonal mesh.

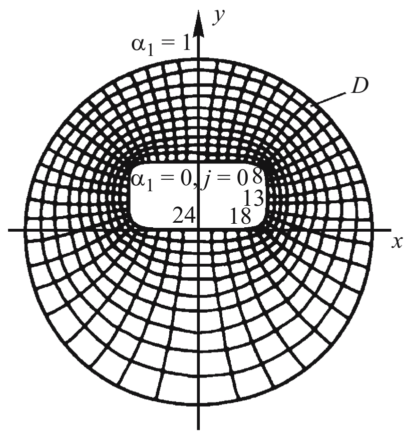

Fig. 2

Formulas (4.2) and cubic splines are then used to calculate the Lamé parameters at mesh nodes. At the second stage, we solve a boundary-value problem for a system of eight partial differential equations describing the two-dimensional geometrically nonlinear deformation of anisotropic plates in arbitrary orthogonal curvilinear coordinates:

$$\frac{\partial \overline{Z}}{\partial \alpha_1} = \overline{G}\left(\alpha_1, \alpha_2, \frac{\partial^k \overline{Z}}{\partial \alpha_2^k}\right) \quad (k = \overline{0, 4}), \tag{4.4}$$

where $\overline{Z} = \{N_1, \hat{S}_1, \hat{Q}_1, M_1, u, v, w, \theta_1\}^{\mathrm{T}}$ is the vector of unknown functions; $\overline{G} = \{g_i\}^{\mathrm{T}}$ $(i = \overline{1, 8})$ is the vector of right-hand side, which is a nonlinear vector function of $\overline{Z}$; N_1, $\widetilde{S}_1$, and $\widetilde{Q}_1$ are the normal, shearing, and transverse forces, M_1 is the bending moment; u, v, and w are the displacements; θ_1 is the angle of rotation of the normal.

The boundary conditions at the edges $\alpha_1 = \text{const}$ for system (4.4) are given by four quantities, one from each of the following pairs:

$$(N_1, u), \quad (\widetilde{S}_1, v), \quad (\widetilde{Q}_1, w), \quad (M_1, \theta_1). \tag{4.5}$$

We use Newton's method to reduce the two-dimensional nonlinear boundary-value problem to a sequence of linear boundary-value problems for the following linearized system of partial differential equations with coefficients varying in two directions:

$$\frac{\partial \overline{Z}^{(s+1)}}{\partial \alpha_1} = \overline{F}\left(\alpha_1, \alpha_2, \frac{\partial^k \overline{Z}^{(s)}}{\partial \alpha_2^k}, \frac{\partial^k \overline{Z}^{(s+1)}}{\partial \alpha_2^k}\right) \quad (k = \overline{0, 4}, s = 0, 1\ldots), \tag{4.6}$$

where $\overline{F} = \overline{G}\left(\alpha_1, \alpha_2, \frac{\partial^k \overline{Z}^{(s)}}{\partial \alpha_2^k}\right) + J\left(\alpha_1, \alpha_2, \frac{\partial^k \overline{Z}^{(s+1)}}{\partial \alpha_2^k}\right)\left(\overline{Z}^{(s+1)} - \overline{Z}^{(s)}\right)$, $J\left(\alpha_1, \alpha_2, \frac{\partial^k \overline{Z}^{(s)}}{\partial \alpha_2^k}\right)$ is the Jacobian matrix of the right-hand side of (4.4), $\overline{F} = \{f_i\}^{\mathrm{T}}$ $(i = \overline{1, 8})$. Let the initial unloaded state of the plate be the initial approximation for the iterative process (4.4). Then the first approximation yields the solution of the linear problem.

Let us reduce the dimension of the boundary-value problem by one. To this end, we represent the unknown functions and the right-hand sides of (4.6) as truncated trigonometric series:

$$X(\alpha_1, \alpha_2) = \sum_{n=0}^{NH} X_n(\alpha_1) \cos n\alpha_2, \qquad Y(\alpha_1, \alpha_2) = \sum_{n=0}^{NH} Y_n(\alpha_1) \sin n\alpha_2, \tag{4.7}$$

where $X = \{N_1, \widetilde{Q}_1, M_1, u, w, \theta_1, f_1, f_3, f_4, f_5, f_7, f_8\}^{\mathrm{T}}$, $Y = \{\widetilde{S}_1, v, f_2, f_6\}$.

TABLE 5

Point number, j	Coordinates			M_2 / q			$wE / qv \quad E / q \cdot 10^{-5}$		
	α_2 / π	x, mm	y, mm	$NH = 10$	$NH = 14$	$NH = 18$	$NH = 10$	$NH = 14$	$NH = 18$
0	0	0	20.00	78.925	83.320	84.375	6.5988	6.5999	6.6001
8	1/3	18.50	18.48	164.81	162.80	159.43	5.2613	5.2562	5.2558
13	13/24	20.00	10.29	116.34	113.00	108.02	6.8027	6.8028	6.8017
18	3/4	18.24	1.328	190.37	189.85	192.09	8.4321	8.4280	8.4281
24	0	0	0	223.78	219.23	219.69	12.343	12.332	12.332

Substituting the series into (4.6) and separating variables, we obtain a system of ordinary differential equations:

$$\frac{\partial \overline{Z}_{in}^{(s+1)}}{\partial \alpha_1} = f_{in}\left(\alpha_1, Z_{im}^{(s)}, \overline{Z}_{kl}^{(s+1)}\right) \quad (i, j, k = 1, 8,\ n, m, l = \overline{0, NH},\ s = 0, 1 \ldots). \tag{4.8}$$

Its order is equal to 8(NH + 1) and dependent on the number of terms retained in the series.

Though the right-hand sides of (4.8) cannot be expressed as explicit functions of the amplitudes of the unknown functions, they can be calculated from these amplitudes. At each iteration, the linear boundary-value problem is solved by discrete orthogonalization, using spline-interpolated tabulated geometrical parameters and the previous approximation. The right-hand sides are calculated by an algorithm based on discrete Fourier series.

The solution can be analyzed for convergence and accuracy by comparing the results obtained with different number NH of terms in (4.7).

Let us consider, as an example, a circular plate of constant thickness h = 1 mm bent by a uniform transverse load q = 0.1 MPa. The plate has a noncentral rectangular hole with rounded corners (Fig. 2). The outer edge is clamped, while the inner edge is free. The boundaries of the plate are described by

$$\varphi_i (x, y) = \left(\frac{x - x_0}{a}\right)^{2p} + \left(\frac{y - y_0}{b}\right)^{2p} - 1 = 0. \tag{4.9}$$

For the inner boundary (i = 0): a = 20, b = 10 mm, $x_0 = 0$, $y_0 = 10$ mm, p = 3. For the outer boundary (i = N): $a = b = 50$ mm, $x_0 = y_0 = 0$, p = 1. We choose a mesh with N = 10 and M = 48.

Table 5 gives the bending moment M_2 and deflection w at several points of the inner boundary of an isotropic plate obtained by solving the linear problem for different values of NH. It can be seen that the number NH of harmonics should be no less than 14 for adequate accuracy.

Figure 3 shows the bending moment M_1 at the outer boundary and the bending moment M_2 and deflection w at the inner boundary of a flexible plate made of an orthotropic material with $E_x = 20 \cdot 10^3$, $E_y = 25 \cdot 10^3$, $G_{xy} = 5 \cdot 10^3$ MPa, v_{xy} = 0.2, $E_x v_{yx} = E_y v_{xy}$.

The stiffness coefficients at each point of the plate are different and dependent on the angle between the tangent to the line $\alpha_2 =$ const and the x-axis.

The dashed line represents the linear solution (first approximation), while the solid line the nonlinear solution (fifth approximation).

5. Cylindrical Shells. Consider cylindrical shells that have thickness varying in two coordinate directions and are subjected to a surface load. The stress–strain of such a shell with arbitrary boundary conditions is determined by solving the

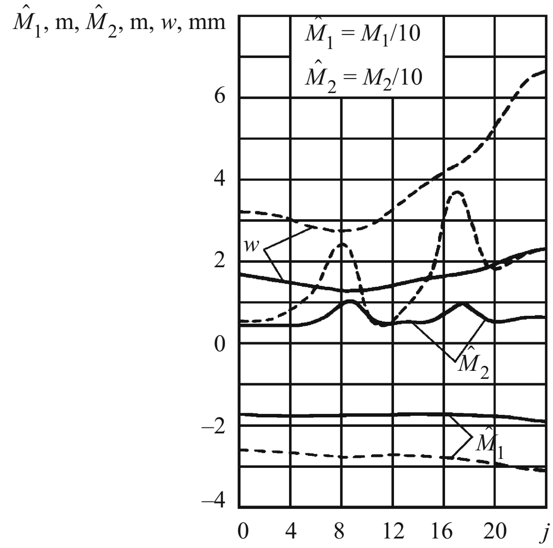

Fig. 3

exact equations of the theory of thin shells [7, 8] using discrete Fourier series [50]. By expanding all the functions into Fourier series in powers of the circumferential coordinate on continuous and discrete sets of points, the two-dimensional boundary-value problem is reduced to a system of ordinary differential equations with appropriate boundary conditions, which can be solved by discrete orthogonalization.

We will use orthogonal coordinates s (arc length) and θ (azimuth) to describe the midsurface of the shell. For the unknown functions, we choose $\hat{Q}_s, N_s, \hat{S}, M_s, w, u, v, \vartheta_s$, where

$$\hat{Q}_s = Q_s + \frac{1}{r}\frac{\partial H}{\partial \theta}, \qquad \hat{S} = S + \frac{2}{r}H, \qquad \vartheta_s = -\frac{\partial w}{\partial s}. \tag{5.1}$$

The governing system of equations becomes

$$\frac{\partial \hat{Q}_s}{\partial s} = \frac{\nu}{r}N_s - \frac{\nu}{r^2}\frac{\partial^2 M_s}{\partial \theta^2} + \frac{1-\nu^2}{r^2}D_N\frac{\partial v}{\partial \theta} - \frac{1-\nu^2}{r^4}\frac{\partial^2}{\partial \theta^2}\left(D_M\frac{\partial v}{\partial \theta}\right)$$

$$+\frac{1-\nu^2}{r^2}D_N w + \frac{1-\nu^2}{r^4}\frac{\partial^2}{\partial \theta^2}\left(D_M\frac{\partial^2 w}{\partial \theta^2}\right) - q_r,$$

$$\frac{\partial N_s}{\partial s} = -\frac{1}{r}\frac{\partial \hat{S}}{\partial \theta} + \frac{4}{r^3}\frac{\partial}{\partial \theta}\left(\frac{D_M}{p_0 D_N}\hat{S}\right) + \frac{2(1-\nu)}{r^3}\frac{\partial}{\partial \theta}\left(\frac{D_M}{p_0}\frac{\partial \vartheta_s}{\partial \theta}\right) - \frac{2(1-\nu)}{r^4}\frac{\partial}{\partial \theta}\left(\frac{D_M}{p_0}\frac{\partial u}{\partial \theta}\right),$$

$$\frac{\partial \hat{S}}{\partial s} = -\frac{\nu}{r}\frac{\partial N_s}{\partial \theta} - \frac{\nu}{r^2}\frac{\partial M_s}{\partial \theta} - \frac{1-\nu^2}{r^2}\frac{\partial}{\partial \theta}\left(D_N\frac{\partial v}{\partial \theta}\right)$$

$$-\frac{1-\nu^2}{r^4}\frac{\partial}{\partial \theta}\left(D_M\frac{\partial v}{\partial \theta}\right) - \frac{1-\nu^2}{r^2}\frac{\partial}{\partial \theta}(D_N w) + \frac{1-\nu^2}{r^4}\frac{\partial}{\partial \theta}\left(D_M\frac{\partial^2 w}{\partial \theta^2}\right),$$

$$\frac{\partial M_s}{\partial s} = \hat{Q}_s - \frac{4}{r^2}\frac{\partial}{\partial \theta}\left(\frac{D_M}{p_0 D_N}\hat{S}\right) - \frac{2(1-\nu)}{r^2}\frac{\partial}{\partial \theta}\left(\frac{D_M}{p_0}\frac{\partial \vartheta_s}{\partial \theta}\right) + \frac{2(1-\nu)}{r^3}\frac{\partial}{\partial \theta}\left(\frac{D_M}{p_0}\frac{\partial u}{\partial \theta}\right),$$

$$\frac{\partial w}{\partial s} = -\vartheta_s, \qquad \frac{\partial u}{\partial s} = \frac{1}{D_N}N_s - \frac{\nu}{r}\frac{\partial v}{\partial \theta} - \frac{\nu}{r}w,$$

$$\frac{\partial v}{\partial s}=\frac{2}{1-\nu}\frac{1}{p_0 D_N}\hat{S}-\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{4}{r^3}\frac{D_M}{p_0 D_N}\frac{\partial u}{\partial \theta}-\frac{4}{r^2}\frac{D_M}{p_0 D_N}\frac{\partial \vartheta_s}{\partial \theta},$$

$$\frac{\partial \vartheta_s}{\partial s}=\frac{1}{D_M}M_s+\frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}-\frac{\nu}{r^2}\frac{\partial v}{\partial \theta}, \qquad p_0=1+\frac{4}{r^2}\frac{D_M}{D_N}=1+\frac{4h^2}{12r^2}, \tag{5.2}$$

where u, v, w, and ϑ_s are the displacements and angle of rotation of the normal; $\hat{Q}_s, N_s$, $\hat{S}$, and M_s are the forces and moment; $D_N=\frac{Eh(s,\theta)}{1-v^2}$ and $D_M=\frac{Eh^3(s,\theta)}{12(1-v^2)}$ are the tangential and flexural stiffnesses. The boundary conditions for the unknown functions are prescribed at the ends $s=0$ and $s=L$.

Because the stiffnesses D_N and D_M depend on the variable θ, it is impossible to separate variables using Fourier series in powers of the circumferential coordinate. Therefore, we introduce the following subsidiary functions that include the terms hindering the separation of variables with respect to the circumferential coordinate:

$$\varphi_1^j=D_M\left\{\frac{\partial v}{\partial \theta},\frac{\partial^2 w}{\partial \theta^2}\right\} \quad (j=1,2), \qquad \varphi_2^j=D_N\left\{w,\frac{\partial v}{\partial \theta}\right\} \quad (j=1,2),$$

$$\varphi_3^j=\frac{D_M}{p_0 D_N}\left\{\hat{S},\frac{\partial u}{\partial \theta},\frac{\partial \vartheta_s}{\partial \theta}\right\} \quad (j=1,2,3), \qquad \varphi_4^j=\frac{D_M}{p_0}\left\{\frac{\partial u}{\partial \theta},\frac{\partial \vartheta_s}{\partial \theta}\right\} \quad (j=1,2),$$

$$\varphi_5=\frac{1}{D_N}N_s, \qquad \varphi_6=\frac{1}{p_0 D_N}\hat{S}, \qquad \varphi_7=\frac{1}{D_M}M_s. \tag{5.3}$$

With (5.3), the governing system (5.2) becomes

$$\frac{\partial \hat{Q}_s}{\partial s}=\frac{\nu}{r}N_s-\frac{\nu}{r^2}\frac{\partial^2 M_s}{\partial \theta^2}+\frac{1-\nu^2}{r^2}\varphi_2^2-\frac{1-\nu^2}{r^4}\frac{\partial^2\varphi_1^1}{\partial \theta^2}+\frac{1-\nu^2}{r^2}\varphi_2^1+\frac{1-\nu^2}{r^4}\frac{\partial^2\varphi_1^2}{\partial \theta^2}-r_r,$$

$$\frac{\partial N_s}{\partial s}=-\frac{1}{r}\frac{\partial \hat{S}}{\partial \theta}+\frac{4}{r^3}\frac{\partial \varphi_3^1}{\partial \theta}+\frac{2(1-\nu)}{r^3}\frac{\partial \varphi_4^2}{\partial \theta}-\frac{2(1-\nu)}{r^4}\frac{\partial \varphi_4^1}{\partial \theta},$$

$$\frac{\partial \hat{S}}{\partial s}=-\frac{\nu}{r}\frac{\partial N_s}{\partial \theta}-\frac{\nu}{r^2}\frac{\partial M_s}{\partial \theta}-\frac{1-\nu^2}{r^2}\frac{\partial \varphi_2^2}{\partial \theta}-\frac{1-\nu^2}{r^4}\frac{\partial \varphi_1^1}{\partial \theta}-\frac{1-\nu^2}{r^2}\frac{\partial \varphi_2^1}{\partial \theta}+\frac{1-\nu^2}{r^4}\frac{\partial \varphi_1^2}{\partial \theta},$$

$$\frac{\partial M_s}{\partial s}=\hat{Q}_s-\frac{4}{r^2}\frac{\partial \varphi_3^1}{\partial \theta}-\frac{2(1-\nu)}{r^2}\frac{\partial \varphi_4^2}{\partial \theta}+\frac{2(1-\nu)}{r^3}\frac{\partial \varphi_4^1}{\partial \theta},$$

$$\frac{\partial w}{\partial s}=-\vartheta_s, \qquad \frac{\partial u}{\partial s}=\varphi_5-\frac{\nu}{r}\frac{\partial v}{\partial \theta}-\frac{\nu}{r}w,$$

$$\frac{\partial v}{\partial s}=\frac{2}{1-\nu}\varphi_6-\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{4}{r^3}\varphi_3^2-\frac{4}{r^2}\varphi_3^3, \qquad \frac{\partial \vartheta_s}{\partial s}=\varphi_7+\frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}-\frac{\nu}{r^2}\frac{\partial v}{\partial \theta}. \tag{5.4}$$

Let us expand all the functions in (5.4) into Fourier series in powers of the coordinate θ:

$$X(s,\theta)=\sum_{n=0}^{\infty}X_n(s)\cos\lambda_n\theta, \quad Y(s,\theta)=\sum_{n=1}^{\infty}Y_n(s)\sin\lambda_n\theta, \quad \lambda_n=\frac{\pi n}{2\pi}=\frac{n}{2}, \tag{5.5}$$

$$X=\{\hat{Q}_s,N_s,M_s,w,u,\vartheta_s,\varphi_1^j,\varphi_2^j,\varphi_5,\varphi_7,q_\gamma\}, \quad Y=\{\hat{S},v,\varphi_3^j,\varphi_4^j,\varphi_6\}.$$

Substituting (5.5) into (5.4) and the boundary conditions and separating variables, we arrive at a boundary-value problem for a system of ordinary differential equations with appropriate boundary conditions for the amplitudes of the functions in (5.4):

$$\frac{\partial \hat{Q}_{s,n}}{\partial s} = \frac{\nu}{r} N_{s,n} + \frac{\nu\lambda_n^2}{r^2} M_{s,n} + \frac{1-\nu^2}{r^2} \varphi_{2,n}^2$$

$$-\frac{(1-\nu^2)\lambda_n^2}{r^4} \varphi_{1,2}^1 + \frac{(1-\nu^2)\lambda_n}{r^2} \varphi_{2,n}^1 - \frac{(1-\nu^2)\lambda_n}{r^4} \varphi_{1,n}^2,$$

$$\frac{\partial N_{s,n}}{\partial s} = -\frac{\lambda_n}{r} \hat{S}_n + \frac{4\lambda_n}{r^3} \varphi_{3,n}^1 + \frac{2(1-\nu)\lambda_n}{r^3} \varphi_{4,n}^2 - \frac{2(1-\nu)\lambda_n}{r^4} \varphi_{4,n}^1,$$

$$\frac{\partial \hat{S}_n}{\partial s} = \frac{\nu\lambda_n}{r} N_{s,n} + \frac{\nu\lambda_n}{r^2} M_{s,n} + \frac{(1-\nu^2)\lambda_n}{r^2} \varphi_{2,n}^2$$

$$+\frac{(1-\nu^2)\lambda_n}{r^4} \varphi_{1,n}^1 + \frac{(1-\nu^2)\lambda_n}{r^2} \varphi_{2,n}^1 - \frac{(1-\nu^2)\lambda_n}{r^4} \varphi_{1,n}^2,$$

$$\frac{\partial M_{s,n}}{\partial s} = \hat{Q}_{s,n} - \frac{4\lambda_n}{r^2} \varphi_{3,n}^1 - \frac{2(1-\nu)\lambda_n}{r^2} \varphi_{4,n}^2 + \frac{2(1-\nu)\lambda_n}{r^3} \varphi_{4,n}^1, \quad \frac{\partial w_n}{\partial s} = -\vartheta_{s,n},$$

$$\frac{\partial u_n}{\partial s} = \varphi_{5,n} - \frac{\nu\lambda_n}{r} v_n - \frac{\nu}{r} w_n, \quad \frac{\partial v_n}{\partial s} = \frac{2}{1-\nu} \varphi_{6,n} + \frac{\lambda_n}{r} u_n + \frac{4}{r^3} \varphi_{3,n}^2 - \frac{4}{r^2} \varphi_{3,n}^3,$$

$$\frac{\partial \vartheta_{s,n}}{\partial s} = \varphi_{7,n} - \frac{\nu\lambda_n^2}{r^2} w_n - \frac{\nu\lambda_n}{r^2} v_n \qquad (n = \overline{0,N}). \tag{5.6}$$

The boundary conditions can be represented as

$$B_1 \bar{Z}(0) = \bar{b}_1, \qquad B_2 \bar{Z}(L) = \bar{b}_2, \tag{5.7}$$

where $\bar{Z}(s) = \{\hat{Q}_{s,n}, N_{s,n}, \hat{S}_n, M_{s,N}, w_n, u_n, v_n, \vartheta_{s,n}\}^{\mathrm{T}}$ is the column vector of unknown functions; B_1 and B_2 are rectangular matrices; and $\bar{b}_1$ and $\bar{b}_2$ are vectors.

The boundary-value problem (5.6), (5.7) is solved by discrete orthogonalization on the interval $0 \le s \le L$. For each n in Eqs. (5.6) with (5.3), we have:

$$\varphi_{1,n}^j = \varphi_{1,n}^j(s, v, w) \quad (j = 1, 2), \qquad \varphi_{2,n}^j = \varphi_{2,n}^j(s, w, v) \quad (j = 1, 2),$$

$$\varphi_{3,n}^j = \varphi_{3,n}^j(s, \hat{S}, u, \vartheta_s) \quad (j = 1, 2, 3), \qquad \varphi_{4,n}^j = \varphi_{4,n}^j(s, u, \vartheta_s) \quad (j = 1, 2),$$

$$\varphi_{5,n}^j = \varphi_{5,n}^j(s, N_s), \qquad \varphi_{6,n}^j = \varphi_{6,n}^j(s, \hat{S}), \qquad \varphi_{7,n}^j = \varphi_{7,n}^j(s, M_s) \qquad (n = \overline{0,N}). \tag{5.8}$$

The functions φ_{in}^j appearing in the coefficients of the Fourier series (5.5) cannot be explicitly expressed in terms of the Fourier coefficients of the unknown functions and are calculated in integrating (5.6) using discrete Fourier series at each step s = const. Formulas (5.8) relate these coefficients and the amplitudes of unknown functions and demonstrate the coupling of Eqs. (5.6).

To demonstrate that the approximate solution converges to the exact one, let us consider a cylindrical shell with circumferentially varying thickness

$$h = h_0(1 + \beta\cos\theta) \qquad (0 \le \theta \le 2\pi). \tag{5.9}$$

TABLE 6

H	Method	R	N	$\frac{WE/10^3 q_0}{N_0/10q_0}$, $\theta/\pi = 0$, $s/L = 0.1$	$\theta/\pi = 0$, $s/L = 0.5$	$\theta/\pi = 1$, $s/L = 0.1$	$\theta/\pi = 1$, $s/L = 0.5$
0.25	1	12	3	$\frac{0.654}{0.794}$	$\frac{2.117}{2.569}$	$\frac{1.887}{0.794}$	$\frac{6.108}{2.571}$
		16	5	$\frac{0.770}{0.963}$	$\frac{2.492}{3.115}$	$\frac{2.098}{0.891}$	$\frac{6.788}{2.884}$
		24	8	$\frac{0.745}{0.917}$	$\frac{2.411}{2.968}$	$\frac{2.146}{0.917}$	$\frac{6.945}{2.969}$
		36	12	$\frac{0.749}{0.926}$	$\frac{2.424}{2.997}$	$\frac{2.157}{0.926}$	$\frac{6.982}{2.998}$
	2	—	—	$\frac{0.748}{0.926}$	$\frac{2.421}{2.996}$	$\frac{2.157}{0.927}$	$\frac{6.981}{3.000}$
0.5	1	12	3	$\frac{0.326}{0.791}$	$\frac{1.055}{2.561}$	$\frac{0.943}{0.794}$	$\frac{3.053}{2.570}$
		16	5	$\frac{0.385}{0.962}$	$\frac{1.245}{3.112}$	$\frac{1.049}{0.891}$	$\frac{3.393}{2.884}$
		24	8	$\frac{0.374}{0.917}$	$\frac{1.211}{2.966}$	$\frac{1.073}{0.917}$	$\frac{3.472}{2.969}$
		36	12	$\frac{0.372}{0.925}$	$\frac{1.206}{2.995}$	$\frac{1.079}{0.926}$	$\frac{3.491}{2.997}$
	2	—	—	$\frac{0.372}{0.922}$	$\frac{1.205}{2.984}$	$\frac{1.078}{0.926}$	$\frac{3.490}{2.998}$

The shell is subjected to a load $q_\gamma = q_0 \sin \pi s / L$ and is hinged at the ends, i.e.,

$$N_s = M_s = w = v = 0 \quad \text{at} \quad s = 0, s = L. \tag{5.10}$$

These boundary conditions allow using two approaches to solve the problem:

(i) use of discrete Fourier series;

(ii) separation of variables with respect to the longitudinal coordinate and solution of a one-dimensional problem by discrete orthogonalization.

The latter approach may be considered exact.

The input data: $R = 30$, $L = 30$, $H = 0.25, 0.50$, $\beta = 0.5$, $\nu = 0.3$. Table 6 presents the values of the deflection w (nominator) and force N_θ (denominator) for some values of s and θ. Here R is the number of terms in the discrete Fourier series, N is the number of terms retained. It can be seen that as R and N increase, the solutions for w and N_θ tend to the exact one. Even at $R = 36$ and $N = 12$, the results agree to three to four significant digits, i.e., the error is several hundredths of a percent, which is indicative of high accuracy of the solution.

If α and β are arbitrary, the weight of the shell remains constant and equals that of the shell with constant thickness H =const ($\alpha = \beta = 0$).

Let us now consider a noncircular cylindrical shell under uniform external pressure $q = -q_0$ [10, 14, 54, 55]. The shell has elliptic cross-section, is closed in the circumferential direction, has length L and constant thickness h. The edges are hinged or clamped.

We choose the functions for which the boundary conditions at the ends are formulated as unknown and reduce the closed-form system of equations of the problem to a system of eight partial differential equations:

$$\frac{\partial N_s}{\partial s} = \frac{\partial}{\partial t}(2\varphi_1 - \tilde{S}_s), \qquad \frac{\partial \tilde{S}_s}{\partial s} = -\frac{\partial N_t}{\partial t} - \varphi_2,$$

$$\frac{\partial \tilde{Q}_s}{\partial s} = -\frac{\partial^2 M_t}{\partial t^2} + \varphi_3 - q, \qquad \frac{\partial M_s}{\partial s} = \tilde{Q}_s - 2\frac{\partial M_{st}}{\partial t},$$

$$\frac{\partial u}{\partial s} = \varepsilon_s, \qquad \frac{\partial v}{\partial s} = C_{66}^{-1}(\tilde{S}_s - 2\varphi_1) - \frac{\partial u}{\partial t}, \qquad \frac{\partial w}{\partial s} = -\theta_s, \qquad \frac{\partial \theta_s}{\partial s} = \kappa_s, \tag{5.11}$$

where

$$\varphi_1 = k_t M_{st}, \quad \varphi_2 = k_t \frac{\partial M_t}{\partial t}, \quad \varphi_3 = k_t N_t, \quad \varphi_4 = k_t v, \quad \varphi_5 = k_t w, \quad \varphi_6 = k_t\left(C_{66}^{-1}\tilde{S}_s - \frac{\partial u}{\partial t}\right),$$

$$\varepsilon_s = C_{11}N_s - v\varepsilon_t, \quad \kappa_s = D_{11}M_s - v\kappa_t, \quad N_t = C_{66}(\varepsilon_t + v\varepsilon_s), \quad M_t = D_{22}(\kappa_t + v\kappa_s),$$

$$\theta_t = -\frac{\partial w}{\partial t} + \varphi_4, \quad \kappa_t = \frac{\partial \theta_t}{\partial t}, \quad \varepsilon_t = -\frac{\partial v}{\partial t} + \varphi_5, \quad M_{st} = 2D_{22}\left(\frac{\partial \theta_s}{\partial t} + \varphi_6\right),$$

N_s, $\tilde{S}_s$, and $\tilde{Q}_s$ are the normal, shearing, and transverse forces; M_s is the bending moment; u, v, and w are the displacements; $\tilde{\theta}_s$ is the angle of rotation of the normal; C_{11} and D_{11} $(l = 1, 2, 6)$ are coefficients dependent on the elastic modulus E, Poisson's ratio v, and the thickness h [8]; $k_t = k_1(t)$ is the curvature of the directrix; and s and t are the longitudinal and circumferential coordinates, respectively.

Let us expand all the functions appearing in (5.11) into series:

$$X(s,t) = \sum_{n=0}^{N} X_n(s)\cos\lambda_n t, \qquad Y(s,t) = \sum_{n=0}^{N} Y_n(s)\sin\lambda_n t, \tag{5.12}$$

where

$$X = \{N_S, \tilde{Q}_s, M_s, u, w, \theta_s, N_t, M_t, \varepsilon_s, \varepsilon_t, \kappa_s, \kappa_t, \varphi_3, \varphi_5, q\},$$

$$Y = \{\tilde{S}_s, v, \theta_t, N_t, M_{st}, M_t, \tilde{Q}_t, \varphi_1, \varphi_2, \varphi_4, \varphi_6\},$$

$\lambda_n - 2\pi n / l$, l is the length of the directrix.

Substituting (5.12) into (5.11), we obtain

$$\frac{dN_{s,n}}{ds} = \lambda_n(2\varphi_{1,n} - \tilde{S}_{s,n}), \qquad \frac{d\tilde{S}_{s,n}}{ds} = \lambda_n N_{t,n} - \varphi_{2,n},$$

$$\frac{d\tilde{Q}_{s,n}}{ds} = \lambda_n^2 M_{t,n} + \varphi_{3,n} - q_n, \qquad \frac{dM_{s,n}}{ds} = \tilde{Q}_{s,n} - 2\lambda_n M_{st,n},$$

$$\frac{du_n}{ds} = \varepsilon_{s,n}, \qquad \frac{dv_n}{ds} = \frac{1-v}{2}D_N(\tilde{S}_{s,n} - 2\varphi_{1,n}) + \lambda_n u_n,$$

TABLE 7

Type	$\frac{\theta}{\pi}$	$w\frac{E}{10^4 q_0}$	$\frac{N_s}{q_0}$	$\frac{N_t}{q_0}$	$\frac{M_s}{q_0}$	$\frac{M_t}{q_0}$
Hinging	0	–5.491	–105.75	–36.20	–3.047	–5.515
	0.1	–4.632	–82.94	–32.04	–2.556	–4.612
	0.2	–2.501	–21.29	–22.48	–1.164	–1.798
	0.3	1.096	58.35	–13.38	0.774	2.614
	0.4	1.789	126.32	–8.44	2.606	7.097
	0.5	2.492	151.35	–7.15	3.322	8.881
Clamping	0	–2.526	–43.11	–37.76	–1.419	–2.501
	0.1	–2.139	–36.58	–33.57	–1.221	–2.113
	0.2	–1.176	–16.66	–23.59	–0.622	–0.889
	0.3	–0.087	13.72	–13.58	0.297	1.110
	0.4	0.783	43.72	–7.74	1.244	3.313
	0.5	1.105	55.50	–6.10	1.629	4.227

$$\frac{dw_n}{ds} = -\theta_{s,n}, \quad \frac{d\theta_{s,n}}{ds} = \kappa_{s,n}, \tag{5.13}$$

where

$$\varepsilon_{s,n} = C_{11}^{-1} N_{s,n} - \nu\varepsilon_{t,n}, \quad \kappa_{s,n} = D_{11}^{-1} M_{s,n} - \nu_{t,n},$$

$$N_{t,n} = C_{22}(\varepsilon_{t,n} + \nu\varepsilon_{s,n}), \quad M_{t,n} = D_{22}(\kappa_{t,n} + \nu\kappa_{s,n}),$$

$$\kappa_{t,n} = \lambda_N(\lambda_n w_n + \varphi_{4,n}), \quad \varepsilon_{t,n} = \lambda_n v_n + \varphi_{5,n},$$

$$M_{st,n} = 2D_{66}(\varphi_{6,n} - \lambda\theta_{s,n}) \quad (n = \overline{0,N}). \tag{5.14}$$

Here for each n, the amplitudes $\varphi_{j,n}$ $(j = \overline{1,6})$, $\varepsilon_{s,n}$, $\kappa_{s,n}$, $N_{t,n}$, $M_{t,n}$, $M_{st,n}$ on the right-hand sides of Eqs. (5.13) depend on all the harmonics of the corresponding unknown functions. To determine these amplitudes, we integrate the system of $8(N+1)$ equations (5.13) proceeding as follows:

We calculate

$$\varphi_4^i = k_t^i \sum_{n=0}^{N} v_n \sin\lambda_n t_i, \quad \varphi_5^i = k_t^i \sum_{n=0}^{N} w_n \cos\lambda_n t_i, \quad \varphi_6^i = k_t^i \sum_{n=0}^{N} \left(\frac{1-\nu}{2} D_N \widetilde{S}_{s,n} + \lambda_n u_n \right) \sin\lambda_n t_i$$

at points t_i $(i = \overline{1,M})$ on the directrix and determine the coefficients $\varphi_{4,n}$, $\varphi_{5,n}$, and $\varphi_{6,n}$.

Next, we use formulas (5.15) to calculate $\varepsilon_{s,n}$, $\kappa_{s,n}$, $N_{t,n}$, $M_{t,n}$, and $M_{st,n}$.

Finally, we determine

$$\varphi_1^i = k_t^i \sum_{n=0}^{N} M_{st,n} \sin\lambda_n t_i, \quad \varphi_2^i = -k_t^i \sum_{n=0}^{N} M_{t,n} \sin\lambda_n t_i, \quad \varphi_3^i = k_t^i \sum_{n=0}^{N} N_{t,n} \cos\lambda_n t_i$$

at the points t_i and find $\varphi_{1,n}$, $\varphi_{2,n}$, and $\varphi_{3,n}$.

The input data: $L = 60$, $h = 0.5$, $\nu = 0.3$, $N = 10$, $M = 40$; the ellipse semiaxes $a = 19.456$, $b = 9.7278$.

Table 7 summarizes the deflection w, forces N_s, N_t, and moments M_s, M_t on the midsurface for different values of θ. The value $\theta = 0$ represents a position on the minor axis of the ellipse. The table compares two types of boundary conditions: hinging and clamping, which demonstrates the influence of boundary conditions on the stress–strain state of the shell.

6. Conical Shells. Consider conical shells that have thickness varying in the circumferential direction and are subjected to a surface load. The stress–strain state is determined for arbitrary boundary conditions using the exact equations of the theory of thin shells [7, 8, 25] and discrete Fourier series [51, 52]. By expanding all the functions into Fourier series in powers of the circumferential coordinate on continuous and discrete sets of points, we reduce the two-dimensional boundary-value problem to a system of ordinary differential equations with appropriate boundary conditions, which can be solved by discrete orthogonalization.

We will use orthogonal coordinates s (arc length) and θ (azimuth) to describe the midsurface of the shell. Then the radius of the directrix is $r(s) = r_0 + \cos\varphi \cdot s$, where r_0 is the radius in the reference plane, φ is the angle between the normal to the shell surface and the z-axis.

Because the stiffnesses D_N and D_M depend on the variable θ, it is impossible to separate variables using Fourier series. Therefore, we introduce subsidiary functions that include terms hindering the separation of variables with respect to the circumferential coordinate [5, 13–15]:

$$\psi_1^j = D_M \left\{ \frac{\partial^2 u_r}{\partial\theta^2}, \frac{\partial^2 u_z}{\partial\theta^2}, \frac{\partial v}{\partial\theta}, \vartheta_s \right\} \quad (j = 1, 2, 3, 4), \quad \psi_2^j = D_N \left\{ u_r, \frac{\partial v}{\partial\theta} \right\} \quad (j = 1, 2),$$

$$\psi_3^j = \frac{D_M}{p_0} \left\{ \frac{\partial u_z}{\partial\theta}, \frac{\partial\vartheta_s}{\partial\theta} \right\} \quad (j = 1, 2), \quad \psi_4^j = \frac{D_M}{p_0 D_N} \left\{ \hat{S}, \frac{\partial u_z}{\partial\theta}, \frac{\partial\vartheta_s}{\partial\theta} \right\} \quad (j = 1, 2, 3),$$

$$\psi_5^j = \frac{1}{D_N} \{N_r, N_z\} \quad (j = 1, 2), \quad \psi_6 = \frac{1}{p_0 D_N} \hat{S}, \quad \psi_7 = \frac{1}{D_M} M_s. \tag{6.1}$$

With (6.1), the governing system becomes

$$\frac{\partial N_r}{\partial s} = -\frac{(1-\nu)\cos\varphi}{r} N_r + \frac{\nu\sin\varphi}{r} N_z - \frac{\cos\varphi}{r} \frac{\partial\hat{S}}{\partial\theta}$$

$$-\frac{\nu\sin\varphi}{r^2} \frac{\partial^2 M_s}{\partial\theta^2} + \frac{(1-\nu^2)\sin^2\varphi}{r^4} \frac{\partial^2\psi_1^1}{\partial\theta^2} + \frac{1-\nu^2}{r^2} \psi_2^1$$

$$-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4} \frac{\partial^2\psi_1^2}{\partial\theta^2} + \frac{1-\nu^2}{r^2} \psi_2^2 - \frac{(1-\nu^2)\sin^2\varphi}{r^4} \frac{\partial^2\psi_1^3}{\partial\theta^2} - \frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^2} \frac{\partial^2\psi_1^4}{\partial\theta^2} - q_r,$$

$$\frac{\partial N_z}{\partial s} = -\frac{\cos\varphi}{r} N_z - \frac{\sin\varphi}{r} \frac{\partial\hat{S}}{\partial\theta} + \frac{4\sin\varphi}{r^3} \frac{\partial\psi_4^1}{\partial\theta}$$

$$+\frac{\nu\cos\varphi}{r^2} \frac{\partial^2 M_s}{\partial\theta^2} - \frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4} \frac{\partial^2\psi_1^1}{\partial\theta^2} + \frac{(1-\nu^2)\cos^2\varphi}{r^4} \frac{\partial^2\psi_1^2}{\partial\theta^2}$$

$$-\frac{2(1-\nu)}{r^4} \frac{\partial\psi_3^1}{\partial\theta} + \frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4} \frac{\partial^2\psi_1^3}{\partial\theta^2} + \frac{(1-\nu^2)\cos^2\varphi}{r^3} \frac{\partial^2\psi_1^4}{\partial\theta^2} + \frac{2(1-\nu)}{r^3} \frac{\partial\psi_3^2}{\partial\theta} - q_z,$$

$$\frac{\partial \hat{S}}{\partial s} = -\frac{\nu\cos\varphi}{r}\frac{\partial N_r}{\partial\theta} - \frac{\nu\sin\varphi}{r}\frac{\partial N_z}{\partial\theta} - \frac{2\cos\varphi}{r}\hat{S} - \frac{\nu\sin\varphi}{r^2}\frac{\partial M_s}{\partial\theta} + \frac{(1-\nu^2)\sin^2\varphi}{r^4}\frac{\partial\psi_1^1}{\partial\theta} - \frac{(1-\nu^2)}{r^2}\frac{\partial\psi_2^1}{\partial\theta}$$

$$-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\frac{\partial\psi_1^2}{\partial\theta} - \frac{1-\nu^2}{r^2}\frac{\partial\psi_2^2}{\partial\theta} - \frac{(1-\nu^2)\sin^2\varphi}{r^4}\frac{\partial\psi_1^3}{\partial\theta} - \frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\frac{\partial\psi_1^4}{\partial\theta} - q_\theta,$$

$$\frac{\partial M_s}{\partial s} = \sin\varphi N_r - \cos\varphi N_z - \frac{4\sin\varphi}{r^2}\frac{\partial\psi_4^1}{\partial\theta} - \frac{(1-\nu)\cos\varphi}{r}M_s - \frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\psi_1^1 + \frac{2(1-\nu)}{r^3}\frac{\partial\psi_3^1}{\partial\theta}$$

$$+\frac{(1-\nu^2)\cos^2\varphi}{r^3}\psi_1^2 + \frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\psi_1^3 - \frac{2(1-\nu)}{r^2}\frac{\partial\psi_3^2}{\partial\theta} + \frac{(1-\nu^2)\cos^2\varphi}{r^2}\psi_1^4,$$

$$\frac{\partial u_r}{\partial s} = \cos^2\varphi\cdot\psi_5^1 + \sin\varphi\cos\varphi\cdot\psi_5^2 - \frac{\nu\cos\varphi}{r}u_r - \frac{\nu\cos\varphi}{r}\frac{\partial v}{\partial\theta} - \sin\varphi\vartheta_s,$$

$$\frac{\partial u_z}{\partial s} = \sin\varphi\cos\varphi\cdot\psi_5^1 + \sin^2\varphi\cdot\psi_5^2 - \frac{\nu\sin\varphi}{r}u_r - \frac{\nu\sin\varphi}{r}\frac{\partial v}{\partial\theta} + \cos\varphi\vartheta_s,$$

$$\frac{\partial v}{\partial s} = \frac{2}{1-\nu}\psi_6 - \frac{\cos\varphi}{r}\frac{\partial u_r}{\partial\theta} + \frac{4\sin\varphi}{r^3}\psi_4^2 - \frac{\sin\varphi}{r}\frac{\partial u_z}{\partial\theta} + \frac{\cos\varphi}{r}v - \frac{4\sin\varphi}{r^2}\psi_4^3,$$

$$\frac{\partial\theta_s}{\partial s} = \psi_7 + \frac{\nu\sin\varphi}{r^2}\frac{\partial^2 u_r}{\partial\theta^2} - \frac{\nu\cos\varphi}{r^2}\frac{\partial^2 u_z}{\partial\theta^2} - \frac{\nu\sin\varphi}{r^2}\frac{\partial v}{\partial\theta} - \frac{\cos\varphi}{r}\vartheta_s, \tag{6.2}$$

where

$$u_r = u\cos\varphi + w\sin\varphi, \quad u_z = u\sin\varphi + w\cos\varphi,$$

$$N_r = N_s\cos\varphi + \hat{Q}_s\sin\varphi, \quad N_z = N_s\sin\varphi + \hat{Q}_s\cos\varphi,$$

$$q_r = q_s\cos\varphi + q_\gamma\sin\varphi, \quad q_z = q_s\sin\varphi + q_\gamma\cos\varphi.$$

Let us expand all the functions in (6.2) into Fourier series in powers of the coordinate θ:

$$X(s,\theta) = \sum_{n=0}^{\infty} X_n(s)\cos\lambda_n\theta, \quad Y(s,\theta) = \sum_{n=1}^{\infty} Y_n(s)\sin\lambda_n\theta, \quad \lambda_n = \frac{\pi n}{2\pi} = \frac{n}{2},$$

$$X = \{N_r, N_z, M_s, u_r, u_z, \vartheta_s, \psi_1^j, \psi_2^j, \psi_5^j, \psi_7, q_r, q_z\}, \quad Y = \{\hat{S}, v, \psi_3^j, \psi_4^j, \psi_6, q_\theta\}. \tag{6.3}$$

Substituting (6.3) into (6.2) and the boundary conditions and separating variables, we arrive at a boundary-value problem for a system of ordinary differential equations for the amplitudes of the functions in (6.2):

$$\frac{dN_{r,n}}{ds} = -\frac{(1-\nu)\cos\varphi}{r}N_{r,n} + \frac{\nu\sin\varphi}{r}N_{z,n}$$

$$-\frac{\cos\varphi}{r}\lambda_n\hat{S}_n + \frac{\nu\sin\varphi}{r^2}\lambda_n^2 M_{s,n} - \frac{(1-\nu^2)\sin^2\varphi}{r^4}\lambda_n^2\psi_{1,n}^1 + \frac{1-\nu^2}{r^2}\psi_{2,n}^1$$

$$+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\lambda_n^2\psi_{1,n}^2 + \frac{1-\nu^2}{r^2}\psi_{2,n}^2 + \frac{(1-\nu^2)\sin^2\varphi}{r^4}\lambda_n^2\psi_{1,n}^3 + \frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^2}\lambda_n^2\psi_{1,n}^4 - q_r,$$

$$\frac{dN_{z,n}}{ds} = -\frac{\cos\varphi}{r}N_{z,n} - \frac{\sin\varphi}{r}\lambda_n\hat{S}_n + \frac{4\sin\varphi}{r^3}\lambda_n\psi_{4,n}^1$$

$$-\frac{\nu\cos\varphi}{r^2}\lambda_n^2 M_{s,n}+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\lambda_n^2\psi_{1,n}^1-\frac{(1-\nu^2)\cos^2\varphi}{r^4}\lambda_n^2\psi_{1,n}^2$$

$$-\frac{2(1-\nu)}{r^4}\lambda_n\psi_{3,n}^1-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\lambda_n^2\psi_{1,n}^3-\frac{(1-\nu^2)\cos^2\varphi}{r^3}\lambda_n^2\psi_{1,n}^4+\frac{2(1-\nu)}{r^3}\lambda_n\psi_{3,n}^2-q_z,$$

$$\frac{d\hat{S}_n}{ds}=\frac{\nu\cos\varphi}{r}\lambda_n N_{r,n}+\frac{\nu\sin\varphi}{r}\lambda_n N_{z,n}-\frac{2\cos\varphi}{r}\hat{S}_n+\frac{\nu\sin\varphi}{r^2}\lambda_n M_{s,n}-\frac{(1-\nu^2)\sin^2\varphi}{r^4}\lambda_n\psi_{1,n}^1+\frac{(1-\nu^2)}{r^2}\lambda_n\psi_{2,n}^1$$

$$+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\lambda_n\psi_{1,n}^2+\frac{1-\nu^2}{r^2}\lambda_n\psi_{2,n}^2+\frac{(1-\nu^2)\sin^2\varphi}{r^4}\lambda_n\psi_{1,n}^3+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\lambda_n\psi_{1,n}^4-q_\theta,$$

$$\frac{dM_{s,n}}{ds}=\sin\varphi N_{r,n}-\cos\varphi N_{r,n}-\frac{4\sin\varphi}{r^2}\lambda_n\psi_{4,n}^1$$

$$-\frac{(1-\nu)\cos\varphi}{r}M_{s,n}-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\psi_{1,n}^1+\frac{2(1-\nu)}{r^3}\lambda_n\psi_{3,n}^1$$

$$+\frac{(1-\nu^2)\cos^2\varphi}{r^3}\psi_{1,n}^2+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\psi_{1,n}^3-\frac{2(1-\nu)}{r^2}\lambda_n\psi_{3,n}^2+\frac{(1-\nu^2)\cos^2\varphi}{r^2}\psi_{1,n}^4,$$

$$\frac{du_{r,n}}{ds}=\cos^2\varphi\cdot\psi_{5,n}^1+\sin\varphi\cos\varphi\cdot\psi_{5,n}^2-\frac{\nu\cos\varphi}{r}u_{r,n}-\frac{\nu\cos\varphi}{r}\lambda_n v_n-\sin\varphi\vartheta_{s,n},$$

$$\frac{du_{z,n}}{ds}=\sin\varphi\cos\varphi\cdot\psi_{5,n}^1+\sin^2\varphi\cdot\psi_{5,n}^2-\frac{\nu\sin\varphi}{r}u_{r,n}-\frac{\nu\sin\varphi}{r}\lambda_n v_n+\cos\varphi\vartheta_{s,n},$$

$$\frac{dv_n}{ds}=\frac{2}{1-\nu}\psi_{6,n}+\frac{\cos\varphi}{r}\lambda_n u_{r,n}+\frac{4\sin\varphi}{r^3}\psi_{4,n}^2+\frac{\sin\varphi}{r}\lambda_n u_{z,n}+\frac{\cos\varphi}{r}v_n-\frac{4\sin\varphi}{r^2}\psi_{4,n}^3,$$

$$\frac{d\theta_{s,n}}{ds}=\psi_{7,n}-\frac{\nu\sin\varphi}{r^2}\lambda_n^2 u_{r,n}+\frac{\nu\cos\varphi}{r^2}\lambda_n^2 u_{z,n}-\frac{\nu\sin\varphi}{r^2}\lambda_n v_n-\frac{\cos\varphi}{r}\vartheta_{s,n}\qquad(n=0,N). \tag{6.4}$$

The boundary conditions are

$$B_1\bar{Z}(0)=\bar{b}_1,\qquad B_2\bar{Z}(L)=\bar{b}_2, \tag{6.5}$$

where $\bar{Z}(s)=\{N_{r,n},N_{z,n},\hat{S}_n,M_{s,n},u_{r,n},u_{z,n},v_n,\vartheta_{s,n}\}^{\mathrm{T}}$ is the column vector of unknown functions; B_1 and B_2 are the rectangular matrices; and $\bar{b}_1$ and $\bar{b}_2$ are vectors.

The boundary-value problem (6.4), (6.5) is solved by discrete orthogonalization on the interval $0\le s\le L$. For each n in Eqs. (5.6) with (5.3), we have:

$$\psi_{1,n}^j=\psi_{1,n}^j(s,u_r,u_z,v,\vartheta_s)\quad(j=1,2,3,4),\qquad\psi_{2,n}^j=\psi_{2,n}^j(s,u_r,v)\quad(j=1,2),$$

$$\psi_{3,n}^j=\psi_{3,n}^j(s,u_z,\vartheta_s)\quad(j=1,2),\qquad\psi_{4,n}^j=\psi_{4,n}^j(s,\hat{S},u_z,\vartheta_s)\quad(j=1,2,3),$$

$$\psi_{5,n}^j=\psi_{5,n}^j(s,N_r,N_z)\quad(j=1,2),\qquad\psi_{6,n}=\psi_{6,n}(s,\hat{S}),\qquad\psi_{7,n}=\psi_{7,n}(s,M_s). \tag{6.6}$$

The functions $\psi_{i,n}^j$ appearing in the coefficients of Fourier series (6.3) cannot be explicitly expressed in terms of the Fourier coefficients of the unknown functions, but can be calculated by integrating system (6.4) and using discrete Fourier series

TABLE 8

θ / π	wE / q_0				
	$R = 20$ $N = 4$	$R = 40$ $N = 6$	$R = 80$ $N = 8$	$R = 100$ $N = 12$	$R = 120$ $N = 15$
0	3539.9	3719.6	3640.9	3658.3	3658.5
0.2	3769.5	3813.7	3840.5	3831.9	3831.7
0.4	4370.6	4200.8	4264.2	4256.3	4256.4
0.6	5113.6	4960.8	4890.1	4903.6	4903.7
0.8	5714.8	5803.4	5779.4	5775.6	5775.4
1.0	5944.3	6179.0	6266.9	6263.2	6263.5

TABLE 9

θ / π	σ_θ^+ / q_0				
	$R = 20$ $N = 4$	$R = 40$ $N = 6$	$R = 80$ $N = 8$	$R = 100$ $N = 12$	$R = 120$ $N = 15$
0	–0.6530	–0.3032	–0.4882	–0.4681	–0.4669
0.2	–0.6347	–0.4928	–0.4540	–0.4547	–0.4556
0.4	–0.6287	–0.8706	–0.7262	–0.7476	–0.7470
0.6	–0.7327	–1.0860	–1.1710	–1.1620	–1.1617
0.8	–0.9605	–0.9225	–0.9818	–0.9647	–0.9655
1.0	–1.1023	–0.7015	–0.5901	–0.6184	–0.6173

TABLE 10

θ / π	σ_θ^- / q_0				
	$R = 20$ $N = 4$	$R = 40$ $N = 6$	$R = 80$ $N = 8$	$R = 100$ $N = 12$	$R = 120$ $N = 15$
0	0.5859	0.2720	0.4380	0.4200	0.4190
0.2	0.5722	0.4443	0.4093	0.4099	0.4109
0.4	0.5740	0.7948	0.6629	0.6825	0.6819
0.6	0.6793	1.0069	1.0857	1.0773	1.0771
0.8	0.9018	0.8661	0.9218	0.9056	0.9065

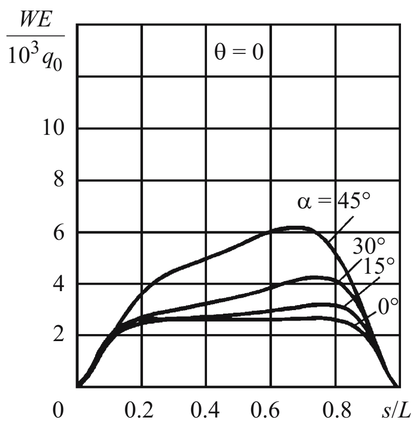

Fig. 4

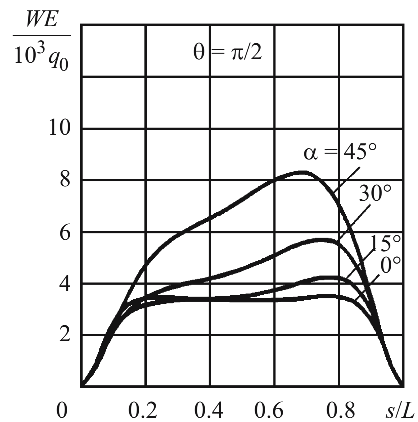

Fig. 5

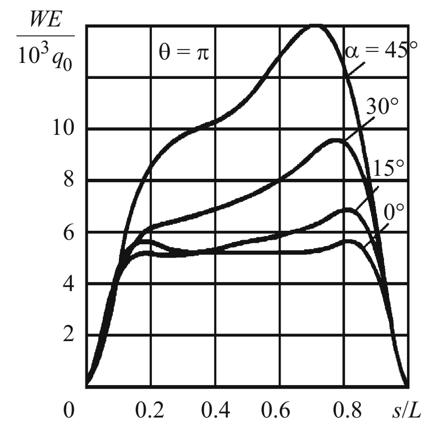

Fig. 6

at each step s = const. Formulas (6.6) relate these coefficients and the amplitudes of the unknown functions and demonstrate the coupling of all the equations in (6.4).

Equations (6.4) are integrated for all harmonics simultaneously using the procedure of Sec. 1.

Let us now examine the influence of the numbers R and N on the results.

Consider a conical shell with cone angle 2α $(\alpha = \pi/2 - \varphi)$ and thickness varying as $h(\theta) = H(1+\beta\cos\theta)$. The shell is clamped at the ends and is subjected to a uniform normal load $q = q_0 =$ const. It is assumed that as the cone angle changes, the generatrix turns about the midradius R_{mid} (radius at $s = L/2$), its length remaining constant. With such a manner of variation in the thickness, the weight of the shell remains constant for any value of β. The input data: $L = 30$, $R_{\text{mid}} = 30$, $H = 0.25$, $\beta = 0.3$, $\alpha = 30°$.

Table 8 collects the deflections in the section $s = L/2$ obtained with different values of R and N. It can be seen that as the number N is changed from 6 to 8, the maximum deflection observed at $\theta = \pi$ increases by almost 1.5%. With further increase in N, however, the difference between deflections decreases and the solution tends to the exact one. As N is increased from 10 to 12, the difference between maximum deflections is 0.03%. The situation is similar for the stresses $\sigma_\theta^{\pm}$ (Tables 9 and 10). Tables 8–10 indicate that the solution converges.

Let us follow this approach to analyze the dependence of the stress–strain state of conical shells with thickness varying in the circumferential direction on the cone angle and thickness.

Let a clamped shell have cone angle 2α $(\alpha = \pi/2 - \varphi)$ and thickness varying as $h(\theta) = H(1+\beta\cos\theta)$. The shell is under a uniform normal load $q = q_0 =$const. It is assumed that as the cone angle is changed, the generatrix turns about the midradius R_{mid} (radius at $s = L/2$), its length remaining constant. With such a manner of variation in the thickness, the weight of the shell remains constant for any value of β. The input data: $L = 30$, $R_{\text{mid}} = 30$, $H = 0.25$, $\beta = 0.0, 0.2, 0.3, 0.4$, $\alpha = 0°, 15°, 30°, 45°$, $R = 100$, $N = 12$.

Figures 4–6 show the variation in the deflection w in the longitudinal direction for $\beta = 0.4$ and $\theta = 0, \pi/2, \pi$ and different values of 2α. It can be seen how the deflection depends on the angle α when θ is kept constant. The deflection near the major base increases. As the thickness changes in the circumferential direction from $\theta = 0$ to $\theta = \pi$, the maximum deflection increases as follows: 1, 1.3, 2.3 for $\alpha = 15°$; 1, 1.4, 2.4 for $\alpha = 30°$, and 1, 1.3, 2.3 for $\alpha = 45°$. That is, the maximum deflections are in similar ratios for all cone angles. The maximum deflection and the deflection of the cylindrical shell are in the ratio 1:1.2:1.6:2.4 for all values of α. The circumferential variation of the deflection can be explained by the fact that the thickness between $\theta = 0$ and $\theta = \pi$ changes from 1.4 to 0.6, i.e., by a factor of 2.3.

Other approaches to solving static problems for conical shells with variable thickness were outlined in [34–36].

7. Flexible Shells with Curved Coordinate Surfaces. Consider shells with curved (Monge) surfaces [26], which form a wide class of surfaces referred to the lines of principal curvatures.

The governing system of partial differential equations describing such shells can be represented in orthogonal coordinates α_1 and α_2 [11, 16, 21]:

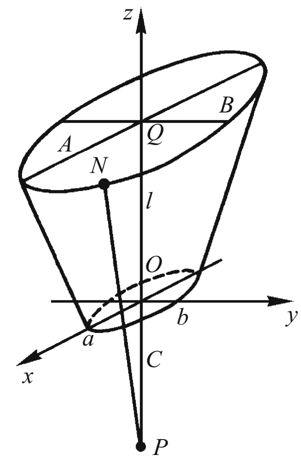

Fig. 7

$$\frac{\partial \bar{Z}(\alpha_1 \alpha_2)}{\partial \alpha_1} = \bar{G}\left(\alpha_1, \alpha_2 \frac{\partial^k \bar{Z}}{\partial \alpha_2^k}\right) \quad (k = \overline{0, 4}), \tag{7.1}$$

where $\bar{Z} = \{N_1, \hat{S}_1, \hat{Q}_1, M_1, u, v, w, \vartheta_1\}^{\mathrm{T}}$, $\bar{Z} = \{Z_i\}(i = \overline{1, 8})$, $\bar{G} = \{g_i\}$ is the vector of right-hand side; N_1, $\hat{S}_1$, and $\hat{Q}_1$ are the normal, shearing, and transverse forces; M_1 is the bending moment; u, v, and w are the displacements; and ϑ_1 is the angle of rotation of the normal. Let the shell be closed in the α_2-direction. Then the solution must satisfy periodicity conditions in α_2 and boundary conditions at the edges $\alpha_1 = $const, which are given by four quantities, one from each of the following pairs:

$$(N_1, u), \quad (\hat{S}_1, v), \quad (\hat{Q}_1, w), \quad (M_1, \vartheta_1). \tag{7.2}$$

We will use linearization to reduce the nonlinear boundary-value problem for the system of equations (7.1) with the boundary conditions (7.2) to a sequence of linear two-dimensional boundary-value problems for the system of equations

$$\frac{\partial \bar{Z}^{(s+1)}}{\partial \alpha_1} = \bar{F}\left(\alpha_1, \alpha_2, \frac{\partial^k \bar{Z}^{(s)}}{\partial \alpha_2^k}, \frac{\partial^k \bar{Z}^{(s+1)}}{\partial \alpha_2^k}\right) \quad (k = \overline{0, 4}, s = 0, 1, 2, \ldots),$$

$$\bar{F} = \bar{G}\left(\alpha_1, \alpha_2 \frac{\partial^k \bar{Z}^{(s)}}{\partial \alpha_2^k}\right) + J\left(\alpha_1, \alpha_2, \frac{\partial^k \bar{Z}^{(s)}}{\partial \alpha_2^k}\right) \left(\bar{Z}^{(s+1)} - \bar{Z}^{(s)}\right), \tag{7.3}$$

where $\bar{F} = \{f_i\}\ (i = \overline{1, 8})$, $J\left(\alpha_1, \alpha_2 \frac{\partial^k \bar{Z}}{\partial \alpha_2^k}\right)$ is the Jacobian matrix of the right-hand side of (7.3).

The boundary conditions can be linearized similarly. The initial condition or the parameter-continuation method is chosen as one sees fit. The two-dimensional linear boundary-value problem is solved at each iteration by expanding the functions on the right-hand side of (7.3) into discrete Fourier series [9, 10]. Then the solution of the boundary-value problem for the system of equations (42) with appropriate boundary conditions is sought in the form of truncated series:

$$X(\alpha_1, \alpha_2) = \sum_{n=0}^{N} X_n(\alpha_1) \cos n\alpha_2, \qquad Y(\alpha_1, \alpha_2) = \sum_{n=1}^{N} Y_n(\alpha_1) \sin n\alpha_2, \tag{7.4}$$

where $X = \{z_i, f_i\}$ and $Y = \{z_j, f_j\}$, $i = 1, 3, 4, 5, 7, 8$, $j = 2, 6$.

Substituting (7.4) into (7.3) and the boundary conditions and performing some transformations, we arrive at a coupled system of 6 + 8N differential equations:

TABLE 11

α_2	Minor base, $\sigma_{vv} \cdot 10^{-2}$, MPa			Major base, $\sigma_{vv} \cdot 10^{-2}$, MPa		
	$s = 1$	$s = 2$	$s = 3$	$s = 1$	$s = 2$	$s = 3$
0	–10.404	–7.494	–7.407	–9.293	–7.989	–7.828
	6.120	4.946	4.898	9.124	8.797	8.706
π/2	6.623	8.614	8.544	4.474	5.079	5.048
	–0.372	–1.221	–1.183	2.385	1.987	1.995

$$\frac{dZ_{i,0}^{(s+1)}}{d\alpha_1} = f_{i,0}\left(\alpha_1, Z_{j,0}^{(s)}, Z_{j,0}^{(s+1)}, Z_{i,m}^{(s)}, Z_{i,m}^{(s+1)}\right),$$

$$\frac{dZ_{k,n}^{(s+1)}}{d\alpha_1} = f_{k,n}\left(\alpha_1, Z_{j,0}^{(s)}, Z_{j,0}^{(s+1)}, Z_{i,m}^{(s)}, Z_{i,m}^{(s+1)}\right),$$

$$(i, j = 1, 3, 4, 5, 7, 8,\ \ k, l = \overline{1, 8},\ \ n, m = \overline{1, N},\ \ s = 0, 1, \ldots). \tag{7.5}$$

The linear boundary-value problem for system (7.5) is solved at each iteration by discrete orthogonalization.

Let us consider, as an example, a flexible elliptic conical shell. Its midsurface is generated by a ray PN (the point P lies on the Oz-axis and the point N on the ellipse) moving along an ellipse with semiaxes A and B (Fig. 7). The minor base of the cone is also an ellipse with semiaxes given by $a = \lambda A$, $b = \lambda B$, $\lambda = c/(l + c)$, $c = PO$, $l = OQ$.

The Cartesian coordinates of the midsurface are expressed as

$$x = A\frac{\alpha_1 + c}{l + c}\sin\alpha_2, \quad y = B\frac{\alpha_1 + c}{l + c}\cos\alpha_2, \quad z = \alpha_1 \quad (0 \le \alpha_1 \le l,\ 0 \le \alpha_2 \le 2\pi).$$

The shell is clamped and subjected to internal pressure q = 1.5 MPa. The input data: E = 70 GPa, $\mu = 0.3$, A = 450, B = 275, c = 500, l = 600, h = 2.5 mm, N = 6.

Table 11 summarizes the normal stresses σ_{vv} for three approximations s and $\alpha_2 = 0$ and $\alpha_2 = \pi/2$. The upper values are the stresses on the outside surface and the lower values are the stresses on the inside surface. The table demonstrates how the nonlinear solution (s = 3) differs from the linear one (s = 1).

8. Shells of Complex Geometry (Linear and Nonlinear Problems). In this section, we address nonlinear boundary-value problems for thin flexible shells with various holes and nontrivially shaped boundaries. Their middle or coordinate surface is described in a nonorthogonal coordinate system that does not coincide with the lines of principal curvatures [18–21].

We use the general tensor equations of the geometrically nonlinear theory of thin shells [31] to derive a governing system of nonlinear partial differential equations in the case of a nonorthogonally parametrized midsurface. The approach to be used to solve nonlinear boundary-value problems involves linearization, reduction of a two-dimensional problem to one-dimensional, and expansion of some functions into discrete Fourier series.

Let us analyze, in invariant tensor form, the stress–strain state of thin flexible shells with midsurface parametrized by two curvilinear (Gaussian) coordinates α^1 and α^2, the lines α^1 = const coinciding with the boundaries of open shells.

The closed-form system of equations includes:

the kinematic equations

$$2\varepsilon_{ij} = e_{ij} + \upsilon_i \upsilon_j, \qquad 2_{ij} = -\nabla_i \upsilon_j - \nabla_j \upsilon_i + b_i^{\alpha} e_{j\alpha} + b_j^{\alpha} e_{i\alpha},$$

$$e_{ij} = -\nabla_i \upsilon_j - b_{ij} w, \qquad \upsilon_i = -\left(\frac{\partial w}{\partial a^i} + b_j^{\alpha} u_{\alpha} \right), \tag{8.1}$$

the equilibrium equations

$$\nabla_{\alpha} T^{\alpha j} - b_{\alpha}^{j} Q^{\alpha} + q^i = 0, \qquad \nabla_{\alpha} Q^{\alpha} + b_{\alpha\beta} Q^{\alpha\beta} + q^3 = 0,$$

$$\nabla_{\alpha} M^{\alpha j} - Q^j - T^{j\alpha} \upsilon_{\alpha} = 0, \tag{8.2}$$

the elastic relations

$$S^{ij} = \frac{Eh}{1-\mu^2} \left[\mu a^{ij} a^{\alpha\beta} + (1-\mu) a^{i\alpha} a^{j\beta} \right] e_{\alpha\beta},$$

$$M^{ij} = -\frac{Eh^3}{12(1-\mu^2)} \left[\mu a^{ij} a^{\alpha\beta} + (1-\mu) a^{i\alpha} a^{j\beta} \right]_{\alpha\beta}, \qquad T^{ij} = S^{ij} - b_{\gamma}^{j} M^{\gamma i}, \tag{8.3}$$

where $i, j = 1, 2$; the summation is over the doubly repeated indices α and β from 1 to 2; a^{ij} are the contravariant components of the first metric tensor a_{ij} of the midsurface; b_{ij} and b_i^j are the covariant and mixed components of the second metric tensor, ∇_i denotes covariant differentiation with respect to the metric a_{ij}; u_i and w are the covariant components of the tangential-displacement vector and the deflection of the midsurface; T^{ij} and M^{ij} are the contravariant components of the tensors of forces and moments; S_{ij} are the components of the symmetric tensor of forces; Q^i are the transverse forces; q^j and q^3 are the components of the external surface load; and E, μ, and $h(\alpha^1, \alpha^2)$ are the elastic modulus, Poisson's ratio, and the thickness of the shell.

At each point of the coordinate line α^1 = const, we choose an orthogonal right-hand coordinate system formed by the unit vectors of the tangential normal $\bar{\nu}$, the tangent $\bar{\tau}$, and the normal $\bar{n}$ to the midsurface.

The boundary conditions for the system of equations (8.1)–(8.3) on each α^1 = const are given by four quantities, one from each of the following pairs:

$$(Q_{\nu\nu}, u_{\nu}), \qquad (Q_{\nu\tau}, u_{\tau}), \qquad (Q_{\nu n}, w), \qquad (M_{\nu\nu}, \vartheta_{\nu}), \tag{8.4}$$

where

$$u_{\nu} = \nu^i u_i, \qquad u_{\tau} = \tau^i u_i, \qquad \upsilon_{\nu} = \vartheta^i u_i,$$

$$Q_{\nu\nu} = T_{\nu\nu} + k_{\nu\tau} M_{\nu\tau}, \qquad Q_{\nu\tau} = T_{\nu\tau} + k_{\tau} M_{\nu\tau}, \qquad Q_{\nu n} = T_{\nu n} + \frac{1}{\sqrt{a_{22}}} \frac{\partial M_{\nu\tau}}{\partial \alpha^2}, \tag{8.5}$$

$T_{\nu\nu} = \nu_i \nu_j T^{ij}$, $T_{\nu\tau} = \nu_i \tau_j T^{ij}$, $T_{\nu n} = \nu_i Q^i$, $M_{\nu\nu} = \nu_i \nu_j M^{ij}$, $M_{\nu\tau} = \nu_i \tau_j M^{ij}$, $\nu_1 = \sqrt{\frac{a}{a_{22}}}$, $\nu_2 = 0$, $\tau_1 = \sqrt{\frac{a_{12}}{a_{22}}}$, $\tau_2 = \sqrt{a_{22}}$, $\nu^1 = \sqrt{\frac{a_{22}}{a}}$, $\nu^2 = \frac{a_{12}}{\sqrt{a_{22}}}$, $\tau^1 = 0$, $\tau^2 = \frac{1}{\sqrt{a_{22}}}$, u_{ν}, u_{τ}, and w are the displacements along $\bar{\nu}$-, $\bar{\tau}$-, and $\bar{n}$; υ_{ν} is the angle of rotation of the normal to the midsurface about the $\bar{\tau}$-axis; $Q_{\nu\nu}, Q_{\nu\tau}, Q_{\nu n}$, and $M_{\nu\nu}$ are the generalized forces associated with the generalized displacements u_{ν}, u_{τ}, w, and ϑ_{ν}, respectively; k_{τ} and $k_{\nu\tau}$ are the normal curvature and geodesic torsion of the midsurface along $\bar{\tau}$; ν_i, τ_i, ν^i, τ^i are the covariant and contravariant components of the unit vectors $\bar{\nu}$ and $\bar{\tau}$. After tedious transformations, the original system of equations (8.1)–(8.3) reduces to a governing system of partial differential equations for the functions in (8.4):

$$\frac{\partial \overline{Z}}{\partial \alpha^1} = \overline{G}\left(\alpha^1, \alpha^2, \frac{\partial^k \overline{Z}}{\partial (\alpha^2)^k} \right) \qquad (k = \overline{0,4}), \tag{8.6}$$

where $\overline{Z}^r = \{Q_{vv}, Q_{v\tau}, Q_{vn}, M_{vv}, u_v, u_\tau, w, \vartheta_v\}$, $\overline{Z}\{Z_i\}$ $(i = \overline{1,8})$ is the vector of unknown functions; $\overline{G}\{g_i\}$ is the vector of the right-hand side, which is a nonlinear vector function of $\overline{Z}$.

We use linearization to reduce the nonlinear boundary-value problem for system (8.6) with the boundary conditions (8.4) to a sequence of linear two-dimensional problems for the system of equations

$$\frac{\partial \overline{Z}^{(s+1)}}{\partial \alpha^1} = \overline{F}\left(\alpha^1, \alpha^2, \frac{\partial^k \overline{Z}^{(s)}}{\partial (\alpha^2)^k} \frac{\partial^k \overline{Z}^{(s+1)}}{\partial (\alpha^2)^k} \right) \qquad (k = \overline{0,4}, s = 0, 1\dots),$$

$$\overline{F} = \overline{G}\left(\alpha^1, \alpha^2, \frac{\partial^k \overline{Z}^{(s)}}{\partial (\alpha^2)^k} \right) + J\left(\frac{\partial^k \overline{Z}^{(s)}}{\partial (\alpha^2)^k} \right)(\overline{Z}^{(s+1)} - \overline{Z}^{(s)}), \tag{8.7}$$

where $\overline{F} = \{f_i\} (i = \overline{1,8})$, $J\left(\frac{\partial^k \overline{Z}}{\partial (\alpha^2)^k} \right)$ is the Jacobian matrix of the right-hand side of system (8.7).

The boundary conditions are linearized similarly. The solution of the linear problem or another approximate solution can be chosen as the initial approximation. To solve the linear boundary-value problem at each iteration, we expand the functions appearing in (8.7) into discrete Fourier series. The solution of the boundary-value problem for the system of equations (8.7) with appropriate boundary conditions is sought in the form

$$X(\alpha^1, \alpha^2) = \sum_{n=0}^{N} X_n(\alpha^1) \cos n\alpha^2, \quad Y(\alpha^1, \alpha^2) = \sum_{n=0}^{N} Y_n(\alpha^1) \sin n\alpha^2, \tag{8.8}$$

where $X = \{Z_i, f_i\}$ and $Y = \{Z_j, f_j\}$, $i = 1, 3, 4, 5, 7, 8$, $j = 2, 6$.

Substituting (8.8) into (8.7) and the boundary conditions and performing some transformations, we arrive at the following coupled system of 6 + 8*N* ordinary differential equations:

$$\frac{dZ_{i,0}^{(s+1)}}{d\alpha^1} = f_{i,0}(\alpha^1, Z_{j,0}^{(s)}, Z_{J,0}^{(s+1)}, Z_{i,m}^{(s)}, Z_{i,m}^{(s+1)}),$$

$$\frac{dZ_{k,n}^{(s+1)}}{d\alpha^1} = f_{k,n}(\alpha^1, Z_{j,0}^{(s)}, Z_{j,0}^{(s+1)}, Z_{l,m}^{(s)}, Z_{l,m}^{(s+1)})$$

$$(i, j = 1, 3, 4, 5, 7, 8, \; k, l = \overline{1,8}, \; n, m = \overline{1,N}, \; s = 0, 1\dots). \tag{8.9}$$

The boundary-value problem for system (8.7) is solved at each iteration by discrete orthogonalization.

Let us discuss solutions for some shell elements obtained with the approach proposed here.

We will solve a linear stress–strain problem for an elbow pipe. The pipe is radiused, has elliptic cross-section in the middle, and is symmetric about the plane $\alpha_1 = \pi/4$ (Fig. 8). Its midsurface is described by

$$x = \left[R + a(\alpha^1)\cos\alpha^2\right]\cos\alpha^1, \quad y = \left[R + a(\alpha^1)\cos\alpha^2\right]\sin\alpha^1, \quad z = b(\alpha^1)\sin\alpha^2. \tag{8.10}$$

For segment *1* with constant circular cross-section, we have

$$a(\alpha^1) = b(\alpha^1) = r \qquad (0 \le \alpha^1 \le \alpha_0^1, 0 \le \alpha^2 \le 2\pi). \tag{8.11}$$

For segment *2* with varying cross-section, we have

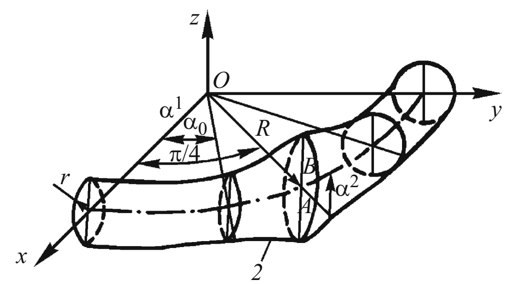

Fig. 8

$$a(\alpha^1)\frac{r}{f(\alpha^1)}-\left[1-\delta(\alpha^1)\right],\qquad b(\alpha^1)\frac{r}{f(\alpha^1)}-\left[1+\delta(\alpha^1)\right]$$

$$(\alpha_0^1 \le \alpha^1 \le \pi/4,\ 0 \le \alpha^2 \le 2\pi),$$

$$f(\alpha^1)=1\frac{\left[\delta(\alpha^1)\right]^2}{4}+\frac{\left[\delta(\alpha^1)\right]^4}{64}+\frac{\left[\delta(\alpha^1)\right]^6}{256},$$

$$\delta(\alpha^1)=\Delta\sin^3\frac{\pi(\alpha^1-\alpha_0^1)}{2(\pi/4-\alpha_0^1)}, \tag{8.12}$$

where $\Delta=(B-A)/(B+A)$ is the eccentricity of the elliptic cross-section; A and B are the semiaxes of the ellipse at $\alpha^1=\pi/4$ (Fig. 8). The current values $a(\alpha^1)$ and $b(\alpha^1)$ of the semiaxes in each cross-section of segment *2* are determined by equating the ellipse perimeter to the perimeter of a circle of radius r. The parametrization based on the function $\delta(\alpha^1)$ in the form (8.10) ensures continuity of the coefficients a_{ij}, b_{ij}, and Γ_{ij}^k at the interface between segments *1* and *2* of the midsurface.

The elbow pipe is designed to be rigidly fixed at the ends and to withstand an internal pressure q. The boundary conditions are

$$\alpha^1=0-u_1=u_2=w=\vartheta_1=0,\qquad \alpha^1=\pi/4-u_1=S_1=Q_1=\vartheta_1=0.$$

The initial data: r = 100 mm, R = 500 mm, h = 5 mm, Δ = 0.25, $\alpha_0^1=\pi/6$, E = 200 GPa, μ = 0.3, N = 10.

Figure 9 shows the stresses $\sigma_{vv}^{\pm}$ and $\sigma_{tt}^{\pm}$ at $\alpha_1=\pi/4$ and the deformed cross-section (not to scale).

These results are indicative of the efficiency of the approach to the stress–strain analysis of thin-walled shells with complex geometry parametrized by nonorthogonal curvilinear coordinates.

Consider a cylindrical tank undergoing geometrically nonlinear deformation. The tank has an off-pole branch pipe in an ellipsoidal head and consists of four shell elements:

axisymmetric cylindrical shell *1*:

$$x=A\sin\alpha^2,\qquad y=A\cos\alpha^2,\qquad z=\alpha^1-L, \tag{8.13}$$

$$0\le\alpha^1\le L,\qquad 0\le\alpha^2\le 2\pi,$$

ellipsoidal shell of revolution *2*:

$$x=A\cos\alpha^1\sin\alpha^2,\qquad y=A\cos\alpha^1\cos\alpha^2,\qquad z=B\sin\alpha^1, \tag{8.14}$$

$$0\le\alpha^1\le\arccos(a/A),\quad 0\le\alpha^2\le 2\pi,$$

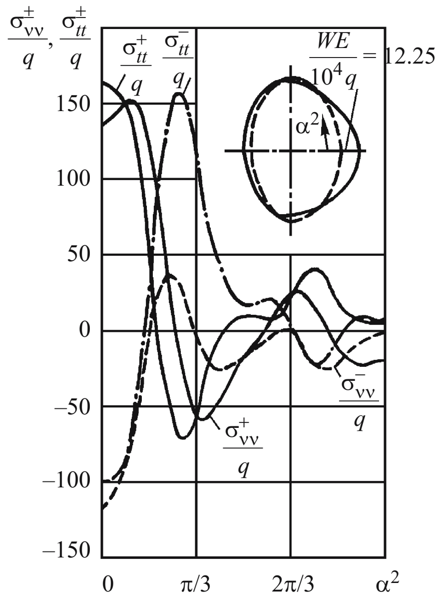

Fig. 9

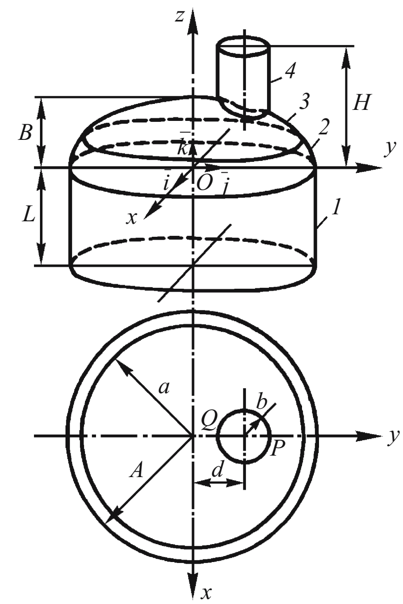

Fig. 10

ellipsoidal shell *3* with off-pole opening; its projection onto a plane perpendicular to the structure axis is a domain bounded by two nonconcentric circles:

$$x = [a + \alpha^1 (b - a)] \sin \alpha^2, \quad y = [a + \alpha^1 (b - a)] \cos \alpha^2 + \alpha^1 d, \tag{8.15}$$

$$z = \frac{B}{A} \sqrt{A^2 - x^2 - y^2}, \quad 0 \le \alpha^1 \le 1, \quad 0 \le \alpha^2 \le 2\pi,$$

cylindrical shell *4* with one edge resulting from the intersection of the ellipsoid and the cylinder:

$$x = b \sin \alpha^2, \quad y = d + b \cos \alpha^2,$$

$$z = (1 - \alpha^1) \frac{B}{A} \sqrt{A^2 - x^2 - y^2} + \alpha^1 H, \tag{8.16}$$

$$0 \le \alpha^1 \le 1, \quad 0 \le \alpha^2 \le 2\pi.$$

where A and B are the semiaxes of the ellipsoid of revolution, L is the length of the cylinder, a is the radius of the interface between elements *2* and *3*, b is the radius of the branch pipe, d is the eccentricity, and H is the distance to the edge of the branch pipe.

The tank is subjected to internal pressure q; the edge of the cylinder is rigidly fixed: $u_\nu = u_\tau = w = \vartheta_\nu = 0$. A uniformly distributed axial force $Q_0 = qb/2 - Q_{\nu\nu}$ is applied to the branch pipe; $u_\tau = w = \vartheta_\nu = 0$.

The input data: $A = 500$, $B = 300$, $L = 400$, $a = 450$, $b = 100$, $d = 200$, $H = 500$, $h = 2.5$ mm, $E = 100$ GPa, $\mu = 0.3$, $q = 2.5$ MPa, $N = 4$.

Table 12 summarizes the dimensionless stresses $\hat{\sigma}_{\nu\nu}$ and $\hat{\sigma}_{rr}$ normal to the lines $\alpha^1 =$ const and $\alpha^2 =$ const, respectively, and the dimensionless deflection $\hat{w}$ at points P and Q in the plane of symmetry of the structure on its head (Fig. 10) on the outside surface (nominator) and inside surface (denominator). Here

$$\hat{\sigma}_{\nu\nu} = \sigma_{\nu\nu} / \sigma_0, \quad \hat{\sigma}_{rr} = \sigma_{rr} / \sigma_0, \quad \hat{w} = w / w_0, \tag{8.17}$$

$$\sigma_0 = \frac{qA}{h}, \quad w_0 = \frac{qA_2}{Eh}\left(1 - \frac{\mu}{2}\right).$$

TABLE 12

Point	$\hat{\sigma}_{vv}$			$\hat{\sigma}_{rr}$			$\hat{w}$		
	$s = 1$	$s = 2$	$s = 3$	$s = 1$	$s = 2$	$s = 3$	$s = 1$	$s = 2$	$s = 3$
P	$\frac{4.141}{-3.491}$	$\frac{3.680}{-3.055}$	$\frac{3.658}{-3.036}$	$\frac{3.540}{-1.248}$	$\frac{3.101}{-1.069}$	$\frac{3.094}{-1.076}$	4.207	3.269	3.196
S	$\frac{1.633}{-1.366}$	$\frac{1.221}{-0.977}$	$\frac{1.185}{-0.945}$	$\frac{2.250}{1.533}$	$\frac{1.878}{1.383}$	$\frac{1.856}{1.382}$	3.418	2.674	2.614

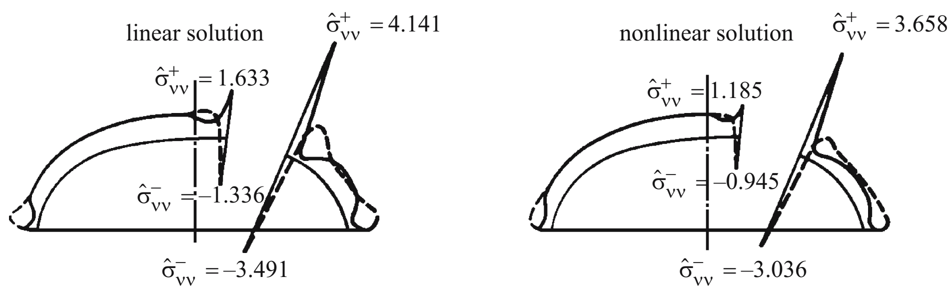

Fig. 11

Figure 11 shows the variation of the dimensionless stresses $\sigma^{\pm}_{vv}$ along the generatrix of the tank head obtained by solving the linear and geometrically nonlinear problems. The index "–" refers to the inside surface of the head (dashed line), and the index "+" to the outside surface (solid line).

These results are indicative of the efficiency of the approach to linear and nonlinear analyses of shells of complex geometry.

9. Noncircular Hollow Inhomogeneous Cylinders. Consider elliptical, longitudinally corrugated circular, and longitudinally corrugated elliptical layered hollow cylinders made of isotropic and orthotropic materials. We will determine their stress state using discrete Fourier series.

Let us first address the general stress–strain problem for longitudinally corrugated elastic hollow layered orthotropic cylinders of constant thickness. We use the equations of the three-dimensional theory of orthotropic elasticity [1, 22, 24, 27]. The first quadratic form is

$$dS^2 = ds^2 + A_2^2(\psi,\gamma)d\psi^2 + d\gamma^2, \tag{9.1}$$

where s, ψ, and γ are orthogonal curvilinear coordinates, s is the longitudinal coordinate, ψ is the polar angle, and γ is the normal coordinate to the reference surface $\gamma = \gamma_0$.

The directrix of the reference surface (Fig. 12) is described by polar coordinates as

$$\rho(\psi) = \frac{a}{(1-e^2\cos^2\psi)^{1/2}} + \alpha\cos m\psi \qquad (0 \le \psi \le 2\pi),$$

$$e = \sqrt{1-(a/b)^2} = 2\sqrt{\Delta}/(1+\Delta), \qquad \Delta = (b-a)/(b+a), \tag{9.2}$$

where α is the amplitude; m is the number of corrugations; a, b, and e are the semiaxes and eccentricity of the ellipse ($b > a$); the point O ($\rho = 0$) is on the intersection of the ellipse axes. Formula (9.2) describes an ellipse if $\alpha = 0$ and a wavy circle if $a = b$. Figure 13 shows an element of the cross-section.

We have

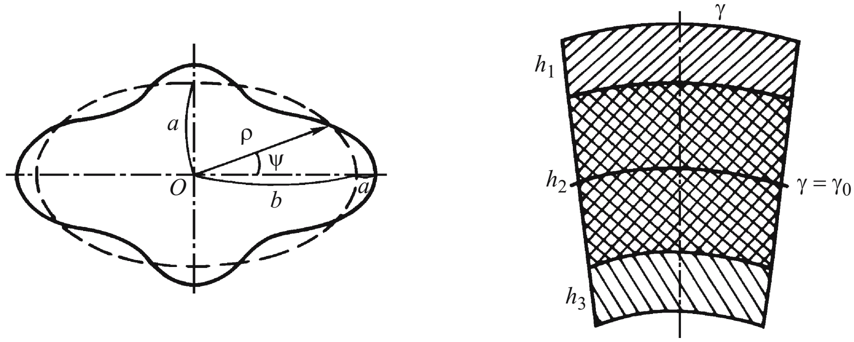

Fig. 12

$$A_2(\psi,\gamma)=H_2(\psi,\gamma)\omega(\psi), \tag{9.3}$$

where

$$H_2(\psi,\gamma)=1+\gamma/R(\psi), \qquad R(\psi)=\frac{[\rho^2+(\rho')^2]^{3/2}}{\rho^2+2(\rho)^2-\rho\rho},$$

$$\omega(\psi)=[\rho^2+(\rho'')^2]^{1/2}, \qquad \rho'=-\left[\frac{ae^2\sin 2\psi}{2(1-e^2\cos^2\psi)^{3/2}}+\alpha m\sin m\psi\right],$$

$$\rho''=-\left\{\frac{ae^2}{2}\left[\frac{2\cos 2\psi}{(1-e^2\cos^2\psi)^{3/2}}-\frac{3e^2\sin^2 2\psi}{2(1-e^2\cos^2\psi)^{5/2}}\right]+\alpha m^2\cos m\psi\right\},$$

$R(\psi)$ is the radius of the curvature of the reference surface.

With (9.1)–(9.3), the original equations describing the equilibrium of the ith layer of the cylinder include

the kinematic equations

$$e_s^i=\frac{\partial u_s^i}{\partial s}, \quad e_\psi^i=\frac{1}{H_2\omega}\frac{\partial u_\psi^i}{\partial\psi}+\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_\gamma^i, \quad e_\gamma^i=\frac{\partial u_\gamma^i}{\partial\gamma},$$

$$e_{s\psi}^i=\frac{1}{H_2\omega}\frac{\partial u_s^i}{\partial\psi}+\frac{\partial u_\psi^i}{\partial s}, \quad e_{s\gamma}^i=\frac{\partial u_\gamma^i}{\partial s}+\frac{\partial u_s^i}{\partial\gamma}, \quad e_{\psi\gamma}^i=H_2\frac{\partial}{\partial\gamma}\left(\frac{u_\psi^i}{H_2}\right)+\frac{1}{H_2\omega}\frac{\partial u_\gamma^i}{\partial\psi}, \tag{9.4}$$

the equilibrium equations

$$H_2\frac{\partial\sigma_s^i}{\partial s}+\frac{1}{\omega}\frac{\partial\tau_{s\psi}^i}{\partial\psi}+\frac{\partial}{\partial\gamma}(H_2\tau_{s\gamma}^i)=0,$$

$$\frac{1}{\omega}\frac{\partial\sigma_\psi^i}{\partial\psi}+\frac{\partial}{\partial\gamma}(H_2\tau_{\psi\gamma}^i)+H_2\frac{\partial\tau_{s\psi}^i}{\partial s}+\frac{\partial H_2}{\partial\gamma}\tau_{\psi\gamma}^i=0,$$

$$\frac{\partial}{\partial\gamma}(H_2\sigma_\gamma^i)+H_2\frac{\partial\tau_{s\gamma}^i}{\partial s}+\frac{1}{\omega}\frac{\partial\tau_{\psi\gamma}^i}{\partial\psi}-\frac{\partial H_2}{\partial\gamma}\sigma_\psi^i=0, \tag{9.5}$$

the generalized Hooke’s low for an orthotropic body

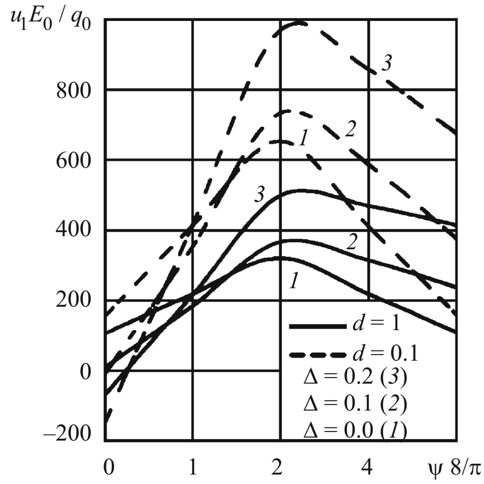

Fig. 13

$$e_s^i = a_{11}^i \sigma_s^i + a_{12}^i \sigma_\psi^i + a_{13}^i \sigma_\gamma^i, \quad i_{\psi\gamma} = a_{44}^i \tau_{\psi\gamma}^i,$$

$$e_\psi^i = a_{12}^i \sigma_s^i + a_{22}^i \sigma_\psi^i + a_{23}^i \sigma_\gamma^i, \quad i_{s\gamma} = a_{55}^i \tau_{s\gamma}^i,$$

$$e_\gamma^i = a_{13}^i \sigma_s^i + a_{23}^i \sigma_\psi^i + a_{33}^i \sigma_\gamma^i, \quad i_{s\psi} = a_{66}^i \tau_{s\psi}^i, \tag{9.6}$$

where

$$a_{11}^i = \frac{1}{E_s^i}, \quad a_{12}^i = -\frac{\nu_{s\psi}^i}{E_\psi^i} = -\frac{\nu_{\psi s}^i}{E_s^i}, \quad a_{13}^i = -\frac{\nu_{s\gamma}^i}{E_\gamma^i} = -\frac{\nu_{\gamma s}^i}{E_s^i},$$

$$a_{22}^i = \frac{1}{E_\psi^i}, \quad a_{23}^i = -\frac{\nu_{\gamma\psi}^i}{E_\psi^i} = -\frac{\nu_{\psi\gamma}^i}{E_\gamma^i}, \quad a_{33}^i = \frac{1}{E_\gamma^i},$$

$$a_{44}^i = \frac{1}{G_{\psi\gamma}^i}, \quad a_{55}^i = \frac{1}{G_{s\gamma}^i}, \quad a_{66}^i = \frac{1}{G_{\psi s}^i}$$

$$(0 \le s \le l,\ 0 \le \psi \le 2\pi,\ \gamma_p \le \gamma \le \gamma_q) \quad (i = \overline{1, T}). \tag{9.7}$$

If the body is isotropic, relations (9.6) become

$$e_s^i = \frac{1}{E^i}(\sigma_s^i - \nu^i(\sigma_\psi^i + \sigma_\gamma^i)), \quad e_{\psi\gamma}^i = \frac{1}{G^i}\tau_{\psi\gamma}^i,$$

$$e_\psi^i = \frac{1}{E^i}(\sigma_\psi^i - \nu^i(\sigma_s^i + \sigma_\gamma^i)), \quad e_{s\gamma}^i = \frac{1}{G^i}\tau_{s\gamma}^i,$$

$$e_\gamma^i = \frac{1}{E^i}(\sigma_\gamma^i - \nu^i(\sigma_s^i + \sigma_\psi^i)), \quad e_{s\psi}^i = \frac{1}{G^i}\tau_{s\psi}^i, \tag{9.8}$$

where u_s^i, u_ψ^i, and u_γ^i are the longitudinal, circumferential, and thickness displacements; $e_s^i, e_\psi^i, e_\gamma^i, e_{s\psi}^i, e_{\psi\gamma}^i, \sigma_s^i, \sigma_\psi^i, \sigma_\gamma^i, \tau_{s\psi}^i, \tau_{\psi\gamma}^i, \tau_{s\gamma}^i$ are the strains and stresses in the ith layer; and $E_s^i, E_\psi^i, E_\gamma^i, G_{s\gamma}^i, G_{s\psi}^i, G_{\psi\gamma}^i, \nu_{s\gamma}^i, \nu_{s\psi}^i, \nu_{\psi\gamma}^i$ are the corresponding elastic moduli, shear moduli, and Poisson's ratios.

Supplementing Eqs. (9.4)–(9.6) with boundary conditions at the ends $s = 0, s = l$ and on the lateral surfaces $\gamma = \gamma_p, \gamma = \gamma_q$ of the cylinder, we obtain a three-dimensional boundary-value problem.

Consider cylinders with simply supported ends [55]:

$$\sigma_s^i = 0, \quad u_\psi^i = 0, \quad u_\gamma^i = 0 \quad \text{at} \quad s = 0, l, \tag{9.9}$$

which means that the ends are closed with diaphragms perfectly rigid in plane and flexible out of plane.

The following boundary conditions are prescribed on the lateral surface of the cylinder:

$$\sigma_\gamma = q_\gamma^-, \quad \tau_{s\gamma} = q_s^-, \quad \tau_{\psi\gamma} = q_\psi^- \quad \text{at} \quad \gamma = \gamma_p,$$

$$\sigma_\gamma = q_\gamma^+, \quad \tau_{s\gamma} = q_s^+, \quad \tau_{\psi\gamma} = q_\psi^+ \quad \text{at} \quad \gamma = \gamma_q. \tag{9.10}$$

Moreover, contacting layers do not separate and slip, i.e.,

$$\sigma_\gamma^i = \sigma_\gamma^{i+1}, \quad \tau_{s\gamma}^i = \tau_{s\gamma}^{i+1}, \quad \tau_{\psi\gamma}^i = \tau_{\psi\gamma}^{i+1},$$

$$u_\gamma^i = u_\gamma^{i+1}, \quad u_s^i = u_s^{i+1}, \quad u_\psi^i = u_\psi^{i+1}. \tag{9.11}$$

Considering these conditions, we choose the stresses and displacements $\sigma_\gamma, \tau_{s\gamma}, \tau_{\psi\gamma}, u_\gamma, u_s, u_\psi$ as unknown functions. After some transformations, Eqs. (9.4)–(9.6) reduce to a governing system of six partial differential equations with variable coefficients:

$$\frac{\partial \sigma_\gamma^i}{\partial \gamma} = (c_2^i - 1)\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\sigma_\gamma^i - \frac{\partial \tau_{s\gamma}^i}{\partial s} - \frac{1}{H_2 \omega}\frac{\partial \tau_{\psi\gamma}^i}{\partial \psi}$$

$$+ b_{22}^i\left(\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\right)^2 u_\gamma^i + b_{12}^i \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\frac{\partial u_s^i}{\partial s} + b_{22}^i \frac{1}{H_2^2}\frac{\partial H_2}{\partial \gamma}\frac{1}{\omega}\frac{\partial u_\psi^i}{\partial \psi},$$

$$\frac{\partial \tau_{s\gamma}^i}{\partial \gamma} = -c_1^i \frac{\partial \sigma_\gamma^i}{\partial s} - \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\tau_{s\gamma}^i - b_{12}^i \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\frac{\partial u_\gamma^i}{\partial s} - b_{11}^i \frac{\partial^2 u_s^i}{\partial s^2}$$

$$-b_{66}^i \frac{1}{H_2 \omega}\frac{\partial}{\partial \psi}\left(\frac{1}{H_2 \omega}\frac{\partial u_s^i}{\partial \psi}\right) - (b_{12}^i + b_{66}^i)\frac{1}{H_2 \omega}\frac{\partial^2 u_\psi^i}{\partial s \partial \psi},$$

$$\frac{\partial \tau_{\psi\gamma}^i}{\partial \gamma} = -c_2^i \frac{1}{H_2 \omega}\frac{\partial \sigma_\gamma^i}{\partial \psi} - \frac{2}{H_2}\frac{\partial H_2}{\partial \gamma}\tau_{\psi\gamma}^i - b_{22}^i \frac{1}{H_2 \omega}\frac{\partial}{\partial \psi}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_\gamma^i\right)$$

$$-(b_{12}^i + b_{66}^i)\frac{1}{H_2 \omega}\frac{\partial^2 u_s^i}{\partial s \partial \psi} - b_{22}^i \frac{1}{H_2 \omega}\frac{\partial}{\partial \psi}\left(\frac{1}{H_2 \omega}\frac{\partial u_\psi^i}{\partial \psi}\right) - b_{66}^i \frac{\partial^2 u_\psi^i}{\partial s^2},$$

$$\frac{\partial u_\gamma^i}{\partial \gamma} = c_4^i \sigma_\gamma^i - c_2^i \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_\gamma^i - c_1^i \frac{\partial u_s^i}{\partial s} - c_2^i \frac{1}{H_2 \omega}\frac{\partial u_\psi^i}{\partial \psi},$$

$$\frac{\partial u_s^i}{\partial \gamma} = a_{55}^i \tau_{s\gamma}^i - \frac{\partial u_\gamma^i}{\partial s}, \quad \frac{\partial u_\psi^i}{\partial \gamma} = a_{44}^i \tau_{\psi\gamma}^i - \frac{1}{H_2 \omega}\frac{\partial u_\gamma^i}{\partial \psi} + \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}u_\psi^i, \tag{9.12}$$

where

$$b_{11}^i = a_{22}^i a_{66}^i / \Omega^i, \quad b_{12}^i = -a_{12}^i a_{66}^i / \Omega^i, \quad b_{22}^i = a_{11}^i a_{66}^i / \Omega^i,$$

$$b_{66}^i = 1/ a_{66}^i, \quad \Omega^i = (a_{11}^i a_{22}^i - a_{12}^{i2}) a_{66}^i,$$

$$c_1^i = -(b_{11}^i a_{13}^i + b_{12}^i a_{23}^i), \quad c_2^i = -(b_{12}^i a_{13}^i + b_{22}^i a_{23}^i), \quad c_4^i = a_{33}^i + c_1^i a_{13}^i + c_2^i a_{23}^i. \tag{9.13}$$

The boundary conditions at the ends allow separation of variables in (9.12) with respect to *s*, thus making the problem two-dimensional. Let us expand the unknown functions and load components into Fourier series in powers of the longitudinal coordinate:

$$X(s, \psi, \gamma) = \sum_{n=1}^{N} X_n(\psi, \gamma) \sin \lambda_n s,$$

$$Y(s, \psi, \gamma) = \sum_{n=0}^{N} Y_n(\psi, \gamma) \cos \lambda_n s,$$

$$X = \{\sigma_\gamma^i, \tau_{\psi\gamma}^i, u_\gamma^i, u_\psi^i, q_\gamma^i, q_\psi^i\}, \quad Y = \{\tau_{s\gamma}^i, u_s^i, q_s^i\}, \quad \lambda_n = \pi n / l$$

$$(0 \le s \le l, \ i = \overline{1, T}). \tag{9.14}$$

Substituting (9.14) into (9.12) and the boundary conditions (9.10), we obtain a two-dimensional boundary-value problem for the *n*th term in (9.14) (the index *i* is omitted for simplicity):

$$\frac{\partial \sigma_{\gamma,n}}{\partial \gamma} = (c_2 - 1)\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\sigma_{\gamma,n} + \lambda_n \tau_{s\gamma,n} - \frac{1}{H_2 \omega}\frac{\partial \tau_{\psi\gamma,n}}{\partial \psi}$$

$$+ b_{22}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\right)^2 u_{\gamma,n} - b_{12}\lambda_n \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma} u_{s,n} + b_{22}\frac{1}{H_2^2}\frac{\partial H_2}{\partial \gamma}\frac{1}{\omega}\frac{\partial u_{\psi,n}}{\partial \psi},$$

$$\frac{\partial \tau_{s\gamma,n}}{\partial \gamma} = -c_1 \lambda_n \sigma_{\gamma,n} - \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma}\tau_{s\gamma,n} - b_{12}\lambda_n \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma} u_{\gamma,n} + b_{11}\lambda_n^2 u_{s,n}$$

$$- b_{66}^i \frac{1}{H_2 \omega}\frac{\partial}{\partial \psi}\left(\frac{1}{H_2 \omega}\frac{\partial u_{s,n}}{\partial \psi}\right) - (b_{12} + b_{66})\lambda_n \frac{1}{H_2 \omega}\frac{\partial u_{\psi,n}}{\partial \psi},$$

$$\frac{\partial \tau_{\psi\gamma,n}}{\partial \gamma} = -c_2 \frac{1}{H_2 \omega}\frac{\partial \sigma_{\gamma,n}}{\partial \psi} - \frac{2}{H_2}\frac{\partial H_2}{\partial \gamma}\tau_{\psi\gamma,n} - b_{22}\frac{1}{H_2 \omega}\frac{\partial}{\partial \psi}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial \gamma} u_{\gamma,n}\right)$$

$$+ (b_{12} + b_{66})\lambda_n \frac{1}{H_2 \omega}\frac{\partial u_{s,n}}{\partial \psi} - b_{22}\frac{1}{H_2 \omega}\frac{\partial}{\partial \psi}\left(\frac{1}{H_2 \omega}\frac{\partial u_{\psi,n}}{\partial \psi}\right) + b_{66}\lambda_n^2 u_{\psi,n},$$

$$\frac{\partial u_{\gamma,n}}{\partial \gamma} = c_4 \sigma_{\gamma,n} - c_2 \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma} u_{\gamma,n} + c_1 \lambda_n u_{s,n} - c_2 \frac{1}{H_2 \omega}\frac{\partial u_{\psi,n}}{\partial \psi},$$

$$\frac{\partial u_{s,n}}{\partial \gamma} = a_{55}\tau_{s\gamma,n} - \lambda_n u_{\gamma,n}, \quad \frac{\partial u_{\psi,n}}{\partial \gamma} = a_{44}\tau_{\psi\gamma,n} - \frac{1}{H_2 \omega}\frac{\partial u_{\gamma,n}}{\partial \psi} + \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma} u_{\psi,n} \quad (n = \overline{0, N}) \tag{9.15}$$

with the boundary conditions

$$\sigma_{\gamma,n}=q^-_{\gamma,n},\quad \tau_{s\gamma,n}=q^-_{s,n},\quad \tau_{\psi\gamma,n}=q^-_{\psi,n}\quad \text{at}\quad \gamma=\gamma_p,$$

$$\sigma_{\gamma,n}=q^+_{\gamma,n},\quad \tau_{s\gamma,n}=q^+_{s,n},\quad \tau_{\psi\gamma,n}=q^+_{\psi,n}\quad \text{at}\quad \gamma=\gamma_q. \tag{9.16}$$

To reduce the dimension of problem (9.15), (9.16) by one, the expressions in (9.15) that hinder the separation of variables with respect to the circumferential coordinate are replaced by subsidiary functions. These functions are expressed in terms of the unknown functions and include geometrical parameters of the cylinder:

$$\phi_1^j=\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\left\{\sigma_{\gamma,n},\tau_{s\gamma,n},u_{\gamma,n},u_{s,n},\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_{\gamma,n}\right\}\quad (j=\overline{1,5}),$$

$$\phi_2^j=\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\{\tau_{\psi\gamma,n},u_{\psi,n}\}\quad (j=\overline{1,2}),$$

$$\phi_3^j=\frac{1}{H_2\omega}\frac{\partial}{\partial\psi}\{\sigma_{\gamma,n},u_{\gamma,n},u_{s,n}\}\quad (j=\overline{1,3}),$$

$$\phi_4^j=\frac{1}{H_2\omega}\left\{\frac{\partial\tau_{\psi\gamma,n}}{\partial\psi},\frac{\partial u_{\psi,n}}{\partial\psi},\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\frac{\partial u_{\psi,n}}{\partial\psi}\right\}\quad (j=\overline{1,3},n=\overline{0,N}),$$

$$\phi_5=\frac{1}{H_2\omega}\frac{\partial\phi_1^3}{\partial\psi},\qquad \phi_6=\frac{1}{H_2\omega}\frac{\partial\phi_3^3}{\partial\psi},\qquad \phi_7=\frac{1}{H_2\omega}\frac{\partial\phi_4^2}{\partial\psi}. \tag{9.17}$$

Substituting functions (9.17) into (9.15), we obtain the following system of equations (*n* is omitted everywhere):

$$\frac{\partial\sigma_\gamma}{\partial\gamma}=(c_2-1)\phi_1^1+\lambda_n\tau_{s\gamma}-\phi_4^1+b_{22}\phi_1^5-b_{12}\lambda_n\phi_1^4+b_{22}\phi_4^3,$$

$$\frac{\partial\tau_{s\gamma}}{\partial\gamma}=-c_1\lambda_n\sigma_\gamma-\phi_1^2-b_{12}\lambda_n\phi_1^3+b_{11}\lambda_n^2u_s-b_{66}\phi_6-(b_{12}+b_{66})\lambda_n\phi_4^2,$$

$$\frac{\partial\tau_{\psi\gamma}}{\partial\gamma}=-c_2\phi_3^1-2\phi_2^1-b_{22}\phi_5+(b_{12}+b_{66})\lambda_n\phi_3^3-b_{22}\phi_7+b_{66}\lambda_n^2u_\psi,$$

$$\frac{\partial u_\gamma}{\partial\gamma}=c_4\sigma_\gamma-c_2\phi_4^2+c_1\lambda_nu_s-c_2\phi_1^3,$$

$$\frac{\partial u_s}{\partial\gamma}=a_{55}\tau_{s\gamma}-\lambda_nu_\gamma,\qquad \frac{\partial u_\psi}{\partial\gamma}=a_{44}\tau_{\psi\gamma}-\phi_3^2+\phi_2^2 \tag{9.18}$$

with the boundary conditions (9.16).

Let us expand all the functions in (9.18) and (9.16) into Fourier series in powers of the coordinate ψ:

$$X(\psi,\gamma)=\sum_{k=0}^{K}X_k(\gamma)\cos k\psi,\qquad Y(\psi,\gamma)=\sum_{k=1}^{K}Y_k(\gamma)\sin k\psi, \tag{9.19}$$

$$X=\{\sigma_\gamma,\tau_{s\gamma},u_\gamma,u_s,\phi_1^j,\phi_4^j,\phi_6,q_\gamma,q_s\},\qquad Y=\{\tau_{\psi\gamma},u_\psi,\phi_2^j,\phi_3^j,\phi_5,\phi_7,q_\psi\}.$$

The number K of terms in the series should be no smaller than the number of terms in the series for q_γ, q_s, and q_ψ.

Substituting (9.19) into (9.18) and (9.16) and separating variables, we arrive at a system of ordinary differential equations for the amplitudes of the functions appearing in (9.18):

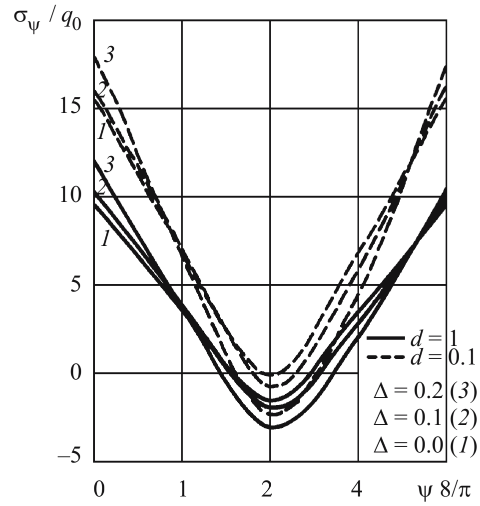

Fig. 14

TABLE 13

ψ	H	α = 2			α = 3		
		Δ = 0	Δ = 0.1	Δ = 0.2	Δ = 0	Δ = 0.1	Δ = 0.2
0	–4	40.2	–34.7	–94.9	32.7	–35.9	–88.7
	–2	34.9	–40.2	–100.8	26.4	–42.2	–95.2
	0	28.5	–45.4	–104.9	19.3	–47.9	–99.8
	2	25.8	–44.5	–102.3	17.8	–47.2	–96.6
	4	26.0	–44.1	–99.4	19.1	–44.9	–93.2
π/4	–4	239.4	275.0	380.3	325.1	360.3	461.2
	–2	239.7	275.7	382.5	327.3	362.9	465.1
	0	238.3	274.7	382.4	328.7	365.0	468.9
	2	229.9	265.9	372.4	319.9	356.0	459.5
	4	224.5	260.1	365.4	313.3	348.9	451.5
π/2	–4	40.2	139.2	276.7	32.7	127.3	261.4
	–2	34.9	133.8	271.2	26.4	120.7	254.4
	0	28.5	126.0	261.9	19.3	112.1	243.9
	2	25.8	120.7	253.5	17.8	108.1	237.1
	4	26.0	119.8	251.4	19.1	108.5	236.4

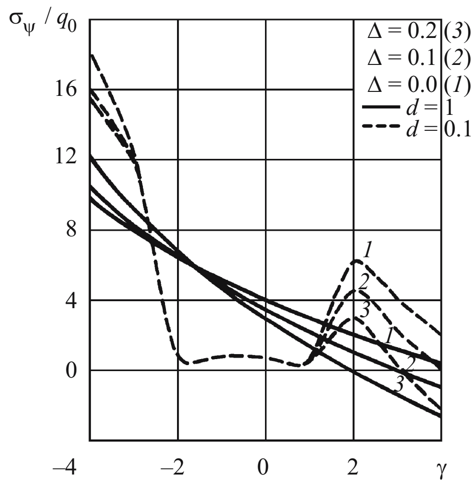

Fig. 15

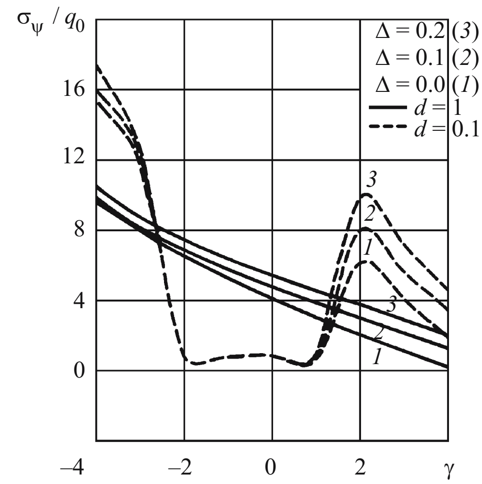

Fig. 16

$$\frac{d\sigma_{\gamma,k}}{d\gamma} = (c_2 - 1)\phi^1_{1,k} + \lambda_n \tau_{s\gamma,k} - \phi^1_{4,k} + b_{22}\phi^5_{1,k} - b_{12}\lambda_n \phi^4_{1,k} + b_{22}\phi^3_{4,k},$$

$$\frac{d\tau_{s\gamma,k}}{d\gamma} = -c_1\lambda_n \sigma_{\gamma,k} - \phi^2_{1,k} - b_{12}\lambda_n \phi^3_{1,k} + b_{11}\lambda_n^2 u_{s,k} - b_{66}\phi_{6,k} - (b_{12} + b_{66})\lambda_n \phi^2_{4,k},$$

$$\frac{d\tau_{\psi\gamma,k}}{d\gamma} = -c_2\phi^1_{3,k} - 2\phi^1_{2,k} - b_{22}\phi_{5,k} + (b_{12} + b_{66})\lambda_n \phi^3_{3,k} - b_{22}\phi_{7,k} + b_{66}\lambda_n^2 u_{\psi,k},$$

$$\frac{du_{\gamma,k}}{d\gamma} = c_4\sigma_{\gamma,k} - c_2\phi^2_{4,k} + c_1\lambda_n u_{s,k} - c_2\phi^3_{1,k}, \quad \frac{du_{s,k}}{d\gamma} = a_{55}\tau_{s\gamma,k} - \lambda_n u_{\gamma,k},$$

$$\frac{du_{\psi,k}}{d\gamma} = a_{44}\tau_{\psi\gamma,k} - \phi^2_{3,k} + \phi^2_{2,k} \tag{9.20}$$

with the boundary conditions

$$\sigma_{\gamma,k} = q^-_{\gamma,k}, \quad \tau_{s\gamma} = q^-_{s,k}, \quad \tau_{\psi\gamma,k} = q^-_{\psi,k} \quad \text{at} \quad \gamma = \gamma_p,$$

$$\sigma_{\gamma,k} = q^+_{\gamma,k}, \quad \tau_{s\gamma,k} = q^+_{s,k}, \quad \tau_{\psi\gamma,k} = q^+_{\psi,k} \quad \text{at} \quad \gamma = \gamma_q \quad (k = \overline{0,K}). \tag{9.21}$$

With (9.17), for each k in Eqs. (9.20), we obtain

$$\phi^j_{1,k} = \phi^j_{1,k}(\gamma, \sigma_{\gamma,l}, \tau_{s\gamma,l}, u_{\gamma,l}, u_{s,l}) \quad (j = \overline{1,5}),$$

$$\phi^j_{2,k} = \phi^j_{2,k}(\gamma, \tau_{\psi\gamma,l}, u_{\psi,l}) \quad (j = \overline{1,2}),$$

$$\phi^j_{3,k} = \phi^j_{3,k}(\gamma, \sigma_{\gamma,l}, u_{\gamma,l}, u_{s,l}) \quad (j = \overline{1,3}),$$

$$\phi^j_{4,k} = \phi^j_{4,k}(\gamma, \tau_{\psi\gamma,l}, u_{\psi,l},) \quad (j = \overline{1,3}),$$

$$\phi_{5,k} = \phi_{5,k}(\gamma, u_{\gamma,l}), \quad \phi_{6,k} = \phi_{6,k}(\gamma, u_{s,l}), \quad \phi_{7,k} = \phi_{7,k}(\gamma, u_{\psi,l}) \quad (l = \overline{0,K}). \tag{9.22}$$

TABLE 14

ψ	H	α = 2			α = 3		
		Δ = 0	Δ = 0.1	Δ = 0.2	Δ = 0	Δ = 0.1	Δ = 0.2
0	–4	7.18	7.58	8.93	9.49	9.60	10.27
	–2	11.00	10.75	11.34	13.19	12.71	13.03
	0	5.91	4.96	4.05	5.74	4.93	4.05
	2	0.61	–0.05	–0.90	–0.20	–0.67	–1.37
	4	–1.01	–1.84	–3.03	–2.46	–2.98	–3.92
π/4	–4	–2.35	–2.70	–3.64	–4.58	–4.91	–5.75
	–2	0.69	0.41	–0.32	–2.21	–2.50	–3.20
	0	5.68	5.82	6.17	4.99	5.06	5.26
	2	4.43	4.70	5.41	5.25	5.48	6.07
	4	6.78	7.28	8.66	9.01	9.50	10.75
π/2	–4	7.18	7.52	8.31	9.49	10.22	11.44
	–2	11.00	12.04	13.53	13.19	14.43	15.97
	0	5.91	6.99	8.13	5.75	6.57	7.26
	2	0.61	1.14	1.58	–0.20	0.08	0.19
	4	–1.01	–0.49	–0.18	–2.46	–2.32	–2.41

Consider a sandwich cylinder. Its face layers are isotropic and have $E_1 = E_3 = E_0$ and $\nu = 0.3$. The core layer is orthotropic and has $E_s = 3.68E_0, E_\psi = 2.68E_0, E_\gamma = 1.1E_0, \nu_{s\psi} = 0.105, \nu_{s\gamma} = 0.405, \nu_{\psi\gamma} = 0.431, G_{s\psi} = 0.5E_0, G_{s\gamma} = 0.45E_0$, $G_{\psi\gamma} = 0.41E_0$.

The results are presented in Tables 13–15 and Figs. 14–16, which show how the displacements u_γ and stresses σ_s and σ_ψ vary throughout the thickness of the cylinder for different values of Δ, α, and ψ. It should be noted (Table 13, Fig. 14) that the maximum displacements u_γ at $\alpha = 2, \Delta = 0.2$ in a cylinder with an orthotropic core, a homogeneous cylinder, and a cylinder with a soft core are in the ratio 1:1.54:3.02; i.e., the cylinders become more compliant in this sequence. Note also that the stress pattern (Tables 14 and 15) is strongly dependent on the orthotropy parameters.

To solve these problems, the values of the subsidiary functions were determined at 80 points to set up discrete Fourier series, and the first 15 harmonics were retained. For stable results, 41 orthogonalization points were used.

Thus, by varying the characteristics of a cylinder, we can choose rational parameters of such structural elements.

This approach was also applied to isotropic and orthotropic elliptic cylinders in [38–40, 44, 49, 53], longitudinally corrugated circular cylinders in [41, 42, 45, 46, 48], and longitudinally corrugated elliptic cylinders in [13, 43, 47].

TABLE 15

ψ	H	α = 2			α = 3		
		Δ = 0	Δ = 0.1	Δ = 0.2	Δ = 0	Δ = 0.1	Δ = 0.2
0	–4	1.04	1.55	2.13	1.78	2.14	2.62
	–2	–1.17	–1.05	–1.14	–0.46	–0.54	–0.63
	0	–0.34	–1.62	–2.85	0.10	–1.23	–2.30
	2	0.09	–0.81	–1.70	–0.10	–0.93	–1.71
	4	–0.25	–1.60	–2.93	–0.70	–1.90	–3.04
π/4	–4	–3.60	–4.10	–5.59	–5.36	–5.89	–7.44
	–2	–6.08	–6.88	–9.33	–8.75	–9.69	–12.52
	0	–0.69	–0.75	–1.06	–1.40	–1.61	–2.36
	2	2.18	2.42	3.05	2.66	2.84	3.30
	4	4.10	4.59	5.98	5.43	5.87	7.05
π/2	–4	1.04	0.51	–0.18	1.78	1.48	1.10
	–2	–1.17	–1.47	–1.95	–0.46	–0.34	–0.08
	0	–0.34	1.18	3.23	0.10	1.92	4.60
	2	0.09	1.12	2.45	–0.10	0.91	2.29
	4	–0.25	1.29	3.30	–0.70	0.76	2.77

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